ST JOHN'S COLLEGE

# ADVANCED PROGRAMME MATHEMATICS

## UPPER V

**July 2013** **Time: 3 Hours** **Marks: 300**

**Examiner: SM**

**Moderator: BH**

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### INSTRUCTIONS AND INFORMATION

**Read the following instructions carefully**

1. This paper consists of 10 pages including this page. Please make sure your paper is complete.
2. A formulae sheet is provided.
3. Read the questions carefully.
4. Plan your time carefully.
   **Section A: Calculus and Algebra (215 marks)**
   **Section B: Financial Mathematics (85 marks)**
5. **Section A and B must be answered in separate books**
6. Answer all questions. Rule off after each question.
7. Number your work exactly as the questions are numbered.
8. You may use an approved non-programmable and non-graphical calculator, unless otherwise stated.
9. Round off your answers to one decimal place, where appropriate.
10. All necessary working must be clearly shown.

## SECTION A

### QUESTION 1 [23 marks]

1.1 Solve for x, **without the use of a calculator:**

$\log_3 x - 4\log_x 3 + 3 = 0$ [7]

1.2 Solve for $x$, give your answer to 2 decimal places

a) $2^{3x}.\, 3^{2x} = 100$ [5]

b) $\dfrac{e^x}{e^x - 1} = 5$ [4]

1.3 Solve the simultaneous equations

$\log(x+y) = 0 \quad \text{and} \quad 2\log x = \log(y+1)$

[7]

### QUESTION 2 [13 marks]

Prove by induction that $6\sum_{r=1}^{n} r(r+2) = n(n+1)(2n+7)$. [13]

### QUESTION 3 [16 marks]

**Without the use of a calculator and showing all working**, solve the following equations:

3.1 $|x|^2 - 5|x| - 14 = 0\ ;\ x \in \Re$ [6]

3.2 $x^4 - 6x^3 + 18x^2 - 30x + 25 = 0$, if $x = 1 + 2i$ is one root of the equation and $x \in \mathbb{C}$ [10]

**QUESTION 4 [ 26 marks]**

Given that $f\left(x\right)=\dfrac{x^2+2x+1}{x-1}$

4.1 Calculate the x and y intercepts. [3]

4.2 Find the equations of all the asymptotes. [4]

4.3 Determine $f'(x)$ and simplify your expression. [5]

4.4 Calculate and determine the nature of all stationary points. [8]

4.5 Sketch the graph, labelling all intercepts, stationary points and asymptotes clearly. [6]

**QUESTION 5 [25 marks]**

Determine the following integrals:

5.1 $\int\left(\cos ec^2\left(2-3x\right)+\dfrac{1}{\sqrt[3]{x}}\right)dx$ [5]

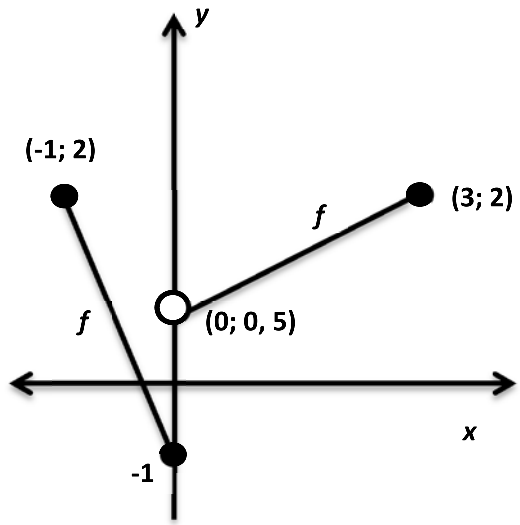
5.2 $\int\dfrac{x}{\sqrt{x^2-2}}dx$ [8]

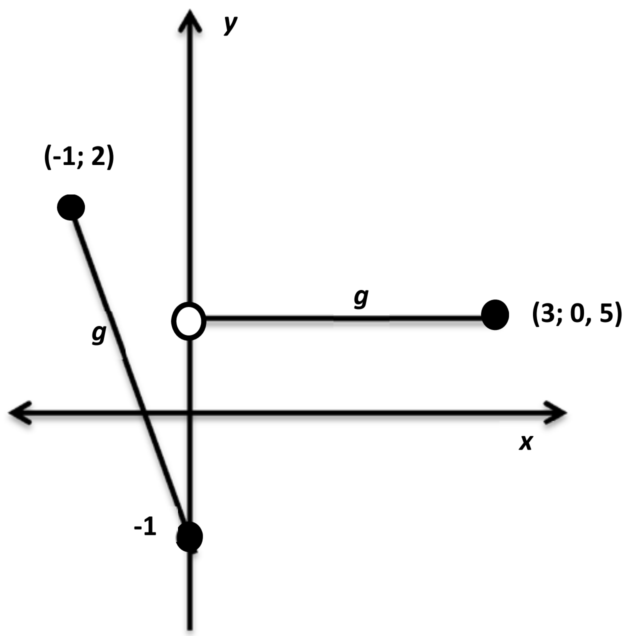
5.3 $\int x\ \sin x\ dx$ [6]

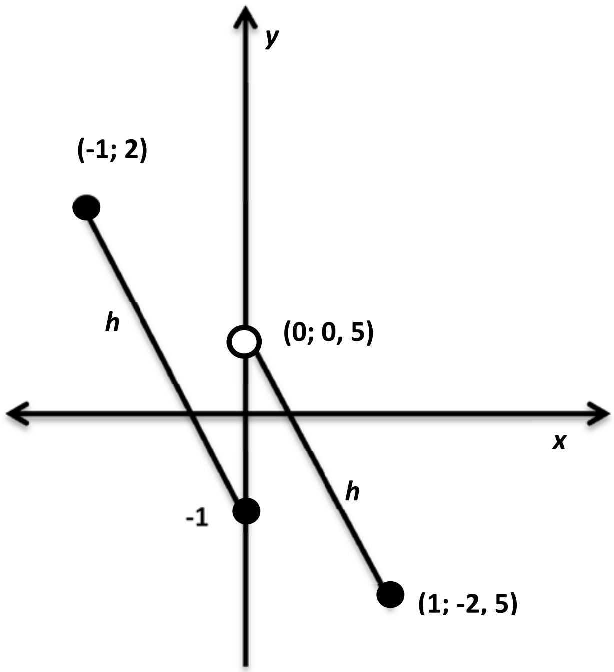
5.4 $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\sin 3x\sin 2x\ dx$ **(Show all workings)** [6]

**QUESTION 6 [21 marks]**

The given sketch graphs represent The functions *f*, g and *h* for the indicated domains. The functions are undefined for any other values of $x$ .




6.1 From the sketches, determine the following limits if they exist.

a) $\lim_{x\to 0^-} f(x)$ [2]

b) $\lim_{x\to 0} g(x)$ [2]

c) $\lim_{x\to 0} h'(x)$ [4]

6.2 Express the following functions in terms of x.

a) $g(x)$ [3]

b) $f(x)$ [3]

c) $f\left(g(x)\right)$ [7]

**QUESTION 7 [20 marks]**

7.1 Prove that $\frac{d}{dx}\left[\sin x.\cos(a-x)\right] = \cos(a-2x)$ [7]

7.2 Find the equation of the tangent of the curve given by the equation:
$x^3-3x^2y+y^3=-1$ at the point $(2;\ 3)$. [13]

**QUESTION 8 [21 marks]**

Given that $f(x)=(2-3x)^{20}$

8.1 Write down

a) $f'(x)$, the first derivative. [3]

b) $f''(x)$, the second derivative. [3]

c) $f'''(x)$, the third derivative. [2]

8.2 Hence, write down the general formula for the n-th derivative,
$f^{(n)}(x)$, of $f(x)$, where n is a natural number. [6]

8.3 Write down, in simplified form

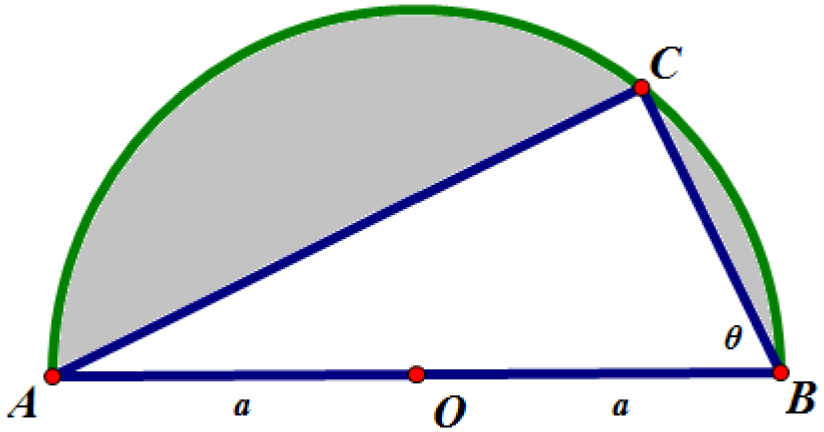
a) The 20th derivative, $f^{(20)}(x)$. [5]

b) The 21st derivative, $f^{(21)}(x)$. [2]

**QUESTION 9 [20 marks]**

In the diagram, ABC is a semicircle, centre O and radius ***a.*** Point C can move along the semicircle such that for any point on the semicircle, $A\hat{C}B = 90^\circ$

$A\hat{B}C = \theta$ radians.




9.1 Find, in terms of ***a*** and $\theta$, the perimeter of the shaded region. [4]

9.2 Show that the area , A, of the shaded region = $a^2\left(\frac{\pi}{2} - \sin 2\theta\right)$ [6]

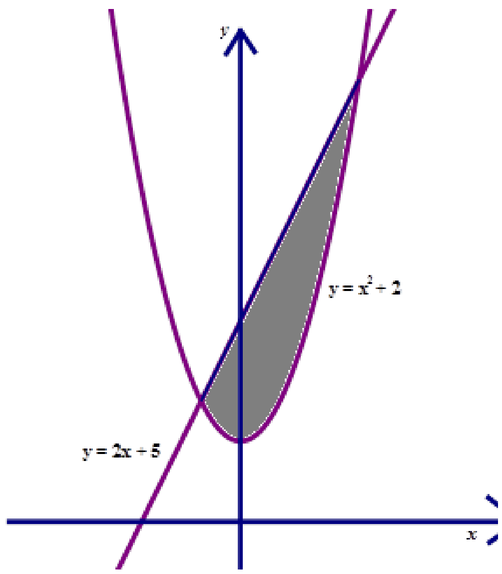
9.3 Write down the domain of $\theta$. [2]

9.4 Find the values of $\theta$ for which the area A is equal to half the area of the semicircle. [8]

**QUESTION 10 [9 MARKS]**

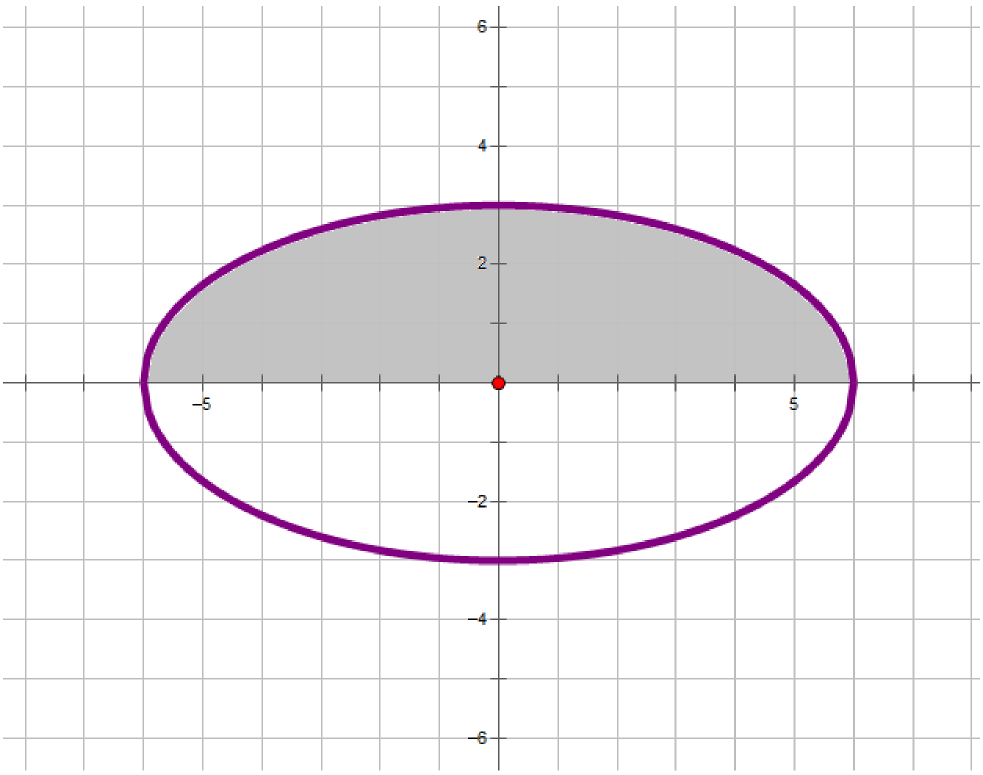
Find the area enclosed by the following curves:

$y = 2x + 5$ and $y = x^2 + 2$



**QUESTION 11 [10 marks]**





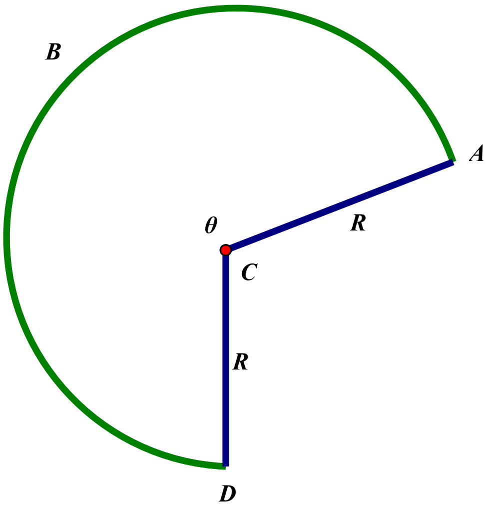

A rugby ball can be modelled by the equation of an ellipsoid. The equation of the ellipsoid is given by $\frac{x^2}{36}+\frac{y^2}{9}=1$.

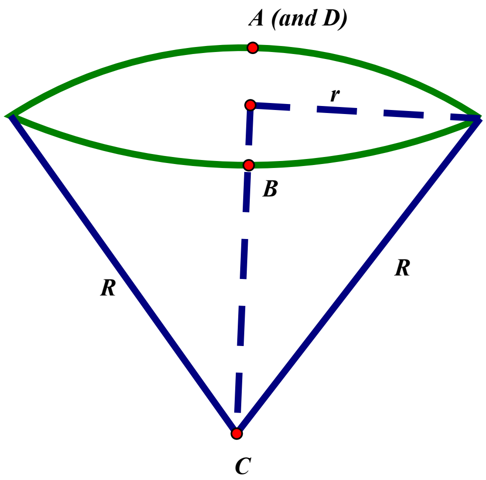
11.1 Express $y^2$ in terms of $x^2$. [2]

11.2 The shaded area is rotated about the $x$-axis to create the rugby ball. Use calculus to find the volume of the rugby ball. [8]

**QUESTION 12 [11 marks]**

The diagram shows a sector of a circular piece of paper of **fixed radius** R. The arc ABD subtends an angle of $\theta$ radians at the centre C.




The second diagram shows the radii AC and DC now joined to form a conical cup with a circular top of radius **r**.

12.1 Write down the length of the major arc ABD in terms of R and $\theta$. [2]

12.2 Show that $r = \dfrac{R\theta}{2\pi}$. [3]

12.3 Show that the volume, V, of the conical cup is given by

$$V = \frac{R^3\theta^2}{24\pi^2}\sqrt{4\pi^2 - \theta^2}.$$ [6]

**[Volume of cone of base radius r and vertical height h is $\frac{1}{3}\pi r^2 h$]**

**END OF SECTION A** **[215]**

**SECTION B**

**FINANCIAL MATHEMATICS**

**ANSWER IN A SEPARATE BOOKLET**

**QUESTION 1 [17 marks]**

James borrows R15 000. He agrees to pay back R5 000 at the end of the first year, a further R8 000 at the end of the third year and a final payment at the end of the fifth year. Interest is calculated at 14% p.a. compounded monthly for the first two years and then reduced to 12.5% p.a. compounded quarterly for the final three years. What is his final payment? [17]

**QUESTION 2 [16 marks]**

Daniel won R1 000 in a raffle during the Easter Rugby Games. He decided to invest this money in a fund paying $i\%$ p.a. compounded monthly. After n years the value of the fund was R1 427, 96 and after another 18 months the value of the fund was R1 541, 23.

2.1 Calculate $i$ [9]

2.2 Calculate n [7]

**QUESTION 3 [24 marks]**

Bradley and Alex each take out a bond of R650 000 to buy properties that they can use while at university.

Bradley decides to pay the bond over 20 years. His monthly payments are calculated at an interest rate of 14, 5% p.a. compounded monthly.

3.1 Show that Bradley's monthly payments are R8 320. [7]

Alex has the same bond over the same period. Alex decides to make his payments every two weeks, his amount each time being half of what Bradley pays each month. His interest rate is 14, 5% p.a. compounded bi-weekly (every two weeks).

3.2 Assuming that a year always has exactly 52 weeks, how long will it take Alex to pay off his bond?(Show your calculations clearly and give your answer in years and months) [9]

3.3 What is Bradley's outstanding balance at the time that Alex's bond is paid off? [8]

**QUESTION 4 [28 marks]**

Franz borrows R35 000 at 15% p.a. compounded monthly and agrees to pay back R800 per month for the first year, R900 per month for the second year and R1 000 per month for the remaining time until the loan is paid back in full (including interest).

4.1 How much will he still owe after the first two years? [12]

4.2 If the balance outstanding after two years is R23 640, 80 how many monthly payments of R1 000 are needed? [8]

4.3 What will be the amount of his final payment, which is less than R1 000? [8]

**END OF SECTION B** **[85]**