Beaulieu College

Mathematics Department

GRADE 12

ADVANCED PROGRAMME MATHEMATICS

Preliminary Examination

Time:	3 Hours		300 marks
Date:	22 July 2013		
Examiner:	Ms Smith	Moderator:	Mrs Richard

PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY

1. This question paper consists of 14 pages, including an answer sheet. An information sheet is also included.
2. Answer all the questions on the folio pages except Question 3.1. Question 3,1 must be answered on the **ANSWER SHEET**.
3. Approved, non-programmable, non-graphical calculators may be used, unless otherwise stated.
4. Diagrams are not drawn according to scale.
5. Work neatly and show all the necessary steps in your calculations.
6. If applicable, calculations should be done using radians and answers should be given in radians.
7. Write your name on the question paper, **ANSWER SHEET** and folio pages.
8. Round off answers as indicated in each question.
9. Good luck!

MODULE 1 CALCULUS AND ALGEBRA

Question 1

Use mathematical induction to prove that:

$1^2+2^2+3^2+...+n^2=\frac{n(n+1)(2n+1)}{6}$ for all $n \in N$.

(16)

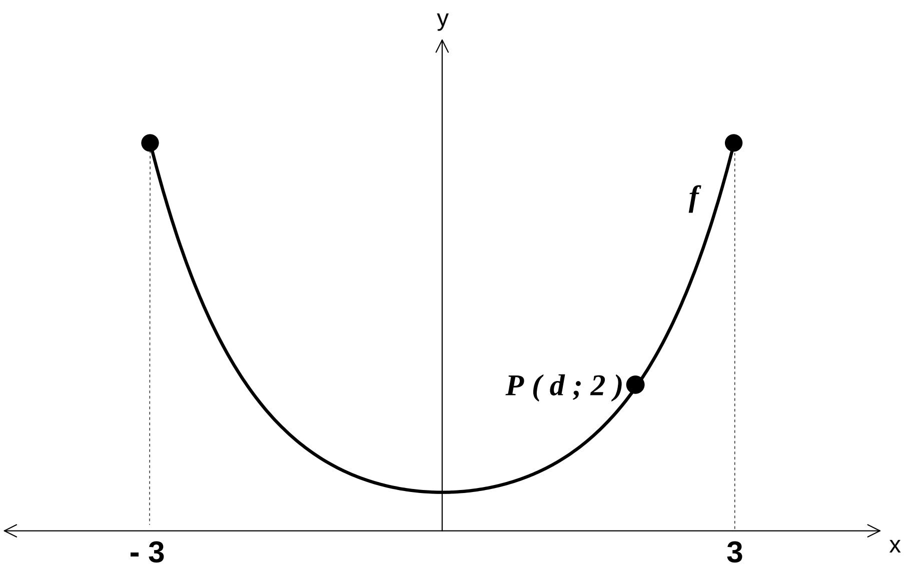
[16]

Question 2

A catenary is the curve formed by a flexible cable with a uniform density if it is suspended between two points. Electricity cables and the cables of suspension bridges are hung in this shape. The equation of a catenary, suspended symmetrically around the line $x=0$, is given as:

$f(x)=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{\frac{-x}{a}}\right)$ where a is the scale factor of the catenary.

An electricity cable, suspended between two poles, 6 metres apart is represented by the catenary sketched below with $a=1$ and $x \in [-3;3]$:



2.1 Determine the minimum height the electricity cable hangs above the ground.

(2)

2.2 Explain the importance of this minimum height in real life.

(2)

2.3 $P(d;2)$ is a point on the catenary. Determine the distance point P lies from the x-axis (d), if the height of the electricity cable at point P is 2 m above the ground. Round off answer to TWO decimal places.

(8)

[12]

Question 3

Given: $f(x)=\left|\ln(x+1)\right|$ and $g(x)=-2x$

3.1 Sketch the graphs of $f(x)$ and $g(x)$ on the graph paper provided on the **ANSWER SHEET**. Clearly indicate the intercepts with the axes and any asymptotes.

(6)

3.2 The graphs of $f(x)$ and $g(x)$ intersect at $x=0$ and one other point. Show that the x-coordinate of the other point of intersection can be determined by solving the equation $e^{2x}=x+1$.

(5)

3.3 Use Newton's method and calculate the x-coordinate of the other point of intersection, rounded off to five decimal places if it is also given that $\dfrac{d}{dx}\left(e^{ax}\right)=a.e^{ax}$.

(9)

3.4 Hence or otherwise, solve for x if: $\left|\ln(x+1)\right|<-2x$

(2)

[22]

Question 4

Given: $g\left(x\right)=x^4-2x^3+14x^2-8x+40$

If $1+3i$ and $1-3i$ are both zeroes of $g\left(x\right)$, factorise $g\left(x\right)$ completely for $x\in C$.

(10)

[10]

Question 5

5.1 $$f\left(x\right)=\begin{cases}\dfrac{\left(x-3\right)}{\left(x+5\right)\left(x-3\right)} & \text{if} \quad x<3 \\ 2^{-x} & \text{if} \quad x\geq 3\end{cases}$$

Determine, with algebraic motivation, whether f is continuous at the following points, and state the type of discontinuity if applicable:

(a) $x=-5$

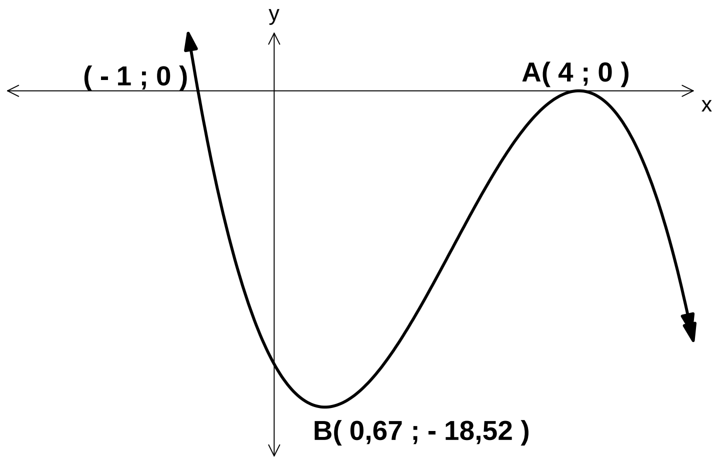
(4)

(b) $x=3$

(6)

(*Please turn over for* ***Question 5.2***.)

5.2 The following sketch shows the graph of $y = g'(x)$. The graph cuts the x-axis at $(-1;0)$ and A$(4;0)$ and it has stationary points at A and B$(0,67;-18,52)$.



The graph of g has two stationary points. Write down the x-values of these points and state, with motivation, the nature of these points. (Remember the sketch represents $g'(x)$)

(7)

[17]

Question 6

6.1 (a) Prove that $$\frac{1}{\sec A - 1} + \frac{1}{\sec A + 1} = 2\cos A.\text{cosec}^2 A$$

(6)

(b) Henceforth, determine $$\lim_{A \to 0}\left(\frac{1}{\sec A - 1} + \frac{1}{\sec A + 1}\right)A^2$$

(5)

6.2 Determine $f'(x)$. It is not necessary to simplify your answers:

(a) $f(x) = 3x^2.\cos 4x$

(5)

(b) $f(x) = \sqrt{x + \sqrt{x}}$

(6)

6.3 $y=\sqrt{4x^2+1}$

(a) Show that $\frac{dy}{dx}=\frac{4x}{y}$

(6)

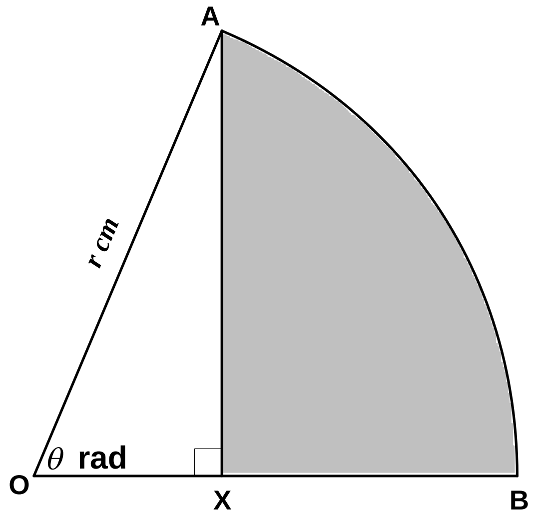
(b) Hence, or otherwise, show that $\frac{d^2y}{dx^2}=\frac{4}{y}-\frac{16x^2}{y^3}$.

(6)

[34]

Question 7

In the diagram, AB is an arc of a circle, centre O and radius r cm and $\mathrm{A\hat{O}B}=\theta$. The point X lies on OB and $\mathrm{AX}\perp\mathrm{OB}$.



7.1 Show that the area, A, of the shaded region AXB is given by

$\mathrm{A}=\frac{1}{2}r^2\left(\theta-\sin\theta\cos\theta\right)$ cm^2. (7)

7.2 In the case where $r=12$ and $\theta=\frac{1}{6}\pi$, find the perimeter of the shaded region AXB, leaving your answer in terms of $\sqrt{3}$ and π.

(8)

[15]

Question 8

Determine the following integrals:

8.1 $\int_0^3 x(x-1)\,dx$

(4)

8.2 $\int \frac{x}{\sqrt{x+4}}\,dx$

(8)

8.3 $\int x\cos 2x\,dx$

(8)

8.4 $\int \cos 5x\cos 8x\,dx$

(7)

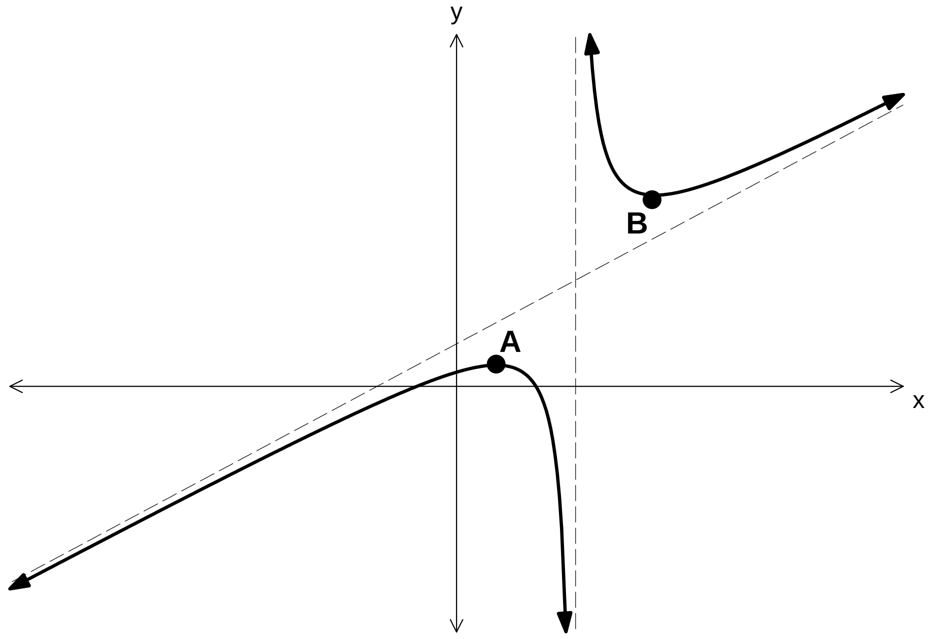
[27]

(*Please turn over for **Question 9**.*)

Question 9

The following sketch shows the graph of $f(x) = \frac{(x+1)(x-2)}{x-3}$.

A and B are the stationary points on the graph.



9.1 Determine the equations of the vertical- and oblique asymptotes.

(8)

9.2 Determine the x-coordinates of the stationary points.

(10)

9.3 Explain why the graph has no points of inflection.

(2)

[20]

(*Please turn over for* ***Question 10***.)

Question 10

Cars are passing over a bridge. It is desirable to set a speed restriction that will ensure an efficient flow of traffic over the bridge.

The "flow-rate", F, is the number of cars passing over the bridge each hour at a velocity of v km/h. The formula for calculating the flow-rate is given as:

$$F = \frac{1\,000v}{0,006v^2 + 35}$$

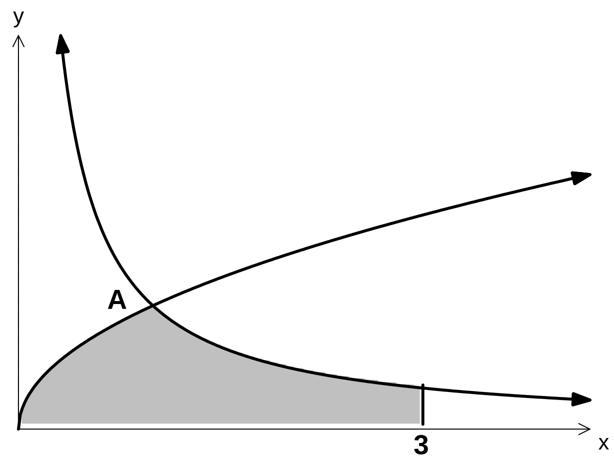
Determine the velocity, rounded off to TWO decimal places, at which the cars should travel in order to maximise the flow-rate of traffic over the bridge.

(12)

[12]

Question 11

The diagram shows parts of the curves $y = \frac{1}{x}$ and $y^2 = x$ which intersect at point A.



11.1 Determine the coordinates of A.

(4)

11.2 The shaded region between the curves, the x-axis and the line $x = 3$ is rotated about the x-axis. Determine the volume of the solid of revolution.

(11)

[15]

Total for Module 1: [200]

MODULE 2 STATISTICS

All answers should be rounded off to four decimal places.

Question 12

Women's heights are said to be normally distributed with a mean of 162 cm and a standard deviation of 6,35 cm. In a random sample, the heights of eight women were measured in centimetres and are given below:

Woman	1	2	3	4	5	6	7	8
Height	154	168	158	174	168	165	162	176

12.1 Conduct a hypothesis test at the 5% significance level to determine whether, based on this sample, it is fair to claim that the mean height of women is 162 cm.

(10)

12.2 Based on this sample determine a 99% confidence interval for the mean height of women.

(8)

[18]

(*Please turn over for **Question 13**.*)

Question 13

A pupil loses his original bivariate data set but finds the following recorded on a piece of paper:

- $\sum x_i = 990$
- $\overline{x} = 66$
- $\sum x_i^2 = 68\,900$
- $\sum y_i = 982$
- $\sum x_i y_i = 67\,680$
- $\sum y_i^2 = 67\,038$

13.1 Show that there are 15 data points in the data set.

(2)

13.2 Determine $\overline{y}$.

(2)

13.3 Show that the least squares regression line of y on x in the form $y = a + bx$ is $y = 12,2971 + 0,8056x$.

(8)

13.4 Use this regression line to estimate the y-value if $x = 54$.

(2)

13.5 Calculate the correlation coefficient by using $r = b \times \frac{S_x}{S_y} = \frac{S_{xy}}{S_x.S_y}$

Given that: $S_x{}^2 = \frac{\sum x_i^2}{n} - \overline{x}^2$ and $S_y{}^2 = \frac{\sum y_i^2}{n} - \overline{y}^2$ and $S_{xy} = \frac{\sum x_i.y_i}{n} - \overline{x}.\overline{y}$

(4)

13.6 Describe the correlation between the variables.

(2)

[20]

Question 14

14.1 At Beaulieu College 20% of all the learners play hockey.
In a random sample of 12 learners find the probability that exactly 7 play hockey.

(9)

14.2 Ms Smith marked 60 Mathematics examination papers and 12 learners failed the examination.
If Mrs Richard selects 5 papers at random to moderate, what is the probability that she will moderate at least one of the papers of the learners that failed?

(8)

[17]

Question 15

A random variable has a probability density function given by:

$$f(x) = \begin{cases} c(x - x^3) & \text{if } 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

15.1 Show that $c = 4$.

(8)

15.2 The mode of a continuous data set is the value of the random variable where it is most dense, i.e. where the density function reaches its maximum value.
Find the mode.

(8)

[16]

(*Please turn over for **Question 16**.*)

Question 16

The two events A and B are such that $\mathrm{P}(\mathrm{A}) = 0,4$, $\mathrm{P}(\mathrm{B}) = 0,24$ and $\mathrm{P}(\mathrm{A}|\mathrm{B}) = 0,25$.

16.1 Prove that the probability that both events occur is $0,06$.

(2)

16.2 Calculate the probability that:

(a) at least one of the events occur.

(2)

(b) exactly one of the events occurs.

(4)

(c) B occurs given A has occurred.

(2)

16.3 Are events A and B independent? Support your answer using calculations.

(3)

[13]

Question 17

17.1 Calculate the number of ways in which 3 girls and 4 boys can be seated in a row of 7 chairs if each arrangement is to be symmetrical.

(10)

17.2 Nine people are to be seated at 3 tables seating 2, 3 and 4 people respectively. In how many ways can the groups at the tables be selected, assuming that the order of seating at the tables does not matter?

(6)

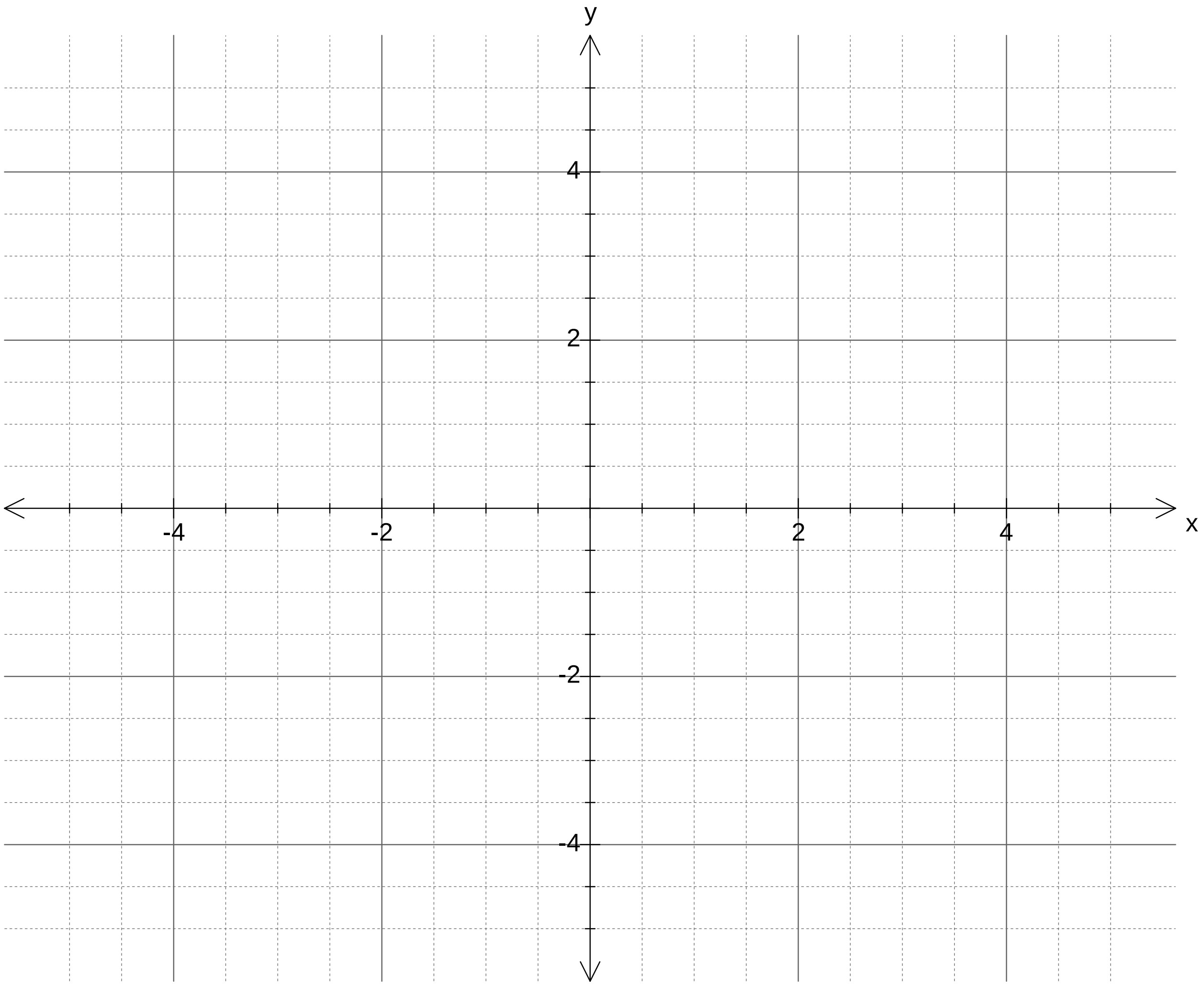
[16]

Total for Module 2: [100]

ANSWER SHEET

Name:____ Grade 12

Question 3.1



(6)