GRADE 12 EXAMINATION
NOVEMBER 2013

ADVANCED PROGRAMME MATHEMATICS

Time: 3 hours 300 marks

PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY

1. This question paper consists of 18 pages, an Answer Sheet of 3 pages and an Information Booklet of 4 pages (i – iv). Please check that your question paper is complete.

2. This question paper consists of FOUR modules:

   MODULE 1: CALCULUS AND ALGEBRA (200 marks) is compulsory.

   Choose **ONE** of the **THREE** optional modules:

   MODULE 2: STATISTICS (100 marks) OR
   MODULE 3: FINANCE AND MODELLING (100 marks) OR
   MODULE 4: MATRICES AND GRAPH THEORY (100 marks)

3. Non-programmable and non-graphical calculators may be used, unless otherwise indicated.

4. All necessary calculations must be clearly shown and writing should be legible.

5. Diagrams have not been drawn to scale.

6. Trigonometric calculations in Module 1 should be done using radians and answers should be given in radians.

7. Answer each module in a separate Answer Book.

8. Round off your answers to two decimal digits, unless otherwise indicated. For the Statistics Module, give probabilities correct to 4 decimal places.

MODULE 1 CALCULUS AND ALGEBRA

QUESTION 1

Prove by induction that for all natural numbers, $n \geq 1$ and $r \neq 1$, that:

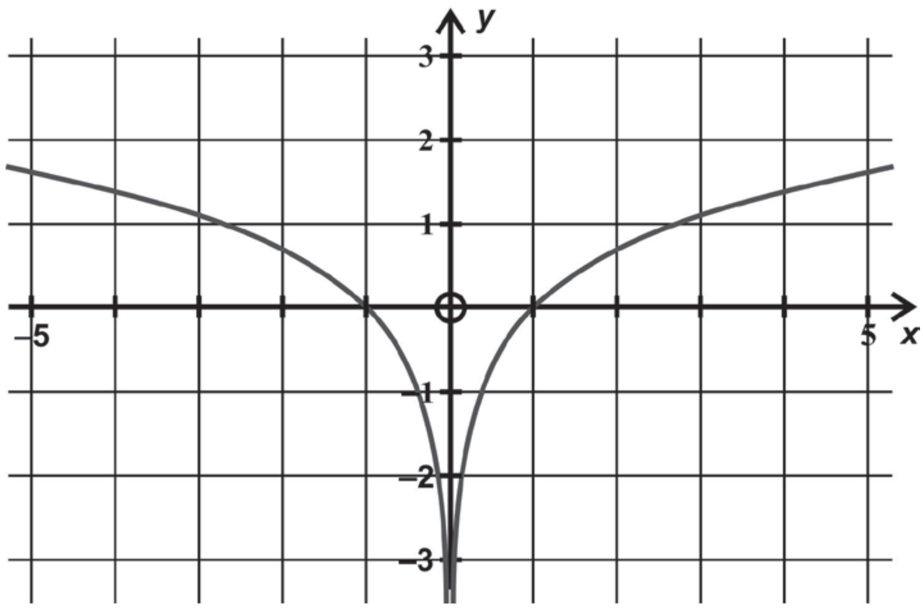
$$\sum_{i=1}^{n} r^{i-1} = \frac{r^n - 1}{r - 1}$$

[14]

QUESTION 2

2.1 Solve for *x*, without the use of a calculator, if $\log_5(x+3) - \log_{\frac{1}{5}}(x-2) - \log_5 2 = 2$ (8)

2.2 Below is the graph of $f(x) = ln|x|$



On the **Answer Sheet** provided draw the graph of $g(x) = |f(x)| - 1$, clearly showing any intercepts with the axes. (10)

2.3 Given: $f(x) = \sqrt{x-2}$ and $g(x) = ln(1 - x^2)$ with $x \in \mathbb{R}$

(a) Find $g(f(x))$. (4)

(b) State the domain $g(f(x))$. (6)

(c) Explain why $g(f(x)) \leq 0$. (4)

[32]

QUESTION 3

Remember $i=\sqrt{-1}$.

3.1 Find the values of a and b, where a and b are real, that satisfy the equation $\dfrac{(a+3i)}{(2-5i)}\times bi=-11-13i$. (8)

3.2 Find the sum $\sum_{n=1}^{4} 2i^n$. (5)

3.3 Find a quadratic equation, with real coefficients, that has $x=\sqrt{3}-i$ as one of its roots. (6)

[19]

QUESTION 4

4.1 Determine, from first principles, the derivative of $f(x)=\dfrac{1}{\sqrt{x}}$. (8)

4.2 Given $f(x)=\begin{cases}\pi-x & \text{if } x\geq\pi\\ \sin x & \text{if } x<\pi\end{cases}$

(a) Show that the function f is continuous at $x=\pi$. (6)

(b) Determine whether the function f is differentiable at $x=\pi$. (6)

[20]

QUESTION 5

Given $f(x) = \dfrac{2x^2 - 7x + 4}{x - 3}$.

5.1 Determine the equation of each asymptote of the function *f*. (8)

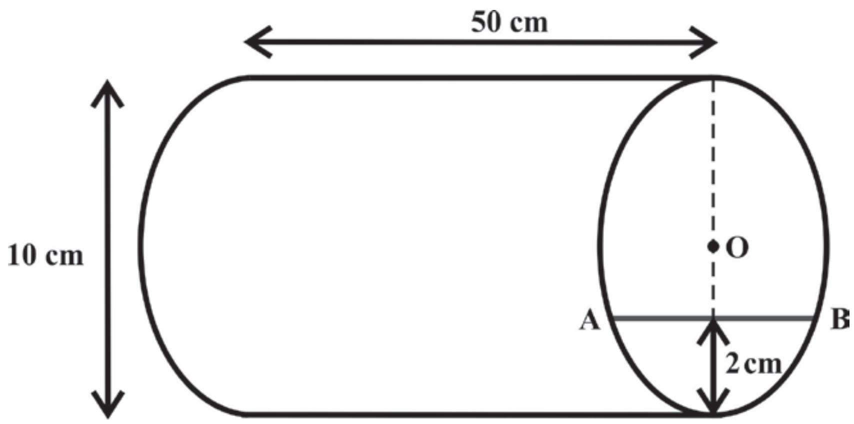
5.2 Determine the coordinates of any turning points of the function *f* (correct to 2 decimal places) and by means of calculation, determine whether they are local maxima or minima. (12)

5.3 Determine the *x* and *y* – intercepts of the function *f*. (6)

5.4 Sketch the graph of $f(x)$ on the axes provided on the Answer Sheet. (8)

[34]

QUESTION 6



A cylindrical pipe with a diameter of 10 cm and length of 50 cm, is lying on its side and there is water inside to a depth of 2 cm.

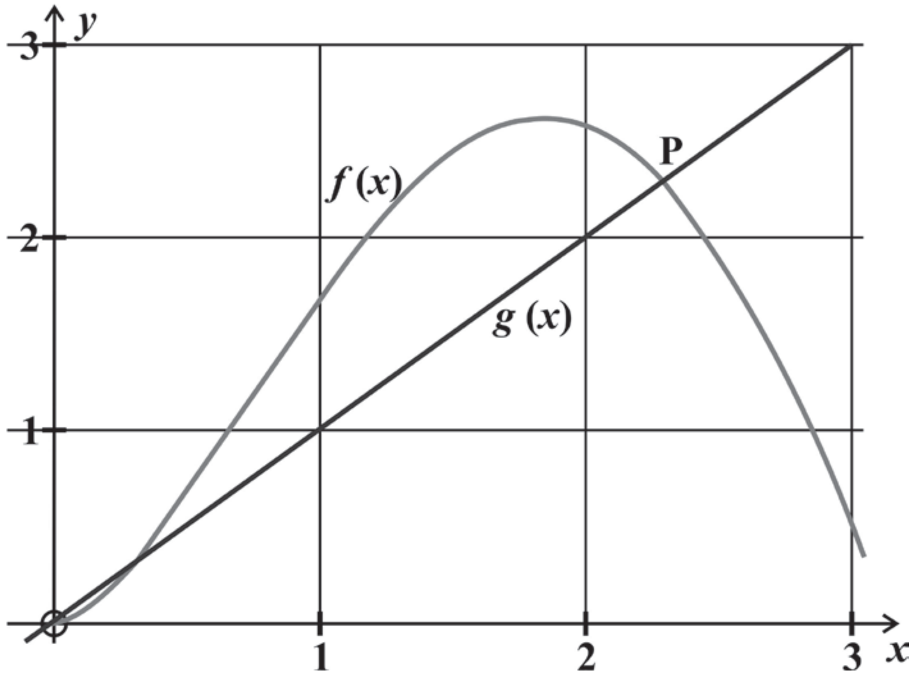
6.1 Show that the width of the surface of the water (the length AB on the diagram) is 8 cm. (4)

6.2 Hence find the volume of water in the pipe. (9)

[13]

QUESTION 7

Below is a sketch of the graphs $f(x) = 2\sqrt{x}\sin x$ and $g(x) = x$. The graphs intersect in three places on the domain $x \in [0;3]$. P is the point of intersection of f and g indicated on the graph.



7.1 Find the derivative $f'(x)$. (6)

7.2 State the iterative formula you would use to solve $f(x) = g(x)$ using the Newton-Raphson Method. (6)

7.3 Determine the x-coordinate of P, correct to 4 decimal places. (4)

[16]

QUESTION 8

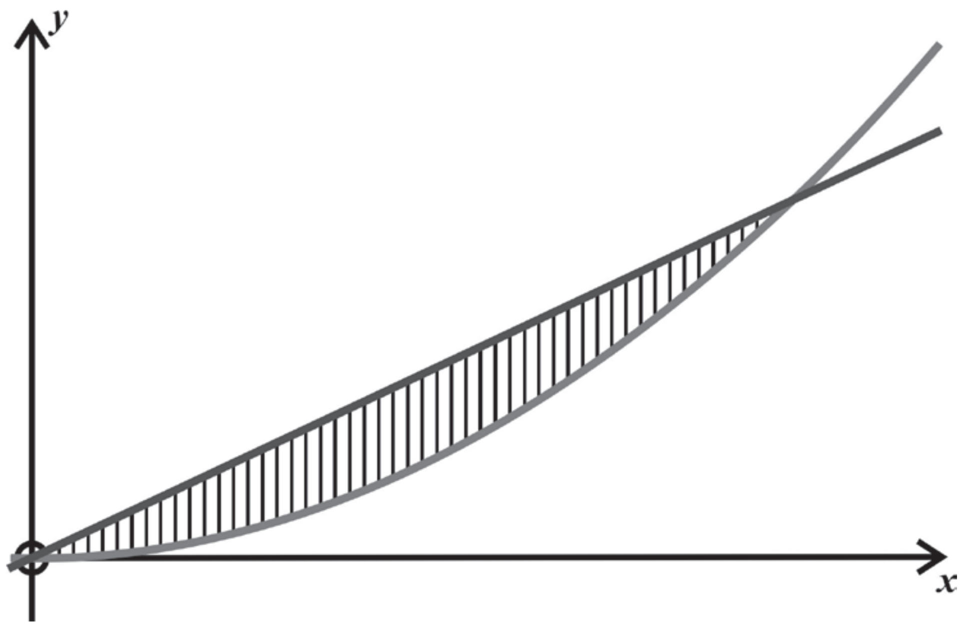
8.1 Find the following integrals:

(a) $\int (2x+1)\cos 2x \; dx$ (10)

(b) $\int x^2 \sec^2(2x^3) \; dx$ (8)

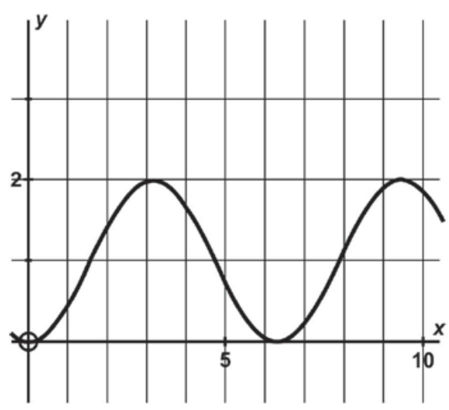
(c) $\int \cot^2 2x.\text{cosec}^2 2x \; dx$ (8)

8.2 Determine the volume of the solid generated by rotating the area bounded by the curves $y = x$ and $y = x^2$ about the x-axis, as indicated by the shaded region below. (10)

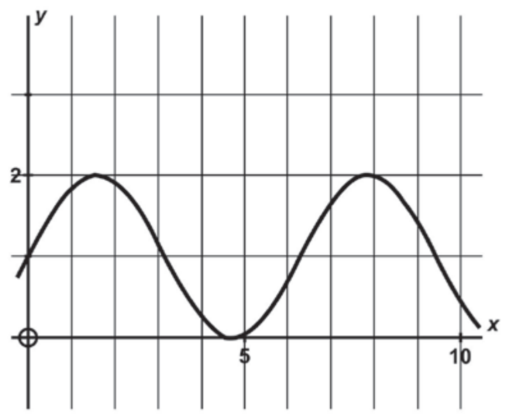


8.3 Below are 4 graphs. Two of the graphs form a pair made up of a function and its integral function. Clearly state the pair and indicate which the function is, and which is the integral function. (6)

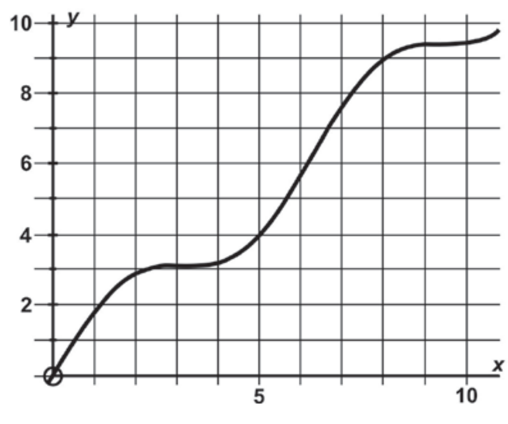
FUNCTION A



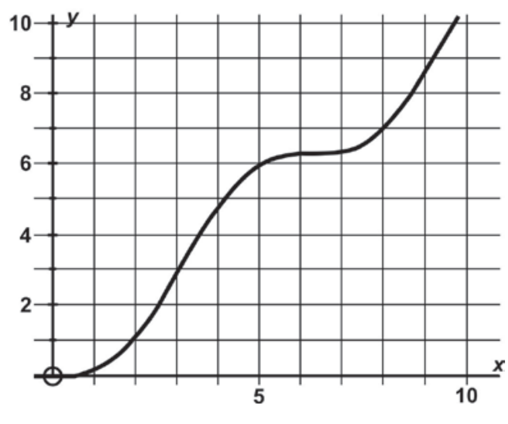
FUNCTION B



FUNCTION C



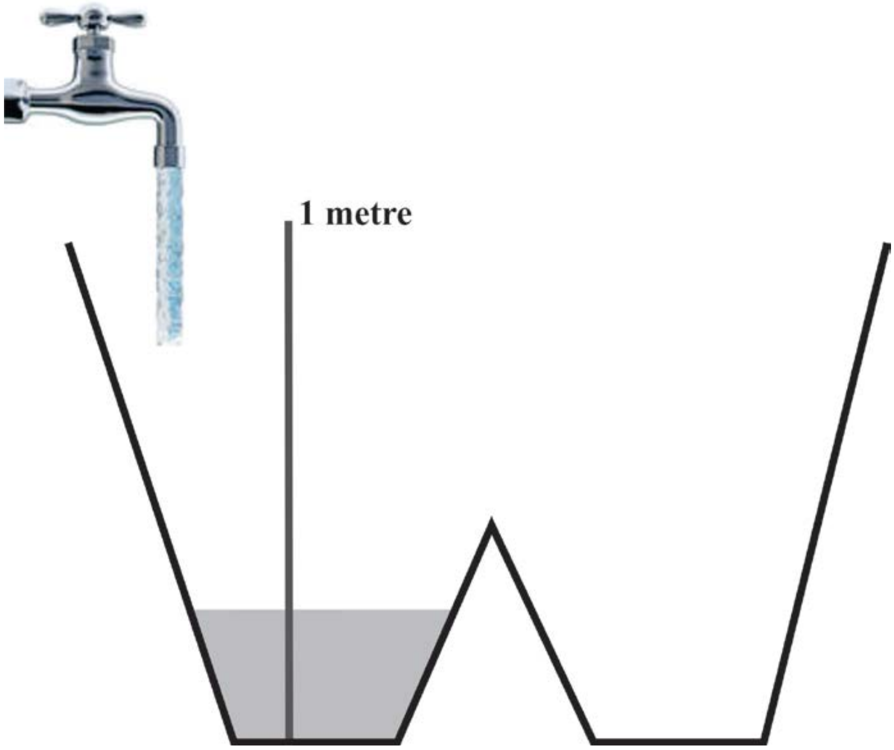
FUNCTION D



[42]

QUESTION 9

There is a small amount of water currently in a W-shaped container shown below. The container is a prism with the W-cross section throughout. A tap is turned on so that the water flows at a constant rate into the left side of the container. Sketch a graph of the height of the water, *h*, as recorded on the meter-stick shown in the left side of the container, as a function of time, *t*. Clearly label your graph showing all salient points and add explanations where you feel it is necessary.



[10]

Total for Module 1: 200 marks

MODULE 2 STATISTICS

QUESTION 1

Ilham and Robyn are analysing the times taken to get to work of a sample of people who travel by car or bus. Their data is summarised in the table.

Car	**Bus**
$\overline{x} = 29{,}6$ min	$\overline{x} = 25{,}2$ min
$\sigma = 5{,}2$ min	$\sigma = 2{,}8$ min
$n = 45$ drivers	$n = 55$ passengers

Ilham concludes on the basis of this data that travelling by bus is quicker on the whole. Robyn is more cautious. She feels that the difference between the means should also be tested at the 4% level of significance.

1.1 Explain why Robyn is wise to do a further test. (3)

1.2 Conduct the appropriate hypothesis test to help clarify Robyn's decision about whether there is a significant difference in the time of travelling by car and bus. (10)

1.3 What is the implication of increasing the level of significance? (2)

[15]

QUESTION 2

2.1 At the matric dance, 12 people (6 couples) are arranged at random in a line for a photograph. Find the probability that each female is standing next to her partner. (6)

2.2 Nine men are eager for selection for a social team of four squash players. Two of them will not play together but each is prepared to play in the absence of the other. In how many different ways can the selection of the team be made? (6)

2.3 Paige and Kerryn are keen archers. The probability that Paige successfully hits a target is 0,7. The probability that Kerryn successfully hits the target is 0,6. Whenever they both independently shoot at the target, what is the probability that:

(a) Only one of them is successful? (6)

(b) At least one of them is successful? (6)

[24]

QUESTION 3

3.1 Laura would like to purchase a second-hand car. She surveyed the ages, x years, and the prices, Ry (in thousands), of ten second-hand vehicles, of a particular type. The data is shown below:

x	6	2	9	5	13	4	7	8	3	10
y	279	599	85	310	65	335	379	135	459	99

(a) Calculate and interpret the correlation coefficient in 'real life' terms. (3)

(b) Calculate the least squares regression line. (4)

(c) Estimate the initial value of a vehicle of this type. (2)

(d) Comment on the validity of the above prediction. (2)

3.2 The regression equation to estimate the number of credit cards a family would use (Y) in terms of the family size (X_1) and the monthly family income (X_2) (in R'000) is given by

$$Y = 0,482 - 0,63X_1 + 0,216X_2$$

What is the likely income of a family of 7 using 3 credit cards? (3)

[14]

QUESTION 4

It is given that the probability density function of the distance x, in metres, that a missile falls from the intended target is:

$$f(x) = \begin{cases} k(x-3) & \text{for } \quad 0 \le x \le 3 \\ 0 & \text{elsewhere} \end{cases}$$

4.1 Show that $k = \frac{-2}{9}$. (5)

4.2 Calculate the probability that the missile will fall within one metre of the intended target. (3)

4.3 Calculate the median distance that a missile falls from the target. (8)

[16]

QUESTION 5

In a certain regiment, the height of the soldiers follows a normal distribution with mean 172 cm and variance of 25 cm^2.

5.1 What percentage of the soldiers have a height between 156 cm and 170 cm? (9)

5.2 If only the shortest 10% of the regiment will be fit for a specific task, what is the maximum height that can be allowed for the task? (6)

[15]

QUESTION 6

A tank contains 10 tagged fish and 50 untagged fish. On each day, 4 fish are selected at random from the tank and placed together in a separate tank for observation. Later the same day the 4 fish are returned to the original tank.

6.1 Show that the probability of selecting no tagged fish on a given day is 0,4723. (6)

6.2 What is the probability of selecting at least one tagged fish on a given day? (4)

6.3 Calculate the probability of selecting no tagged fish every day for seven given days. (6)

[16]

Total for Module 2: 100 marks

MODULE 3 FINANCE AND MODELLING

QUESTION 1

Lindiwe takes out a R2 000 loan from a loanshark, who charges her a weekly interest rate. She agrees to pay back the loan in weekly instalments until the loan has been fully repaid.

This situation can be expressed as a first order recursive equation:

$T_{n+1} = 1{,}08.T_n - 200$, with $T_0 = 2\ 000$

1.1 Give the value of Lindiwe's weekly instalments. (1)

1.2 Give the weekly interest rate charged. (2)

1.3 Calculate the outstanding balances just after her 19th and 20th payments. (3)

1.4 Calculate how much of her 20th payment is interest on the outstanding balance. (3)

1.5 Calculate how much she would have paid the loanshark in total once the loan is fully repaid. (4)

[13]

QUESTION 2

A company immediately starts making monthly contributions into a sinking fund to replace outdated machinery in **eight** years' time. The replacement cost of the machinery is expected to be R2 500 000 and the fund earns an effective interest rate of 8,73% per annum.

2.1 Calculate the percentage increase in the value of the machinery from its current value to its expected replacement value in eight years' time, if the annual inflation rate for machinery is 12,14%. (5)

2.2 Calculate the value to which the sinking fund needs to accrue, if provision has to be made for an annual maintenance contract of R50 000 for the existing machinery at the end of each year for the first **seven** years. (8)

[13]

QUESTION 3

A married couple plans to invest R10 000 every quarter in a bank account, **starting immediately**. The account yields 6,5% interest per annum, compounded quarterly, for the first 12 years, and 6,82% per annum, compounded quarterly, for the next six years. Their last payment will be made nine months before the end of the 18 years.

3.1 Calculate the value to which their investments will accrue. (14)

3.2 The couple intends giving equal amounts to each of their three children (Jen, Ken and Len) on each one's 18th birthday. Jen is two-and-a-half-years old, Ken is one year old and Len has just been born. Calculate what each child will receive, assuming that the couple's investments accrue to R1 375 000 at the end of 18 years. (10)

[24]

QUESTION 4

A logistic population model is described by $\frac{\Delta P}{P} = -0{,}000\,147\,P + 0{,}103$

4.1 Give the value of the intrinsic growth rate for this model. (2)

4.2 Calculate the value of the carrying capacity (correct to the nearest integer value) for this model. (4)

4.3 Determine the population size (correct to the nearest integer value) when the growth rate is 0,095. (4)

4.4 Calculate the population after 15 cycles for this model, if $P_0 = 12$. Use a carrying capacity of 700, and round the final answer to the nearest integer value. (6)

[16]

QUESTION 5

The Kgalagadi Transfrontier National Park straddles the border between the Northern Cape and Botswana.

There are currently 24 000 meerkats living in the Park, which is 80% of the Park's carrying capacity. Females form about 70% of the meerkat population, but only alpha females (which are 15% of the females) mate. Alpha females have 4 litters per year, and each litter has an average size of 3 kits. About 60% of the kits survive until maturity.

The main predator of meerkats is the Black-Back Jackal. There are currently 1 000 jackals in the Park, half of whom are female. Female jackals have one litter of pups per year.

The parameters $b = 0{,}00056$ and $f = 0{,}1116$ are given.

5.1 Calculate the carrying capacity of the Park for the meerkat population. (2)

5.2 Calculate how many meerkats it is predicted the jackals will kill this year. (4)

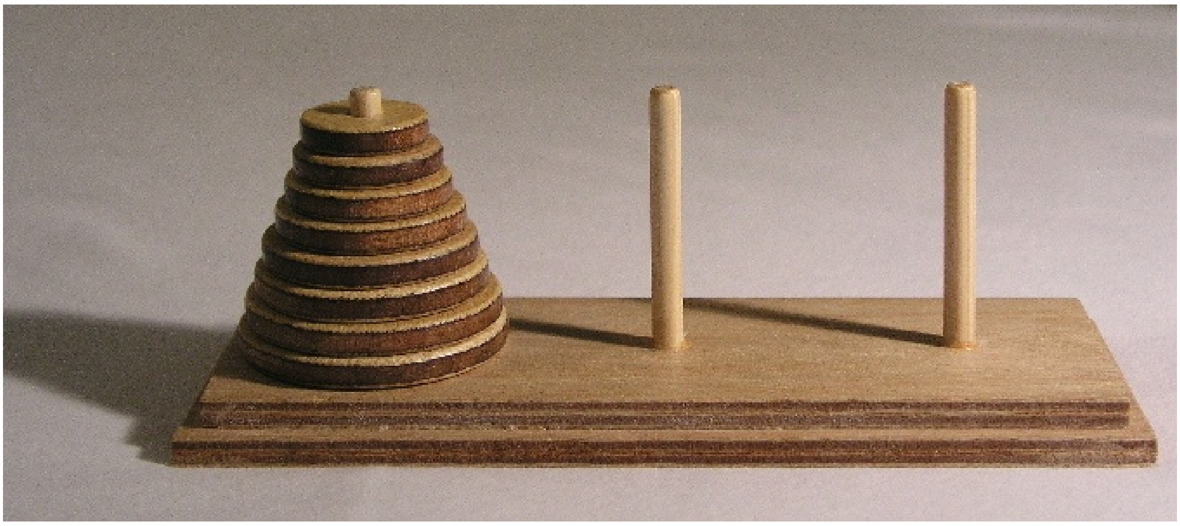
5.3 Calculate the annual intrinsic growth rate, correct to three decimals, for the meerkats. (6)

5.4 Calculate how many surviving pups there are to an average litter of a female jackal. (8)

[20]

QUESTION 6

The **Tower of Hanoi** is a well-known mathematical puzzle popularised by Edward Lucas in the late 19^{th} century. All the rings on the left peg are to be moved, one at a time, and placed on one of the other two pegs. In the process of moving the rings, and in the final product, a larger ring may never be placed on a smaller ring.



If the set consists of three rings, a minimum of 7 moves are necessary.
If the set consists of four rings, a minimum of 15 moves are necessary.
If the set consists of five rings, a minimum of 31 moves are necessary.
If the set consists of six rings, a minimum of 63 moves are necessary.

6.1 State the minimum number of moves necessary for a set of seven rings. (2)

6.2 Set up a **first** order recursive formula that will generate the minimum number of moves required. (4)

6.3 Set up a **second** order recursive formula that will generate the minimum number of moves required. (8)

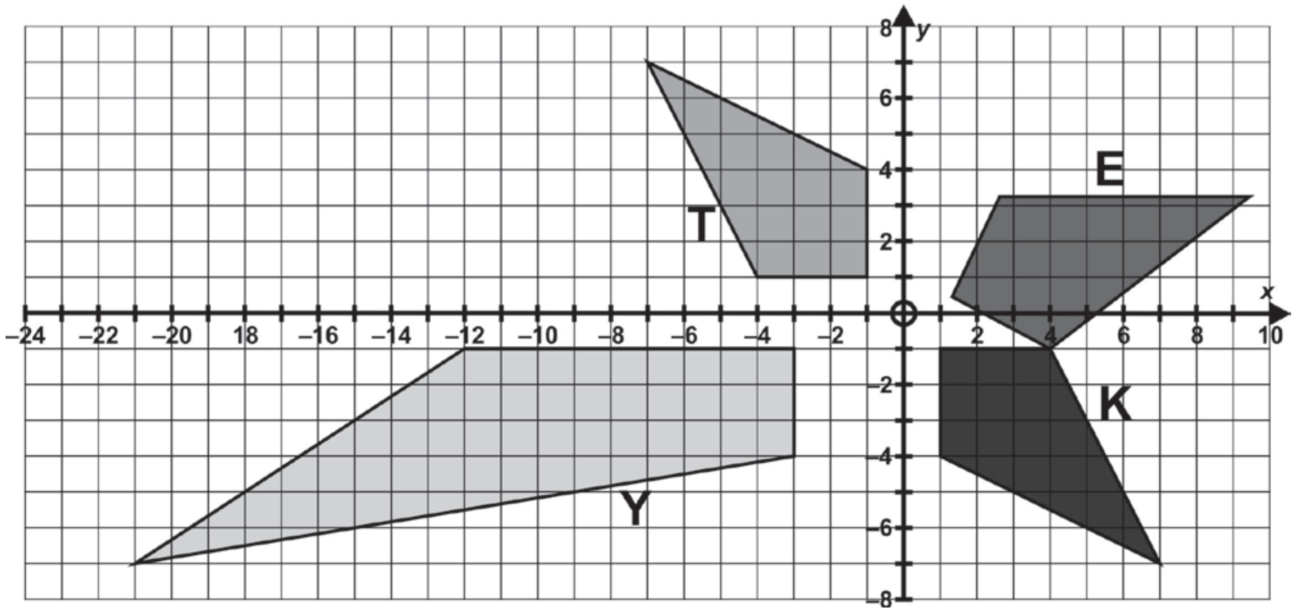
[14]

Total for Module 3: 100 marks

MODULE 4 MATRICES AND GRAPH THEORY

QUESTION 1

1.1 In the diagram below, figure K has been transformed in three different ways.



(a) Name two different ways of transforming K, so that its image is T. (4)

(b) Quote the matrix that would map K onto Y. (4)

(c) K has been rotated through the origin by 63,13° so that its image is E. Use a matrix equation to calculate the coordinates (correct to two decimal places) of the image of the point (1; – 4). (6)

1.2 On the **Answer Sheet** provided, sketch the following transformations of K:

(a) An enlargement, centre the origin, by a scale factor of – 2. (4)

(b) A shear of scale factor 2, with the *y*-axis as the invariant line. (4)

[22]

QUESTION 2

2.1 Give the numerical value of the determinant of the following matrices:

(a) An identity matrix (2)

(b) A matrix in which one column consists only of zeroes (2)

(c) A matrix in which one row is a multiple of another row (2)

2.2 A three-by-three matrix has a determinant of *k*. Give the value of the determinant of the matrix under the following circumstances in terms of *k*:

(a) The transpose of the matrix (2)

(b) Any two columns in the matrix are interchanged (2)

(c) Each element in the matrix is multiplied by a factor of 2 (4)

[14]

QUESTION 3

3.1 The set of simultaneous equations is to be solved by using matrix multiplication:

$2x + y + 2z = 10 \qquad 4x + y + z = 4 \qquad 3y - z = -2$

STEP 1: $$\begin{pmatrix} 2 & 1 & 2 \\ 4 & 1 & 1 \\ 0 & 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (\mathrm{A})$$

STEP 2: $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{20}(\mathrm{B})(\mathrm{A})$$

STEP 3: $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathrm{C}$$

Write down the elements of the matrices A, B and C. (10)

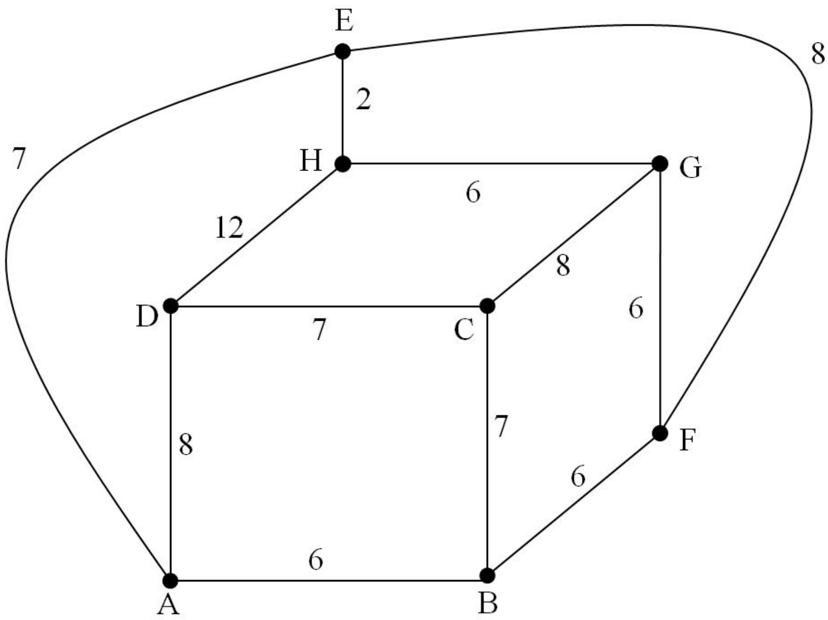
3.2 A matrix M is said to be **idempotent** if it satisfies the identity $\mathrm{M} = \mathrm{M}^2 = \mathrm{M}^3 = \ldots \mathrm{M}^n$.

Calculate the integer value(s) of *b* if the matrix $\begin{pmatrix} \mathrm{b} & \mathrm{b} & 2\mathrm{b} \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix}$ is idempotent. (8)

[18]

QUESTION 4

Consider the given graph and answer the questions that follow:



4.1 Is this a regular graph? Give a reason for your answer. (2)

4.2 It is clear that an Eulerian **circuit** does NOT exist in the graph, as all the vertices are of odd degree. Determine whether an Eulerian **path** exists in the graph. Give a reason for your answer. (2)

4.3 Starting and ending at vertex A, find an upper bound for the graph, using the Nearest Neighbour Algorithm. Clearly indicate the order in which edges are chosen, as well as the girth of this upper bound. (10)

4.4 It is generally accepted that a 'good' Hamiltonian circuit forms about 80% of the girth of the upper bound starting and ending at the same vertex as the circuit required. Find a good Hamiltonian circuit for this graph, starting at A, by inspection. State the circuit, and its length. (8)

[22]

QUESTION 5

Three graphs are given in the form of five-by-five adjacency matrices.

Matrix G

	G_1	G_2	G_3	G_4	G_5
G_1			5		
G_2				3	8
G_3	5				
G_4		3			6
G_5		8		6	

Matrix H

	H_1	H_2	H_3	H_4	H_5
H_1					5
H_2			5		
H_3		8			8
H_4					6
H_5	5		8	6	

Matrix K

	K_1	K_2	K_3	K_4	K_5
K_1		3		5	4
K_2	3				
K_3				8	
K_4	5		8		6
K_5	4			6	

5.1 Draw a graph of matrix K. Clearly indicate the weight of each edge in the graph. (6)

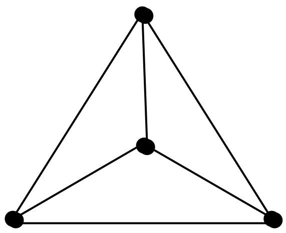
5.2 State which matrix represents a graph that is connected, but not simple. Give a reason for your answer. (4)

5.3 State which matrix represents a graph that is not connected. Give a reason for your answer. (4)

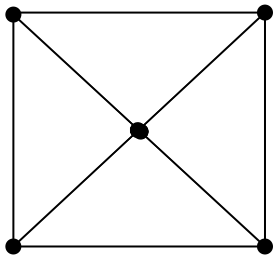
[14]

QUESTION 6

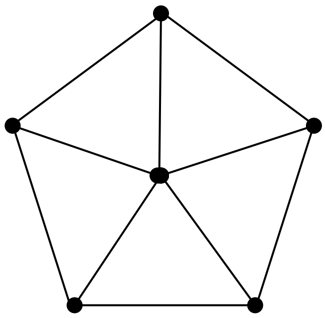
Wheel graphs consist of a central vertex, which is surrounded by a ring of peripheral vertices. Each peripheral vertex is directly connected to the central vertex, as well as to only two adjacent peripheral vertices. Four wheel graphs have been sketched below:



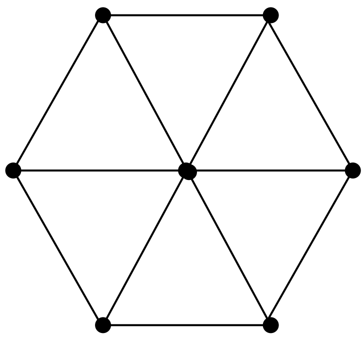
W_4: 4 vertices
6 edges
3 internal regions



W_5: 5 vertices
8 edges
4 internal regions



W_6: 6 vertices
10 edges
5 internal regions



W_7: 7 vertices
12 edges
6 internal regions

6.1 For a wheel graph W_n , where n is the number of vertices, state in terms of n:

(a) The number of internal regions (2)

(b) The number of edges in the graph (2)

(c) The minimum number of edges that need to be added to create an Eulerian circuit for odd values of n (2)

6.2 Sketch the graph W_3, clearly showing the correct number of vertices, edges and internal regions. (4)

[10]

Total for Module 4: 100 marks

Total: 300 marks