GRADE 12 EXAMINATION
NOVEMBER 2014

ADVANCED PROGRAMME MATHEMATICS

Time: 3 hours 300 marks

PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY

1. This question paper consists of 18 pages and an Information Booklet of 4 pages (i – iv). An Answer Sheet is also provided for use with Questions 2.1 (a) and 2.2 (a) of Module 1, Question 5.2 of Module 3 and Question 5.1 of Module 4. This should be handed in with your Answer Book. Please check that your question paper is complete.

2. This question paper consists of FOUR modules:

   MODULE 1: CALCULUS AND ALGEBRA (200 marks) is compulsory.

   Choose **ONE** of the **THREE** optional modules:

   MODULE 2: STATISTICS (100 marks) OR
   MODULE 3: FINANCE AND MODELLING (100 marks) OR
   MODULE 4: MATRICES AND GRAPH THEORY (100 marks)

3. Non-programmable and non-graphical calculators may be used.

4. All necessary calculations must be clearly shown and writing should be legible.

5. Diagrams have not been drawn to scale.

6. If applicable, calculations should be done using radians and answers should be given in radians.

7. **Rounding of final answers.** Answers to financial questions should be given to 2 decimal places. Probabilities should be given to 4 decimal places or left as fractions in simplest form. All other answers should be given to 2 decimal places, unless a different level of accuracy is specified in the question.

MODULE 1 CALCULUS AND ALGEBRA

QUESTION 1

Use mathematical induction to prove that $1+3+5+...+(2n-1)=n^2$ for $n \in N$.

[12]

QUESTION 2

2.1 (a) Sketch $f(x)=4e^{1-x}$ and $g(x)=7$ on the axes provided. Show all intercepts with the axes, and any other significant features. (8)

(b) **Hence**, solve for *x*, correct to 2 decimal places, if $4e^{1-x}>7$. (6)

2.2 (a) Sketch the graph of $f(x)=|2x+3|+x+5$ on the axes provided. (8)

(b) Given $g(x)=kx$. For what values of k will the graphs f and g have no intersection? (4)

[26]

QUESTION 3

3.1 The complex numbers $z=5-2i$ and $w=6i-1$ are given.

Determine, in simplest form: $2z-iw$. (6)

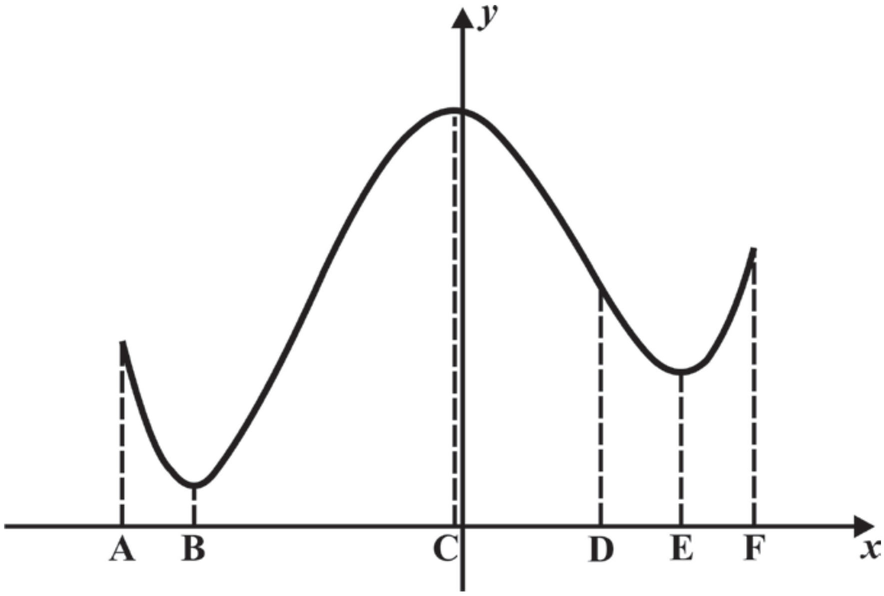
3.2 Decompose $\dfrac{x-10}{2x^2+5x-3}$ into its partial fractions. (10)

3.3 Evaluate: $\lim\limits_{n\to\infty}\left(\dfrac{\sum_{k=1}^{n}(4k-3)}{n^2}\right)$. (8)

[24]

QUESTION 4

The graph f' (the derivative of f) is given below.



4.1 Giving clear reasons, state at which of the marked values of x (A to F) $f(x)$ will have

(a) its greatest value. (4)

(b) points of inflection. (6)

4.2 Is there sufficient information to determine whether the y-intercept of f is positive or negative? Give a reason. (4)

4.3 Determine whether $f(x)$ has a turning point. Motivate your answer. (4)

[18]

QUESTION 5

5.1 Find the derivative of $\dfrac{sin2x}{(2-x)^3}$. (You do not need to simplify your answer.) (6)

5.2 Given the equation $x^3 + y^3 - xy^2 = 5$.

(a) Show that the point (1 ; 2) lies on the curve. (2)

(b) Determine an expression for $\dfrac{dy}{dx}$. (10)

(c) Hence, determine the equation of the tangent to the curve at the point $(1\,;\,2)$. (4)

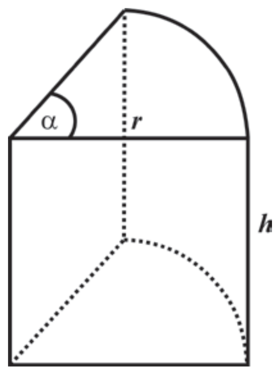
5.3 (a) Find expressions for $\dfrac{dy}{dx}$; $\dfrac{d^2y}{dx^2}$ and $\dfrac{d^3y}{dx^3}$ if $y = \dfrac{1}{bx+c}$. (6)

(b) Hence, determine the formula for the n^{th} derivative of $y = \dfrac{1}{bx+c}$. (6)

[34]

QUESTION 6

A school is having a competition to design crazy containers. **The container is closed at the top and must hold 1 litre.** Wayne designs the Wedge container, which is a sector of a cylinder as shown below with height *h*, radius *r* and angle of sector α. Wayne decides that the height and the radius of the container are equal.



6.1 Prove that an expression for the surface area of the container is $A = 2r^2(\alpha + 1)$. (8)

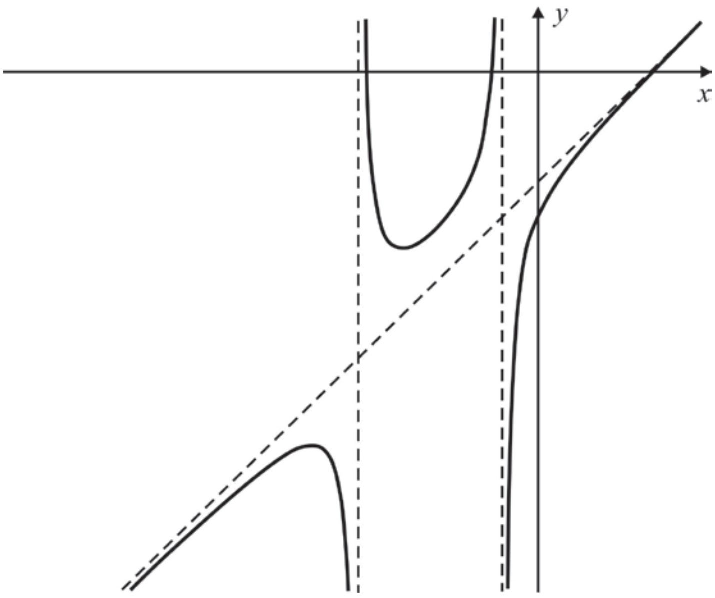
6.2 Given that the volume must be 1 000 cm^3, prove that $\alpha = \dfrac{2000}{r^3}$. (4)

6.3 Hence, determine the value of r and α such that the surface area will be a minimum for the given volume required. (8)

[20]

QUESTION 7

7.1 Below is the graph of the function $f(x)=\dfrac{2x^3+6x^2-25x-39}{x^2+6x+5}$.

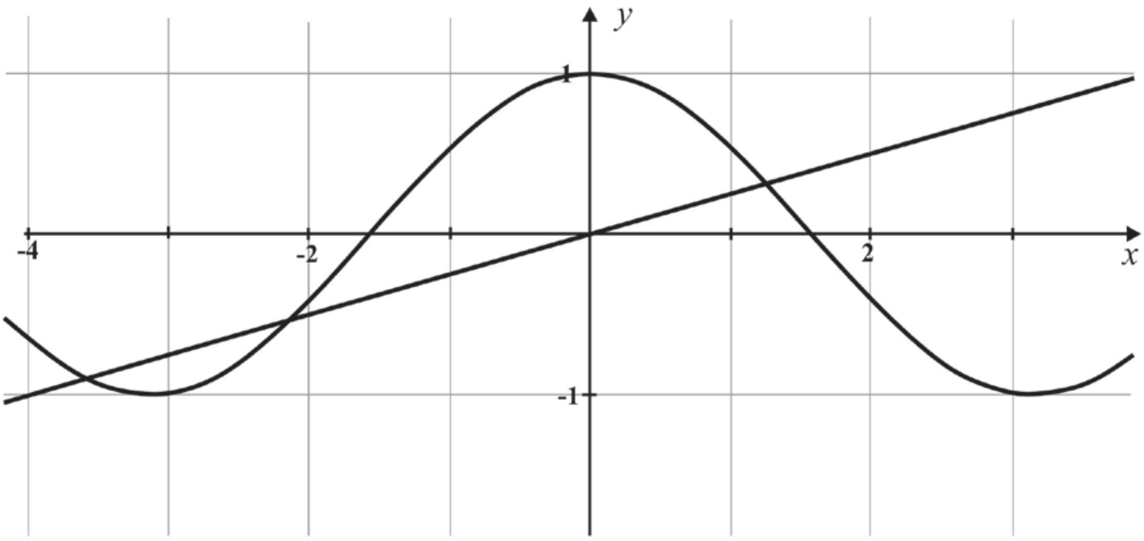


(a) Determine the equations of the vertical asymptotes. (4)

(b) Determine the equation of the oblique asymptote. (8)

(c) Considering the domain $x<0$, use the graph to determine the interval(s) where $f''(x)>0$. (Do not attempt to find $f''(x)$ algebraically.) (4)

7.2 Below is a graph showing $f(x) = 0,25x$ and $g(x) = \cos x$.



(a) There are three solutions to the equation $0,25x = \cos x$ on the interval $x \in (-4; 2)$.

Use Newton's method to find the solution closest to $x = -2$, giving your answer correct to 4 decimal places. (8)

(b) Explain why Newton's method will not work to find a root if the initial estimate for x is $x = \sin^{-1}\left(-\frac{1}{4}\right)$. (2)

[26]

QUESTION 8

Determine the following integrals:

8.1 $\int \cos^2 3x \; dx.$ (8)

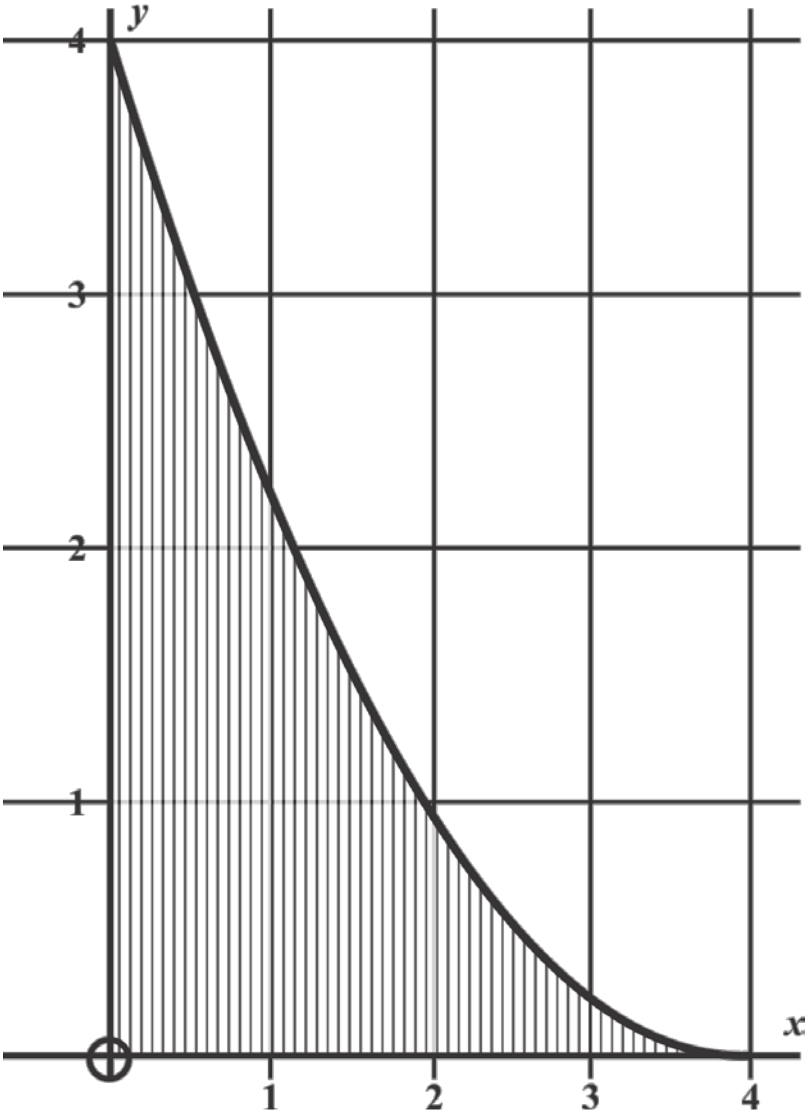
8.2 $\int \cos 2\theta \sin 5\theta \; d\theta.$ (6)

8.3 $\int \frac{5x}{\sqrt{2-x}} \; dx.$ (8)

[22]

QUESTION 9

The shaded region below is the area bounded by the function $f(x)=\dfrac{(x-4)^2}{4}$ and the x-axis and y-axis.



9.1 Determine the volume of revolution if the area is rotated about the x-axis. (8)

9.2 Will the volume of revolution be the same if the above area is rotated about the x-axis or the y-axis? Give a reason to support your answer. (4)

9.3 Calculate the value of a, with $a>0$, such that the volume of revolution of $f(x)$ about the x-axis, between $x=0$ and $x=a$ is 10π units^3. (6)

[18]

Total for Module 1: 200 marks

MODULE 2 STATISTICS

QUESTION 1

1.1 Dr Johnson's research stated that during sleep, the reduction of a person's oxygen consumption has a normal distribution with a mean of 38,4 mℓ/min and a standard deviation of 4,6 mℓ/min.

(a) Determine the probability that during sleep a person's oxygen consumption will be reduced by anywhere from 30 to 40 mℓ/min. (8)

(b) If 90% of people reduce their oxygen consumption by more than k mℓ/min, determine k. (8)

1.2 The proportion of people who watch 'medical dramas' on television for more than 7 hours a week is believed to be 0,32 of the total population.

(a) What size sample should be chosen to estimate the proportion to within 10%, with a 98% confidence interval? (6)

(b) Hence, determine this 98% confidence interval. (4)

[26]

QUESTION 2

2.1 The following statements are written in terms of null and alternate hypotheses. State in each case whether the stated hypothesis is in the correct written notation, where μ is the population mean; σ^2 is the population variance and $\overline{x}$ is the sample mean. (Answer: True if acceptable, False if unacceptable)

(a) $H_0 : \overline{x} = 10 \qquad H_1 : \overline{x} > 10$

(b) $H_0 : \mu = 71 \qquad H_1 : \mu \neq 71$

(c) $H_0 : \mu = 20 \qquad H_1 : \mu \neq 40$

(d) $H_0 : \sigma^2 = 1 \qquad H_1 : \sigma^2 < 1$

(e) $H_0 : \mu_x = \mu_y \qquad H_1 : \mu_x > \mu_y$

(f) $H_0 = 100 \qquad H_1 \neq 100$ (6)

2.2 The manager, George, of Hayden's Ice Cream claims that the average fat content of his product is less than 8%. To verify his claim a random sample of 30 boxes of ice cream was selected and showed an average fat content of 7,92% and a standard deviation of 0,2%. Test George's theory at a 4% level of significance. (10)

[16]

QUESTION 3

3.1 Describe, the relationship between the events A and B, as indicated by each of the following statements:

(a) $P(A|B) = P(A)$ (2)

(b) $P(A|B) = 0$ (2)

3.2 Draw a Venn diagram depicting $P(A|B) = 1$. (4)

3.3 (a) Consider the word TRAPEZIUM. Find the probability that, if four letters are chosen at random, all four letters chosen are consonants. (8)

(b) Four letters are to be selected from the letters in the word RIGIDITY. How many different options are there? (8)

3.4 It is known that 40% of students taking a Bachelor of Arts degree are male. If six students are chosen at random from those students taking a Bachelor of Arts degree, calculate the probability that more than four of the students are **female**. (8)

[32]

QUESTION 4

4.1 Consider the function defined by:

$$P(X = x) = \frac{1}{24}(x+6) \text{ for } x \in \{1\ ;\ 2\ ;\ 3\}$$

(a) Show that $P(X = x)$ is a probability mass function. (4)

(b) Calculate the mean value of *X*. (5)

4.2 In a multiple choice question, five possible answers are given. Only one of the options is correct. The probability that a student will know the correct answer is 0,65. Assume that, if a student does not know the answer, he will guess at random.

(a) What is the total probability that the correct option will be selected? (4)

(b) Given that Sarah selected the correct answer, what is the probability that she guessed? (4)

[17]

QUESTION 5

A statistician, Andrea, discovers the following part of an old research paper, which has some of the information missing.

> giving us mean values $\overline{x} = 21$ and $\overline{y} =$
> The regression line of y on x may be written as $y = -7x + 163$
> Treating x as the independent variable, we calculate that when x is known to be 5, an estimate for the value of y would be

5.1 (a) Calculate $\overline{y}$. (2)

(b) Calculate the estimated y value. (2)

5.2 Using a formula she knows, Andrea works out that the correlation coefficient satisfies $49r^2 - 1 = 0$. Hence, calculate the correlation coefficient correct to 3 decimal places and interpret your findings. (3)

5.3 Comment on the reliability of the estimate calculated above in Question 5.1. (2)
[9]

Total for Module 2: 100 marks

MODULE 3 FINANCE AND MODELLING

QUESTION 1

1.1 A loan of R150 000 is repaid in quarterly instalments of R11 300 at the end of each quarter. Interest is charged at 9% per annum, compounded quarterly. Write down a recursive formula representing the outstanding balance after each payment. (5)

1.2 Office equipment depreciates on a reducing balance to half its value over a period of four years. Calculate how many full years it will take for the equipment to be worth less than one-tenth of its current value. (9)

[14]

QUESTION 2

Chi-Wun repays a bond over 20 years in equal monthly instalments of R8 932,75 at the end of each month. The initial annual interest rate is 6,48%, compounded monthly.

2.1 Calculate her outstanding balance just after her 189^{th} payment. (6)

2.2 Once minor bank charges have been added, assume the outstanding balance just after her 189^{th} payment to be R400 000. At this point in time Chi-Wun's bank changes the interest rate to 6,74% per annum, compounded monthly. Furthermore, she is unable to make the 190^{th}, 191^{st} and 192^{nd} payments. Calculate what her new monthly payments ought to be so that she will still be able to pay back her loan in the remaining time. (8)

2.3 Assuming that the initial value of her bond was R1 200 000, calculate how much interest she will have paid the bank by the time her bond is amortised. (6)

[20]

QUESTION 3

Buhle takes out a home loan for R850 000. Due to the nature of his employment, the bank will allow him to pay off the loan in equal **quarterly** instalments over a period of 15 years. Interest will be charged at 5,8% per annum, compounded quarterly.

However, Buhle realises he can save a considerable amount of money by paying a **monthly** instalment of R7 300 (which is about a third of the quarterly instalment), and that he will pay the loan off in less time as well.

3.1 Calculate, as a percentage correct to four decimal places, the equivalent annual interest rate, compounded monthly. (8)

3.2 Calculate how many months earlier Buhle will amortise his bond, if he decides on monthly payments. (10)

[18]

QUESTION 4

A farmer allows tourists to fish in a large reservoir on his land as a recreational activity. The fish have a birth rate of 0,6 per annum, and a death rate of x % per annum. In addition to these rates, about 2 500 fish are removed from the reservoir each year due to fishing, and the farmer restocks the reservoir with y fish per year.

A Malthusian model describes the fish population of the reservoir:
$F_{n+1} = 0{,}95.F_n + 500, \; F_0 = 5\ 000$

4.1 Calculate the annual death rate and the amount by which the reservoir is restocked every year. (6)

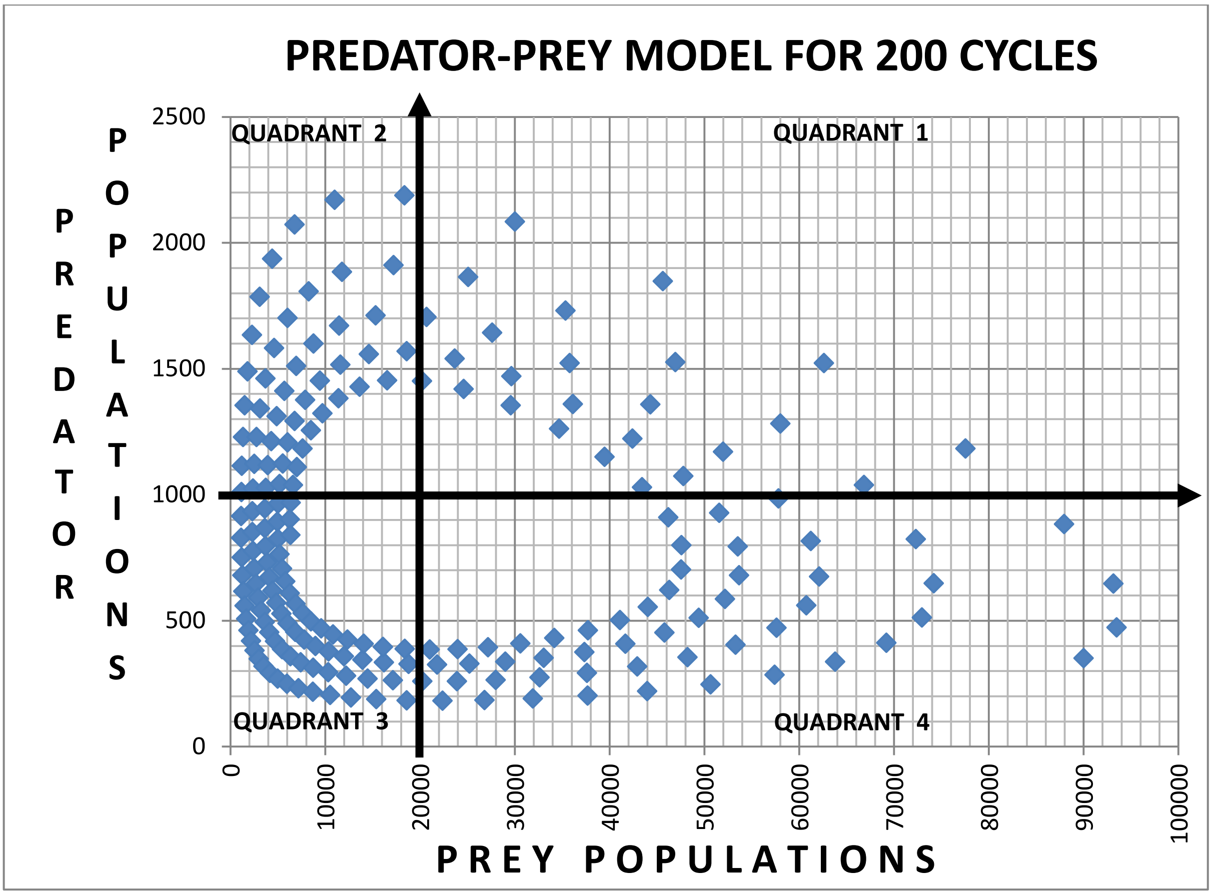
4.2 Determine the population of the fish at the end of 20 years, and discuss the general trend in the population over these years. (5)

4.3 For a first order recursive formula of the form $F_{n+1} = k.F_n + c,$ the value of a general term can be found with the formula $F_n = k^n.F_0 + \dfrac{c[k^n - 1]}{k - 1}$
Use this formula to calculate the population after 100 years. (5)

[16]

QUESTION 5

A phase plot of a predator-prey model, divided into four quadrants, is given below.



5.1 Answer the following questions by analysing the phase plot:

(a) Read off the expected equilibrium point of each species. (2)

(b) Give the range (as an interval) of the prey population. (2)

(c) State in which quadrant the prey population is increasing, while the predator population is decreasing. (2)

(d) Refer to the Lotka-Volterra formulae on the Formulae Sheet. Explain the meaning of the parameter b. (2)

(e) The parameter b is doubled. What possible impact could this have on the prey population? Explain your answer fully. (4)

5.2 Using the phase plot, plot the population of only the prey for the first 15 cycles. Use the **ANSWER SHEET** provided. (8)

[20]

QUESTION 6

Political analysts are quick to study voting trends in order to predict how political parties will fare at election time. Analysts have prepared the following report for two parties P and Q.

From one election to the next, it seems as if P retains 80% of its voters. They also gain 6% of Q's previous voters, in addition to 600 000 first-time voters.

From one election to the next, Q seems to retain 90% of its voters, as well as gain 12% of P's previous voters. They only attract 400 000 first-time voters.

P's voting pattern can be modelled with the formula $P_{n+1} = 0,8P_n + 0,06Q_n + 600\,000.$

6.1 Design a similar model for the voting pattern of Q. (4)

6.2 Over the years the voting patterns of the two parties are tending to an equilibrium. Calculate the number of votes each party will receive at the equilibrium point. (8)

[12]

Total for Module 3: 100 marks

MODULE 4 MATRICES AND GRAPH THEORY

QUESTION 1

1.1 Consider the matrices $A=\begin{pmatrix} a & 2 \\ -1 & 4 \end{pmatrix}$ and $B=\begin{pmatrix} 2 & -1 \\ -2 & b \end{pmatrix}$

(a) Give the value of *a* so that matrix A has no inverse. (4)

(b) Give the inverse of matrix B, where $b=4$. The elements of the matrix must be integer values. (4)

(c) Calculate the values of a and b if $A-2B=\begin{pmatrix} -7 & 4 \\ 3 & 4a \end{pmatrix}$. (6)

1.2 Three matrices M, N and I are given.
M has dimensions $p \times q$; N has dimensions $q \times r$; and I is an identity matrix. Answer the following questions in terms of *p*, *q* and/or *r*.

(a) Give the dimensions of the matrix M^T, the transpose of M. (2)

(b) Give the dimensions of the matrix product MN. (2)

(c) Give the dimensions of I, if the product NI can be calculated. (2)

[20]

QUESTION 2

2.1 Write down a single matrix for each of the following transformations of a figure in a Cartesian plane:

(a) a reflection in the line $y=-x$. (2)

(b) the figure is firstly sheared by a factor of 3, with the *y*-axis as invariant line, followed by a reflection in the line $y=-x$. (6)

2.2 Point T (–1 ; 2) is to be rotated in an anticlockwise direction repeatedly about the origin so as to create the vertices of a regular octagon in a Cartesian plane. Calculate the coordinates (correct to three decimal places) of the first vertex that T will create. (6)

2.3 The transformation matrix $\begin{pmatrix} -0,8 & -0,6 \\ -0,6 & 0,8 \end{pmatrix}$ represents a reflection in a line through the origin. Calculate the equation of the line of reflection. (10)

[24]

QUESTION 3

Consider the matrices $P = \begin{pmatrix} 4 & -1 & 0 \\ 2 & 2 & 1 \\ 6 & 2 & 3 \end{pmatrix}$ and $Q = \begin{pmatrix} 4 & 0 & a \\ -3 & b & 14 \\ -1 & 4 & c \end{pmatrix}$

3.1 Write down the determinant of P. (2)

3.2 Write down the transpose of P. (2)

3.3 Calculate the values of a, b and c, if it is given that Q is P's matrix of minors. (6)

[10]

QUESTION 4

Consider the four situations described below and then choose the algorithm that is best suited to the given situation. An algorithm may be chosen more than once, or not at all.

Algorithms: Fleury, Nearest Neighbour, Prim, Dijkstra.

4.1 Tebogo uses her GPS to determine the shortest route (in terms of time) from one end of Johannesburg to another in rush hour traffic. She needs to get from her work to her home. (2)

4.2 Chloe is a technician and needs to connect the computer terminals in a network, using the least possible amount of cable to minimise costs. Each terminal does not have to be directly connected to every other terminal. (2)

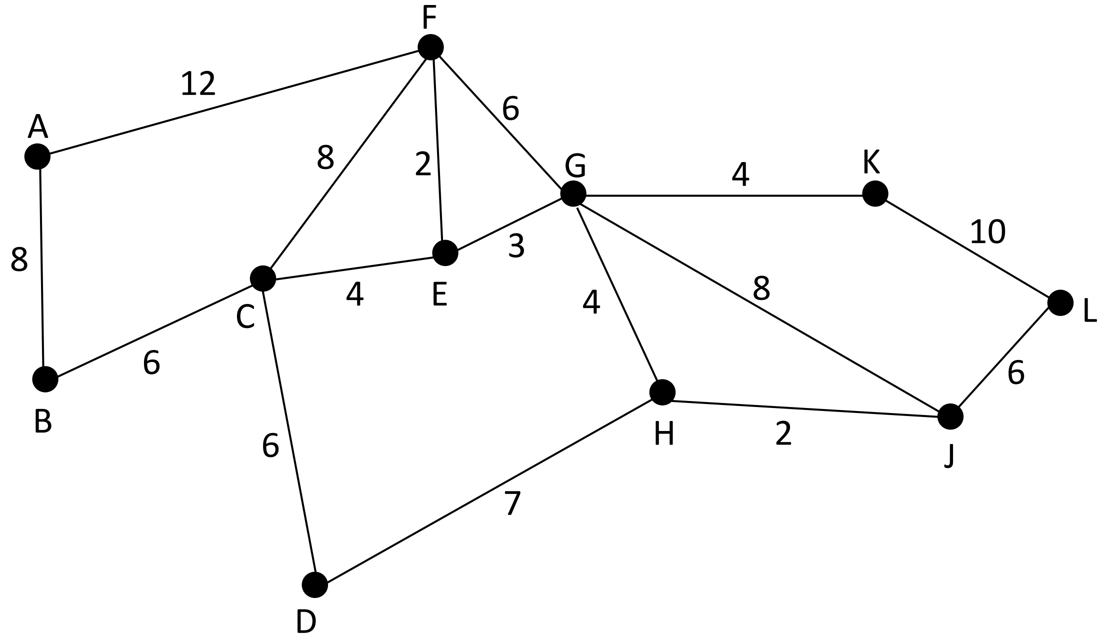
4.3 Pranav needs to regularly inspect every water pipe in his municipal area for leaks. By now he has worked out the shortest route necessary to look at every pipe. (2)

4.4 Matthew is a businessman and needs to visit several designated towns before returning home. He wants to minimise the distance he needs to travel between the towns. (2)

[8]

QUESTION 5

Study the following graph, and then answer the questions which follow.



5.1 On the **ANSWER SHEET** an incomplete adjacency matrix of the graph has been drawn. Six edges are still missing. Fill in the weight of those edges on the **ANSWER SHEET**. (6)

5.2 Use Dijkstra's Algorithm to determine the shortest route to travel, starting from vertex A and ending at vertex L. Clearly indicate the process (e.g. in-out box or table or tree diagram) by which you have chosen the edges. State the route that you have chosen, as well as the length of this route. (12)

5.3 State which two edges need to be **removed** from the original graph, so that the edges which remain form an Eulerian circuit. (4)

5.4 **Dirac's theorem** states: A Hamiltonian circuit can be found in a graph consisting of more than three vertices, if every vertex in the graph is directly connected to at least half the remaining vertices.

(a) Show that the given graph does NOT statisfy Dirac's Theorem. (2)

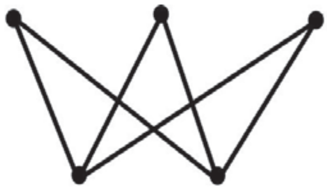
(b) Despite the fact that in this graph Dirac's Theorem is not satisfied, a Hamiltonian circuit can indeed be found. Explain the apparent anomaly in this situation. (2)

[26]

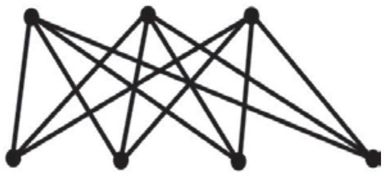
QUESTION 6

Three examples of **bipartisan** graphs are sketched below. These graphs are all of the form $K_{m,n}$ where *m* and *n* are the number of vertices in each subset of the graph.

$K_{1,2}$ with 3 vertices and 2 edges



$K_{3,2}$ with 5 vertices and 6 edges



$K_{3,4}$ with 7 vertices and 12 edges

6.1 Give in terms of *m* and *n* the number of

(a) vertices in $K_{m,n}$. (2)

(b) edges in $K_{m,n}$. (2)

6.2 $K_{m,n}$ has 10 edges. How many vertices can there potentially be? (4)

6.3 Sketch the bipartisan graph $K_{4,2}$. Clearly show the grouping of vertices, and the manner in which the edges are connected. (4)

[12]

Total for Module 4: 100 marks

Total: 300 marks