GRADE 12 EXAMINATION
NOVEMBER 2013

ADVANCED PROGRAMME MATHEMATICS

MARKING GUIDELINES

Time: 3 hours 300 marks

These marking guidelines are prepared for use by examiners and sub-examiners, all of whom are required to attend a standardisation meeting to ensure that the guidelines are consistently interpreted and applied in the marking of candidates' scripts.

The IEB will not enter into any discussions or correspondence about any marking guidelines. It is acknowledged that there may be different views about some matters of emphasis or detail in the guidelines. It is also recognised that, without the benefit of attendance at a standardisation meeting, there may be different interpretations of the application of the marking guidelines.

MODULE 1 CALCULUS AND ALGEBRA

QUESTION 1

Let $n = 1$

$\text{LHS} = \sum_{i=1}^{1} r^{i-1} = r^0 = 1$

$\text{RHS} = \dfrac{r^1 - 1}{r - 1} = 1$

$\therefore$ true for $n = 1$

Assume true for $n = k$ $\therefore \sum_{i=1}^{k} r^{k-1} = \dfrac{r^k - 1}{r - 1}$

Let $n = k + 1$

$\text{LHS} = \sum_{i=1}^{k+1} r^{i-1} = \dfrac{r^k - 1}{r - 1} + r^{k-1+1} = \dfrac{r^k - 1 + r^{k+1} - r^k}{r - 1}$

$= \dfrac{r^{k+1} - 1}{r - 1}$

$\therefore$ by the principle of Mathematical Induction the statement is true $\forall n \geq 1$ with $r \neq 1$

[14]

QUESTION 2

2.1 $\log_5 \dfrac{(x+3)(x-2)}{2} = 2$

$\therefore \dfrac{(x+3)(x-2)}{2} = 5^2 = 25$

$\therefore x^2 + x - 6 = 50$

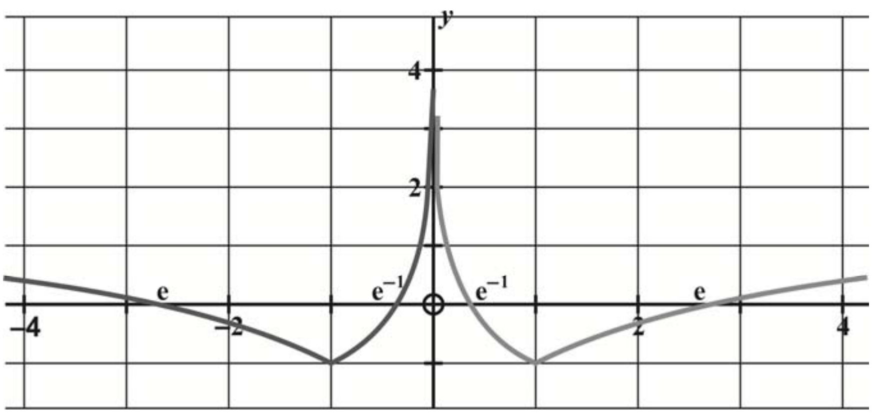
$\therefore x^2 + x - 56 = 0$

$\therefore (x+8)(x-7) = 0$

$\therefore x = -8$ or $x = 7$ but $x \neq -8$

Thus only solution is $x = 7$ (8)

2.2



(marks: basic shape for cusp at (–1;-1) and for cusp at (1:-1)

$x = \pm$ e for $x = \pm$ e^{-1} for vertical asymptote at $x = 0$ (10)

2.3 (a) $g\left(f(x)\right)=\ln\left(1-\left(\sqrt{x-2}\right)^2\right)$ (4)

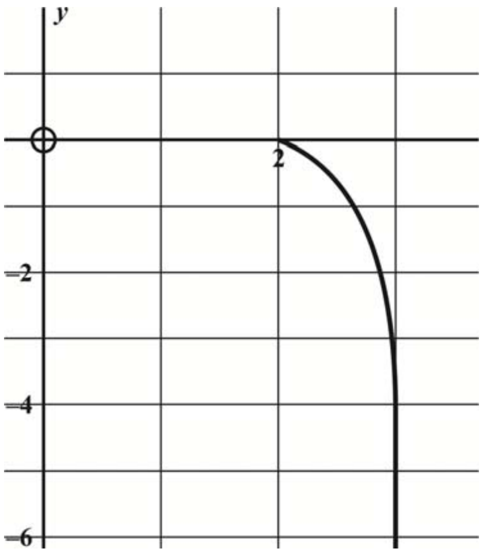
(b) Domain: $x-2\geq 0\ \therefore x\geq 2$ but also $\left(1-\left(x-2\right)\right)>0\ \therefore x<3$

Thus the domain is $2\leq x<3$ (6)

(c) Range : if $x=2$ then $\ln\left(1-0\right)=\ln 1=0$

As $x\to 3\ \ \ln\left(3-x\right)\to -\infty$ (Graph is not necessary but shown below)

Thus $g\left(f(x)\right)\leq 0$ on its domain in $\mathbb{R}[x]$



(4)

[32]

QUESTION 3

3.1 $\text{abi}+3\text{bi}^2=(-11-13\text{i})(2-5\text{i})$

$\text{abi}-3\text{b}=-22+55\text{i}-26\text{i}+65\text{i}^2$

$\text{abi}-3\text{b}=-87\ \ +29\text{i}$

$\therefore \text{ab}=29$ and $-3\text{b}=-87$

$\therefore \text{b}=29$ and $\therefore \text{a}=1$ (8)

3.2 $\sum_{n=1}^{4}2\text{i}^n=2\text{i}+2\text{i}^2+2\text{i}^3+2\text{i}^4$

$=2\text{i}\ -2-2\text{i}+2=0$ (5)

3.3 $\left(x-\left(\sqrt{3}-\text{i}\right)\right)\left(x-\left(\sqrt{3}+\text{i}\right)\right)=\left(x-\sqrt{3}\right)^2-\text{i}^2$

$=x^2-2\sqrt{3}x+3+1$

$\therefore x^2-2\sqrt{3}x+4=0$ is a quadratic equation. Any other permissible one will get full marks.

OR

$x^2-(\sqrt{3}+i+\sqrt{3}-i)x+(\sqrt{3}-i)(\sqrt{3}+i)$

$\therefore x^2-2\sqrt{3}x+4=0$ (6)

[19]

QUESTION 4

4.1 $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \dfrac{1}{h}\left(\dfrac{1}{\sqrt{x+h}} \quad - \dfrac{1}{\sqrt{x}} \quad \right)$

$= \lim_{h\to 0} \dfrac{1}{h}\left(\dfrac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x}\sqrt{x+h}} \right) \times \left(\dfrac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}} \right)$

$= \lim_{h\to 0} \dfrac{1}{h}\left(\dfrac{x - h - x}{\left(\sqrt{x}\sqrt{x+h}\right)\left(\sqrt{x} + \sqrt{x+h}\right)} \right)$

$= \lim_{h\to 0} \left(\dfrac{-1}{\left(\sqrt{x}\sqrt{x+h}\right)\left(\sqrt{x} + \sqrt{x+h}\right)} \right)$

$= \dfrac{-1}{2x\sqrt{x}}$ correct use of limit notation (8)

4.2 (a) $\lim_{x\downarrow\pi} f(x) = \pi - \pi = 0$

$\lim_{x\uparrow\pi} f(x) = \sin\pi = 0$

$f(\pi) = 0$ $\therefore$ continuous at $x = \pi$ (6)

(b) $\lim_{x\downarrow\pi} f'(x) = -1$

$\lim_{x\uparrow\pi} f'(x) = \cos x = -1$

Thus function is continuous and the gradient from the left is tending to the same value as the gradient from the right at $x = \pi$ thus the function is differentiable at $x = \pi$. (6)

[20]

QUESTION 5

5.1 Vertical asymptote $x = 3$

Oblique asymptote $2x^2 - 7x + 4 = (x-3)(2x-1) + 1$

$\therefore f(x) = 2x - 1 \; + \dfrac{1}{x-3}$

Thus the oblique asymptote is $y = 2x - 1$ (8)

5.2 Using $f(x) = 2x - 1 + \dfrac{1}{x-3}$

Then $f'(x) = 2 \; -(x-3)^{-2}$

$\therefore \text{TP}: 2 - (x-3)^{-2} = 0$

$\therefore (x-3)^2 = \dfrac{1}{2}$

$\therefore x = 3 \pm \dfrac{1}{\sqrt{2}}$

$f''(x) = \dfrac{2}{(x-3)^3}$ thus $f''\left(3 + \dfrac{1}{\sqrt{2}}\right) > 0 \;\; \therefore \text{min}$ and $f''\left(3 - \dfrac{1}{\sqrt{2}}\right) < 0 \;\; \therefore \text{max}$

Co-ords $(3,71;7,83)$ Min and $(2,29;2,17)$ Max (12)

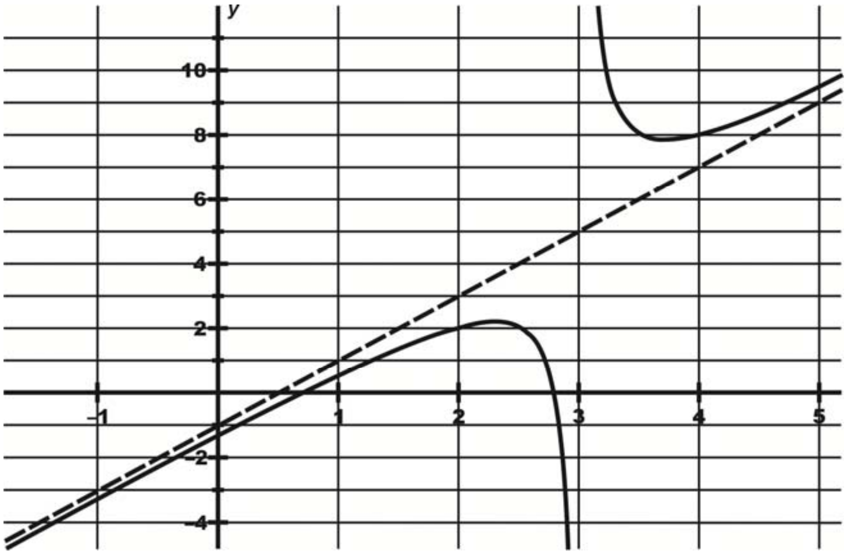
5.3 y-intercept: $f(0) = -1 - \frac{1}{3} = -\frac{4}{3}$

x-intercept: $2x^2 - 7x + 4 = 0$

$$x = \frac{7 \pm \sqrt{17}}{4}$$

(or $x = 2.78$ or $x = 0.72$) (6)

5.4

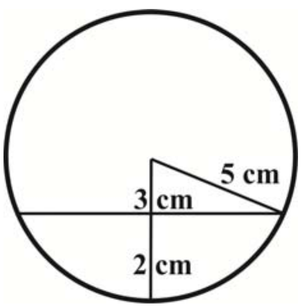


Mark allocation: vertical asymptote
oblique asymptote
basic shape
intercepts with axes
turning points (8)

[34]

QUESTION 6

6.1



Thus 3: 4: 5 triangle so width is 2 × 4 = 8 cm (4)

6.2 $\theta = \cos^{-1}\left(\frac{3}{5}\right) = 0.9273$

X-sectional area $= \frac{1}{2}5^2(2 \times 0.9273) - \frac{1}{2}5^2 \sin(2 \times 0.9273)$

$= 11.1825 \text{ cm}^2$

$\therefore$ Volume $= 50 \times 11.1825 = 559.13 \text{ cm}^3$ or (ml) (9)

[13]

QUESTION 7

7.1 $f'(x)=\frac{1}{\sqrt{x}}\quad \sin x \;+2\sqrt{x}\;\cos x$ (6)

7.2 Require $\mathrm{h}(x)=f(x)-g(x)=0$

Newton Raphson formula: $x_{n+1}=x_n-\dfrac{2\sqrt{x_n}\sin x_n-x_n}{\frac{1}{\sqrt{x_n}}\sin x_n+2\sqrt{x_n}\cos x_n-1}$ with $x_0=2.5$ (6)

7.3 $x=2.2848$ correct to 4DP (4)

[16]

QUESTION 8

8.1 (a) $2\int x\cos 2x\,\mathrm{d}x+\int\cos 2x\,\mathrm{d}x$

Let $\mathrm{g}=x$ then $g'=1$ and $f'=\cos 2x$ then $f=\frac{\sin 2x}{2}$

Thus $2\int x\cos 2x\,\mathrm{d}x+\int\cos 2x\,\mathrm{d}x=2\left(\frac{x\sin 2x}{2}-\int\frac{\sin 2x}{2}\mathrm{d}x\right)+\frac{\sin 2x}{2}+\mathrm{c}$

$=x\sin 2x+\frac{\cos 2x}{2}+\frac{\sin 2x}{2}+\mathrm{c}$

OR

$g=2x+1\quad\therefore g'=2$

$f'=\cos 2x\quad\therefore f=\frac{\sin 2x}{2}$

$\therefore\int(2x+1)\cos 2x\,dx=(2x+1).\frac{\sin 2x}{2}-\int 2.\frac{\sin 2x}{2}dx$

$=(2x+1).\frac{\sin 2x}{2}+\frac{\cos 2x}{2}+c$ (10)

(b) Let $\mathrm{u}=2x^3$ then $\mathrm{du}=6x^2\mathrm{d}x$

Thus $\int x^2\sec^2\left(2x^3\right)\mathrm{d}x=\frac{1}{6}\int\sec^2\mathrm{u}\;\mathrm{du}=\frac{1}{6}\tan\mathrm{u}+\mathrm{c}=\frac{1}{6}\tan(2x^3)+\mathrm{c}$ (8)

(c) Let $u=\cot 2x$ then $du=-\operatorname{cosec}^2 2x.2dx$

Thus $\int\cot^2 2x\operatorname{cosec}^2 2x\,dx=-\frac{1}{2}\int u^2du$

$=-\frac{1}{2}\frac{u^3}{3}+c$

$=-\frac{\cot^3 2x}{6}+c$ (8)

8.2 Volume $= \int_0^1 \pi x^2 \, dx - \int_0^1 \pi x^4 \, dx$

$$\left[\frac{\pi x^3}{3}\right]_{x=0}^{x=1} - \left[\frac{\pi x^5}{5}\right]_{x=0}^{x=1}$$

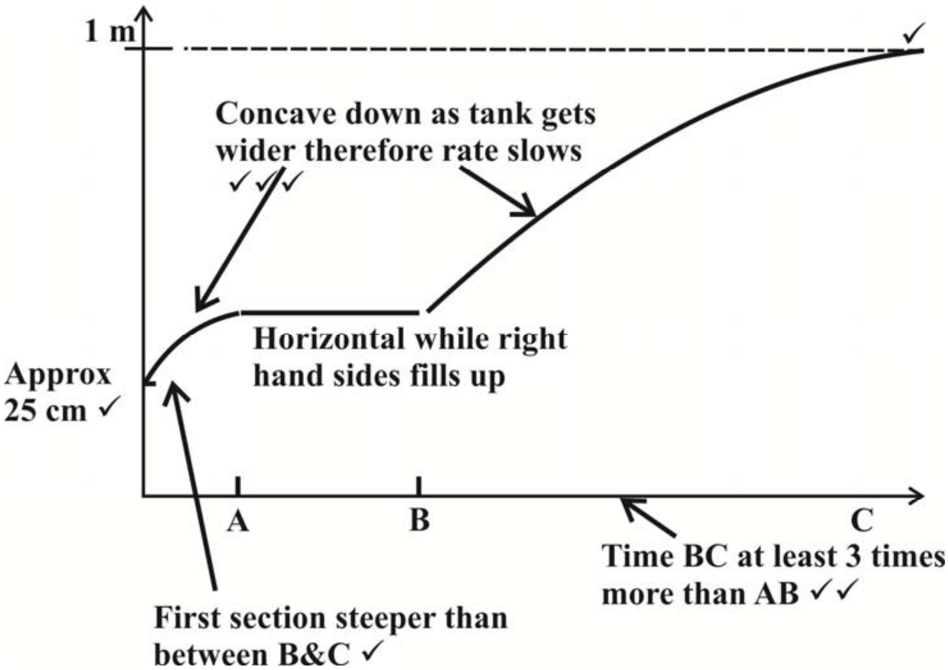
$$\frac{\pi}{3} - \frac{\pi}{5}$$

$$= \frac{2\pi}{15} \text{ units}^3$$ (10)

8.3 Function D is the integral of Function A (6)

[42]

QUESTION 9



[10]

Total for Module 1: 200 marks

MODULE 2 STATISTICS

QUESTION 1

1.1 Travelling by car might have the higher mean, but the bus statistic has a high standard deviation. Random samples have a degree of variability. (3)

1.2 Let x be the car commuters and y the bus commuters.

$H_0 : \mu_y - \mu_x = 0$

$H_1 : \mu_y - \mu_x < 0$

Rejection Region:

Reject H_0 if $Z < -1,75$

Test statistic:

$$z = \frac{(25,2 - 29,6)}{\sqrt{\frac{(2,8)^2}{55} + \frac{(5,2)^2}{45}}} = -5,103$$

Conclusion: We reject H_0 at the 4% level of significance and suggest sufficient evidence to support the claim. (10)

1.3 Margin of error increases. In other words you could reject a null hypothesis when in fact it is true. (2)

[15]

QUESTION 2

2.1 $\frac{6! \times (2!)^6}{12!} = 0,000096$ (6)

2.2 $\binom{7}{4} + 2 \times \binom{7}{3} = 105$ (6)

2.3 (a) $P(P \cap K') + P(P' \cap K) = (0,7)(0,4) + (0,6)(0,3) = 0,46$ (6)

(b) $$\begin{aligned} P(P \cup K) &= P(P) + P(K) - P(P \cap K) \\ &= (0,7) + (0,6) - (0,7)(0,6) \\ &= 0,88 \end{aligned}$$ (6)

[24]

QUESTION 3

3.1 (a) $r = -0,9028$

There is a strong negative linear correlation implying as the age of a vehicle increases so its value decreases. (3)

(b) $y = 591,12488 - 47,25744x$ (4)

(c) R591 124,88 (2)

(d) Unreliable as a new car would not fit the model for second hand cars. (2)

3.2 $3 = 0,482 - 0,63(7) + 0,216(x_2)$

x_2 = R32 074,07 (3)

[14]

QUESTION 4

4.1 $\int_0^3 (\mathrm{k}x - 3\mathrm{k})\mathrm{d}x = 1$

$\left[\frac{\mathrm{k}x^2}{2} - 3\mathrm{k}x\right]_0^3 = 1$

$\frac{9\mathrm{k}}{2} - 9\mathrm{k} = 1$

$\mathrm{k} = \frac{-2}{9}$

OR $\left[\frac{\mathrm{k}(x-3)^2}{2}\right]_0^3 = 1$

$0 - \frac{9\mathrm{k}}{2} = 1$

$\mathrm{k} = -\frac{2}{9}$ (5)

4.2 $\frac{-2}{9}\int_0^1 (x-3)\mathrm{d}x = \frac{5}{9}$ (3)

4.3 $\frac{-2}{9}\int_0^{\mathrm{m}} (x-3)\mathrm{d}x = 0,5$

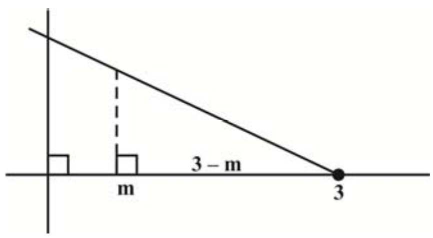
$\left[\frac{x^2}{2} - 3x\right]_0^{\mathrm{m}} = \frac{-9}{4}$

$\frac{\mathrm{m}^2}{2} - 3\mathrm{m} = \frac{-9}{4}$

$2\mathrm{m}^2 - 12\mathrm{m} + 9 = 0$

$\mathrm{m} \neq 5,12$ or $\mathrm{m} = 0,8787$

OR by similar triangles



Ratio of areas = 1 : 2

Ratio of sides = $1 : \sqrt{2}$

$\therefore \frac{3-m}{3} = \frac{1}{\sqrt{2}}$

$\therefore$ m = 0,8787 (8)

[16]

QUESTION 5

5.1 $P(156 < x < 170) = P\left(\frac{156-172}{5} < z < \frac{170-172}{5}\right)$

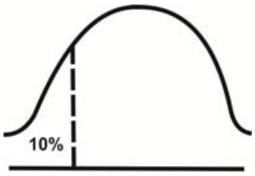
$= P(-3,2 < z < -0,4)$

$= 0,49931 - 0,1554$

$= 0,34391$

$\therefore 34\%$ (9)

5.2



$-1,28 = \frac{x-172}{5}$

$x = 165,6$

(6)

[15]

QUESTION 6

6.1 $P(4\,\text{untagged}) = \frac{50}{60} \times \frac{49}{59} \times \frac{48}{58} \times \frac{47}{57} = 0.4723$

OR $\frac{\binom{50}{4}\binom{10}{0}}{\binom{60}{4}} = 0,4723$ (6)

6.2 $P(\text{at least one}) = 1 - P(\text{none})$

$= 1 - \frac{\binom{10}{0}\binom{50}{4}}{\binom{60}{4}}$ OR $1 - 0,4723$

$= 0,5277$ $= 0,5277$ (4)

6.3 $\binom{7}{7}(0,4723)^7(0,5277)^0 = 0,0052$

(6)

[16]

0 0052

= ,

Total for Module 2: 100 marks

MODULE 3 FINANCE AND MODELLING

QUESTION 1

1.1 **R200** (1)

1.2 **8%** (2)

1.3 **R342,15** and **R169,52** (3)

1.4 342,15 – 169,52 = 172,63
200 – 172,23 = **27,37** (3)

1.5 20 × 200 + 169,52 (1 + 0,08) = **4 183,08** (4)
[13]

QUESTION 2

2.1 $2\ 500\ 000 = A(1 + 0{,}1214)^8$ A = 999 667,55

$$\frac{2\ 500\ 000 - 999\ 667{,}75}{999\ 667{,}75} = 1{,}50083 = 150\%$$ (5)

2.2 $$Fv = \frac{50\,000 \quad (1+0{,}0873) \quad \left[(1+0{,}0873)^7 - 1\right]}{0{,}0873} = 496\ 058{,}21$$

2 500 000 + 496 058,21 = **2 996 058,21**

OR

$$Pv = \frac{50\,000 \quad (1+0{,}0873)^8 \quad \left[1-(1+0{,}0873)^{-7}\right]}{0{,}0873} = 496\ 058{,}21$$

2 500 000 + 496 058,21 = **2 996 058,21** (8)
[13]

QUESTION 3

3.1
$$\frac{10\ 000 \checkmark \left[\left(1+\frac{0,065}{4}\right)^{48}-1\right]\left(1+\frac{0,0682}{4}\right)^{24}\left(1+\frac{0,065}{4}\right)}{\frac{0,065}{4}}=1\ 095\ 834,76$$

$$\frac{10\ 000 \checkmark \left[\left(1+\frac{0,0682}{4}\right)^{22}-1\right]\left(1+\frac{0,0682}{4}\right)^{3}}{\frac{0,0682}{4}}=277\ 996,32$$

1 095 834,76 + 277 996,32 = **1 373 831,08**

OR

$$\frac{10\ 000 \left[\left(1+\frac{0,065}{4}\right)^{49}-1\right]\left(1+\frac{0,0682}{4}\right)^{24}}{\frac{0,065}{4}}=1\ 110\ 839,05$$

$$\frac{10\ 000 \left[\left(1+\frac{0,082}{4}\right)^{21}-1\right]\left(1+\frac{0,0682}{4}\right)^{3}}{\frac{0,0682}{4}}=262\ 992,03$$

1 110 839,05 + 262 992,03 = **1 373 831,08** (14)

3.2
$$x\left(1+\frac{0,0682}{4}\right)^{10}+x\left(1+\frac{0,0682}{4}\right)^{4}+x=1\ 375\ 000$$

***x* = 422 536,25**

OR

$$\left[(1\ 375\ 000-x)\left(1+\frac{0,0682}{4}\right)^{-4}-x\right]\left(1+\frac{0,0682}{4}\right)^{-6}-x=0$$

***x* = 422 536,25** (10)

[24]

QUESTION 4

4.1 r = **0,103** (2)

4.2 $K = -\frac{r}{m} = -\frac{0,103}{-0,000147}$ = 700,68 ≈ **701** (4)

4.3 0,095 = – 0,000 147.P + 0,103 P = 54,42 ≈ **54** (4)

4.4 $P_{n+1} = P_n + 0,103P_n \left(1 - \frac{P_n}{700}\right)$, $P_0 = 12$

P_{15} = 49,62 ≈ **50** (6)

[16]

QUESTION 5

5.1 24 000 × 5/4 = **30 000** (2)

5.2 kills = 0,000 56 × 24 000 × 1 000 = **13 440** (4)

5.3 0,7 × 0,15 × 4 × 3 × 0,6 = **0,756** (6)

5.4 0,1116 × 0,00056 × 1 000 × 24 000 = $\frac{1}{2}$ × 1 000 × pups

Pups = 3 (8)

[20]

QUESTION 6

6.1 127 moves (2)

6.2 $T_n = 2.\ T_{n-1} + 1$, $\mathbf{T_3 = 7}$

OR

$T_{n+1} = T_n + 2^n$, $T_3 = 7$

OR

$T_n = T_{n-1} + 2^{n-1}$, $\mathbf{T_3 = 7}$ (4)

6.3 63 = a.31 + b.15 and 31 = a.15 + b.7

a = 3 b = –2

$\mathbf{T_n = 3.T_{n-1} - 2.T_{n-2}}$ $\mathbf{T_3 = 7}$ **and** $\mathbf{T_4 = 15}$

OR

Each time adding 2, 4, 6, 8 … that is 2^{n-1}

$T_n = T_{n-1} + 2.T_{n-2} + 2$, $\mathbf{T_3 = 7}$ **and** $\mathbf{T_4 = 15}$ (8)

[14]

Total for Module 3: 100 marks

MODULE 4 MATRICES AND GRAPH THEORY

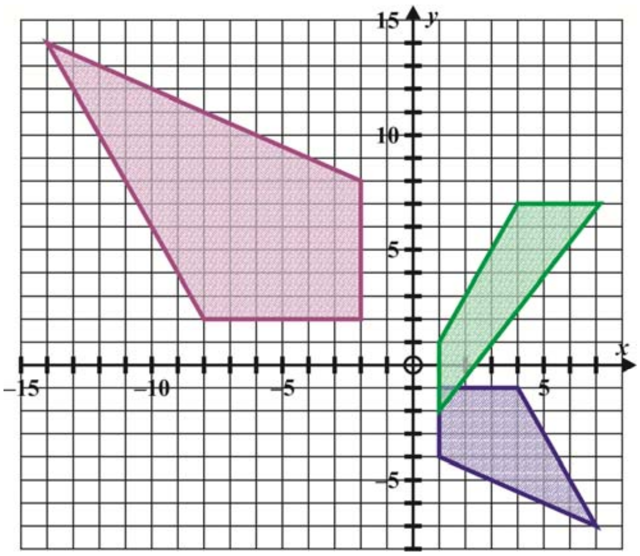
QUESTION 1

1.1 (a) rotation of 180^o about the origin
reflection about $y = x$ (4)

(b) $\begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix}$ stretch factor matrix (4)

(c) $\begin{pmatrix} \cos 63,13 & -\sin 63,13 \\ \sin 63,13 & \cos 63,13 \end{pmatrix} \begin{pmatrix} 1 \\ -4 \end{pmatrix} = \begin{pmatrix} 4,02 \\ -0,92 \end{pmatrix}$ (6)

1.2



(a) working sketch (4)

(b) working sketch (4)

[22]

QUESTION 2

2.1 (a) 1 (2)

(b) 0 (2)

(c) 0 (2)

2.2 (a) k (2)

(b) – k (2)

(c) $2a_{11}.2a_{22}.2a_{33} + 2a_{12}.2a_{23}.2a_{31} + ... - 2a_{13}.a_{22}.a_{31}$
$= 8\left[a_{11}.a_{22}.a_{33} + ... - a_{13}.a_{22}.a_{31}\right]$
$= 8k$ (4)

[14]

QUESTION 3

3.1 $A = \begin{pmatrix} 10 \\ 4 \\ -2 \end{pmatrix}$ $B = \begin{pmatrix} -4 & 7 & -1 \\ 4 & -2 & 6 \\ 12 & -6 & -2 \end{pmatrix}$ $C = \begin{pmatrix} -0,5 \\ 1 \\ 5 \end{pmatrix}$ (10)

3.2 $\begin{pmatrix} b & b & 2b \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix} \bullet \begin{pmatrix} b & b & 2b \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} b & b & 2b \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix}$

$b = b^2$ OR $2b = 2b^2$ **b = 0, 1** (8)

[18]

QUESTION 4

4.1 Yes, all vertices of the same degree. (2)

4.2 No, more than 1 pair of odd vertices. (2)

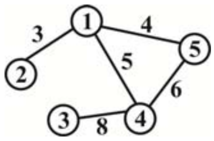
4.3 A B F G H E HD C BA = 60 (10)

4.4 A E H G F B C D A = 49
8 vertices
Hamiltonian circuit
girth ≤ 50
Several options possible. (8)

[22]

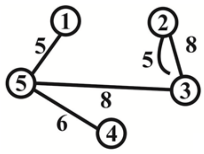
QUESTION 5

5.1 edges connected correctly
edge weight



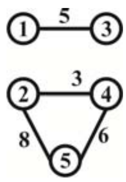
(6)

5.2 H: matrix is not symmetrical
OR double edge between two vertices
OR a sketch



(4)

5.3 G: $G_1 - G_3$ not connected to other edges **OR** a sketch (4)



[14]

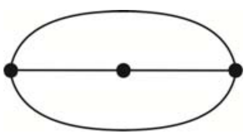
QUESTION 6

6.1 (a) n – 1 (2)

(b) 2n – 2 (2)

(c) (n – 1)/2 (2)

6.2



3 vertices
4 edges
sketch (4)

[10]

Total for Module 4: 100 marks

Total: 300 marks