REDHILL HIGH SCHOOL

PRELIMINARY EXAMINATION JULY 2017

ADVANCED PROGRAMME MATHEMATICS

Examiner: Mr R. Rhodes-Houghton　　Moderator: Prof J Ridley

Time: 3 Hours　　300 marks

PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY

1. This paper consists of 7 pages and an Information Booklet of 4 pages (i – iv). Please check that your question paper is complete.

2. This question paper consists of TWO modules which must be answered **SEPARATELY**:
   MODULE 1: CALCULUS AND ALGEBRA (210 marks)
   MODULE 2: STATISTICS (90 marks)
   Plan your time carefully.

3. Non-programmable and non-graphical calculators may be used but not the SOLVE button on the CASIO *fx-991ZA PLUS*.

4. All necessary calculations must be clearly shown and writing should be legible.

5. Diagrams have not been drawn to scale.

6. If applicable, calculations should be done using radians and answers should be given in radians.

7. **Rounding of final answers (where necessary)**. Probabilities should be given correct to 4 decimal places. All other answers should be given correct to 2 decimal places, unless otherwise specified.

MODULE 1: CALCULUS AND ALGEBRA (210 marks)
(ANSWER THIS MODULE SEPARATELY FROM MODULE 2)

QUESTION 1

Use mathematical induction to prove that $2^{3n}-1$ is divisible by 7 for all $n \in \mathbb{N}$.

[14]

QUESTION 2

2.1 (a) Sketch the graphs $y = 2e^{x+1}+1$ and $y = 9$ on the same set of axes. On your sketch, show one point with integer coordinates on the graph of the exponential function, the values of all intercepts with the axes and any asymptote with its equation. (9)

(b) Determine the solution of $2e^{x+1}+1 \geq 9$. (6)

2.2 Solve for *x* without using a calculator: $\log_e x = \log_x e$ (8)

[23]

QUESTION 3

The functions f and g are defined by $f(x) = \dfrac{1}{x-4}$ and $g(x) = |x|$.

3.1 Determine each of the following functions and their domains.

(a) $(g \circ f)(x)$ (4)

(b) $(f \circ g)(x)$ (6)

3.2 On separate sets of axes, showing all asymptotes and intercepts with the axes, sketch the graphs:

(a) $y = (g \circ f)(x)$ (6)

(b) $y = (f \circ g)(x)$ (9)

3.3 (a) Prove that $f \circ g$ is continuous at $x = 0$. (6)

(b) Prove that $f \circ g$ is *not* differentiable at $x = 0$. (10)

[41]

QUESTION 4

In this question i is the complex number with the property $i^2 = -1$.

4.1 Determine the real numbers a and b for which $(3+bi)(a+3i) = 6bi$ (10)

4.2 Solve the equation $z^3 + z^2 + 3z - 5 = 0$ for $z \in \mathbb{C}$. (10)

4.3 Solve the equation $(1+i)x^2 = 2x$ for $x \in \mathbb{C}$, giving the answers in the form $a+bi$ where a and b are real numbers.
Why are the roots not complex conjugates of each other. (10)

[30]

QUESTION 5

The rational functions f, g and h are defined by

$$f(x) = \frac{x-2}{x^2-4x+4}$$

$$g(x) = \frac{x^2-4}{2x^2-4x}$$

$$h(x) = \frac{8-x^3}{x^2-4}$$

Determine the following limits, if they exist. Show all working.

5.1 (a) $\lim_{x \to 2} f(x)$ (3)

(b) $\lim_{x \to 2} g(x)$ (3)

(c) $\lim_{x \to 2} h(x)$ (3)

5.2 (a) $\lim_{x \to \infty} f(x)$ (3)

(b) $\lim_{x \to \infty} g(x)$ (3)

(c) $\lim_{x \to \infty} h(x)$ (3)

5.3 $\lim_{x \to -\infty} \frac{\sqrt{x^2}}{x}$ (3)

[21]

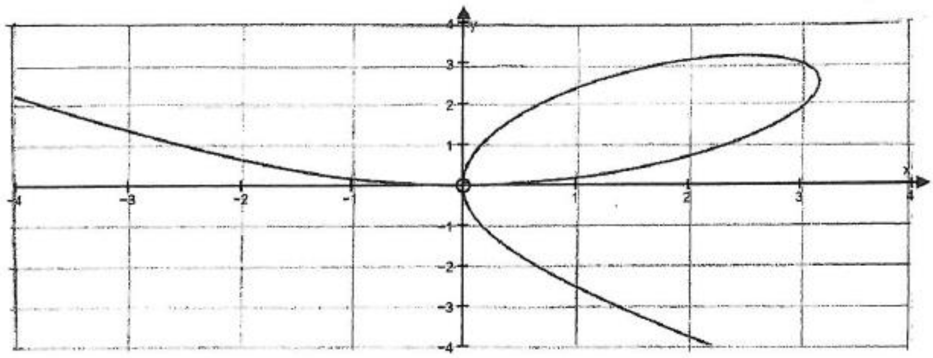
PTO

QUESTION 6

6.1 (a) Determine $f'(x)$ if $f(x)=\sqrt{x+\sqrt{x}}$ (6)

(b) Determine $\dfrac{dy}{dx}$ if $y=\sin x.\cos(a-x)$ and a is a constant. Write your answer as a single trigonometric ratio. (7)

6.2 The graph shown in the diagram below is known as the folium of Descartes and is defined by the equation $x^3+y^3=6xy$.

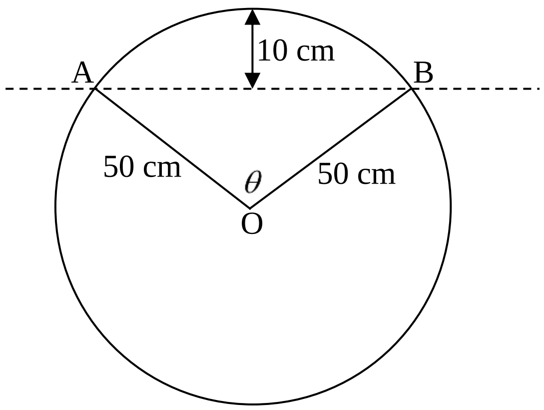


(a) Show that $\dfrac{dy}{dx}=\dfrac{2y-x^2}{y^2-2x}$ (10)

(b) Determine the equation of the normal line to the curve at the point $(3;3)$. (5)

[28]

QUESTION 7



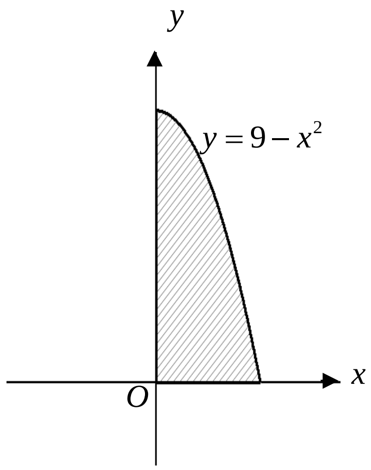
The diagram shows a cross-section of a cylindrical wooden log floating in water. O is the centre of the circle and the radius of the circle is 50 cm. The top of the log is 10 cm above the surface of the water, as shown. $\text{A}\hat{\text{O}}\text{B}=\theta$.

(a) Show that $\theta=1,287$ radians . (6)

(b) Calculate the area of the cross-section of the log which is below the water surface. (8)

[14]

QUESTION 8



Use a Riemann sum to determine the area of the region bounded by the graph of $f(x) = 9 - x^2$, the x – axis and the y – axis, as indicated.

[14]

QUESTION 9

Determine the following integrals

9.1 $\int \left(2x - 3\sqrt{x} + \frac{1}{x^2} \right) dx$ (4)

9.2 $\int \frac{x}{\left(x^2+1\right)^2} dx$ (4)

9.3 $\int \sec^5 \theta \tan \theta \, d\theta$ (4)

[12]

QUESTION 10

10.1 Decompose $\frac{2x^2+x+1}{(x+1)(x^2+1)}$ into partial fractions. (9)

10.2 Given that $\int \frac{1}{x} dx = \ln|x| + c$, use your answer in 10.1 to determine

$\int \frac{2x^2+x+1}{(x+1)(x^2+1)} dx$ (4)

[13]

END OF MODULE 1

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MODULE 2: STATISTICS (90 MARKS)
(ANSWER THIS MODULE SEPARATELY FROM MODULE 1)

QUESTION 1

1.1 Two events A and B are such that:

$$P(A|B)=0,4;\ P(B|A)=0,25 \text{ and } P(A\cap B)=0,12.$$

(a) Calculate $P(B)$. (4)

(b) Are A and B independent events? Explain. (4)

(c) Calculate $P(A\cap B')$. (7)

1.2 Events A, B, and C of a sample space are such that the events A and C are mutually exclusive, whereas events A and B are independent.
Given that $P(A)=\frac{2}{5}$, $P(C)=\frac{1}{3}$ and $P(A\cup B)=\frac{5}{8}$, determine

(a) $P(A\cup C)$ (4)

(b) $P(B)$ (6)

[25]

QUESTION 2

2.1 20% of a certain population is known to be infected with a particular virus. A screening test, which does not always work perfectly, is used to detect the presence of the virus. Previous results have shown that when the virus is actually present, there is an 80% chance that the test will be positive (indicate the presence of the virus). However, when the virus is not present, there is a 5% chance that the test will also be positive.

(a) Calculate the probability that a randomly selected person from the population will test positive for the virus. (8)

(b) If a person's test is negative, what is the probability that the person is infected with the virus? (5)

2.2 Murray and Ben play a game in which they each toss a fair coin in turn until someone throws a head. The person who throws the head wins the game. Murray starts the game.

(a) What is the probability that Ben will win? (6)

(b) Is this a fair game? Explain. (2)

2.3 Rosy plays a game where she throws a certain number of unbiased normal six sided playing dice hoping to obtain at least one six.

(a) What is the probability that Rosy obtains at least one six when she tosses 5 dice? (5)

(b) How many dice must Rosy throw so that the probability of obtaining at least one six is 0,99 or better? (8)

[34]

QUESTION 3

3.1 The probability that, with each throw, Teagan puts the ball through the hoop in a game of netball is 0,7.

(a) Calculate the probability that Teagan has exactly 2 successes out of 3 attempts to put the ball through the hoop. (6)

(b) Complete the following table (probability distribution) where x is the number of successes and $P(X=x)$ is the probability of x successes out of 3 attempts:

x	0	1	2	3
$P(X=x)$				

(4)

(c) Calculate $\sum_{x=0}^{3} P(X=x)$. Are you surprised by your answer? Explain. (3)

(d) The *expectation*, $E(x)$, of a discrete distribution with n trials is defined to be

$E(x)=\sum_{x=0}^{n} x.P(X=x)$ and gives the expected number of successes (the mean) in n trials. Calculate the expectation of the above distribution for the 3 attempts. (2)

3.2 Three sisters Ann, Betty and Carly enter a triathlon with 6 additional competitors. If each competitor has an equal chance of winning and a gold medal is awarded to each of the three people who cross the winning line first (irrespective of order), what is the probability that at least two of the sisters will win a medal? (6)

[21]

QUESTION 4

4.1 The letters of the word MINIMUM are arranged in a line at random. What is the probability that the arrangement begins or ends with MMM? (5)

4.2 A student council consists of 8 boys and 2 girls. In how many ways can a committee of 5 be chosen if it must include at least one girl? (5)

[10]