ST. DAVID'S MARIST INANDA

ADVANCED PROGRAMME MATHEMATICS
COMPULSORY MODULE
CALCULUS & ALGEBRA

GRADE 12
PRELIMINARY EXAMINATION
31 AUGUST 2017

EXAMINER: MRS C KENNEDY MODERATOR: MRS S RICHARD	MARKS: 200 TIME: 2 hours.

NAME: MEMO

INSTRUCTIONS:

- This paper consists of 20 pages and a separate information sheet. Please check that your paper is complete.
- Please answer all questions in this booklet.
- A Casio FX 991 may be used, unless otherwise indicated.
- **CHECK THAT YOUR CALCULATOR IS IN RADIANS**
- All necessary calculations must be clearly shown.
- Work neatly. Do NOT answer in pencil.

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	TOTAL
35	6	16	14	8	33	28	30	22	8	200

QUESTION 1 **[35 marks]**

a) Solve for p and q if $(3 + pi)^2 = 30i - 8q$ (7)

$$9 + 6pi + p^2 i^2 = 30i - 8q$$

$$9 + 6pi - p^2 = 30i - 8q$$

$6pi = 30i$

$6p = 30$

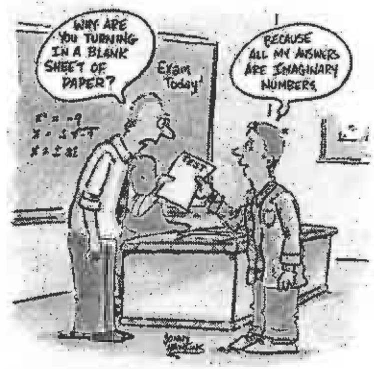
$p = 5$

$9 - p^2 = -8q$

$9 - (5)^2 = -8q$

$-16 = -8q$

$2 = q$



b) Solve for x in each of the following, rounding off to TWO decimal places where necessary:

i) $x|x+1| + 4 = x$ (7)

if $x + 1 \geq 0$

$x \geq -1$

$x(x+1) + 4 = x$

$x^2 + x + 4 = x$

$x^2 = -4$

No soln.

if $x + 1 < 0$

$x < -1$

$x(-x-1) + 4 = x$

$-x^2 - x + 4 = x$

$0 = x^2 + 2x - 4$

$$x = \frac{-2 \pm \sqrt{4 - 4(1)(-4)}}{2}$$

$$x = \frac{-2 \pm \sqrt{20}}{2}$$

$x = 1,24$ or $x = -3,24$

N.V

ii) $e^{2x} - 4e^{x} = 5$ (5)

let $k = e^{x}$

$k^2 - 4k - 5 = 0$

$(k-5)(k+1) = 0$

$e^{x} = 5$ or $e^{x} = -1$

$\ln 5 = x$ No soln.

$1{,}61 = x$

c) The points P(3 ; p) and Q(6 ; q) lie on the curve: $y = 3\ln x$.
Determine the gradient of the line PQ and give the answer in log form. (5)

$P(3\ ;\ 3\ln 3)$ $Q(6\ ;\ 3\ln 6)$

$$m_{PQ} = \frac{3\ln 6 - 3\ln 3}{6-3}$$

$$= \ln 6 - \ln 3$$

$$= \ln 6/3$$

$$= \ln 2$$

d) Given $f(x) = 2\ln(x-1)$

i) Determine the equation of $f^{-1}(x)$, the inverse of $f(x)$. (4)

$$x = 2\ln(y-1) \checkmark$$

$$\frac{x}{2} \overset{\checkmark}{=} \ln(y-1)$$

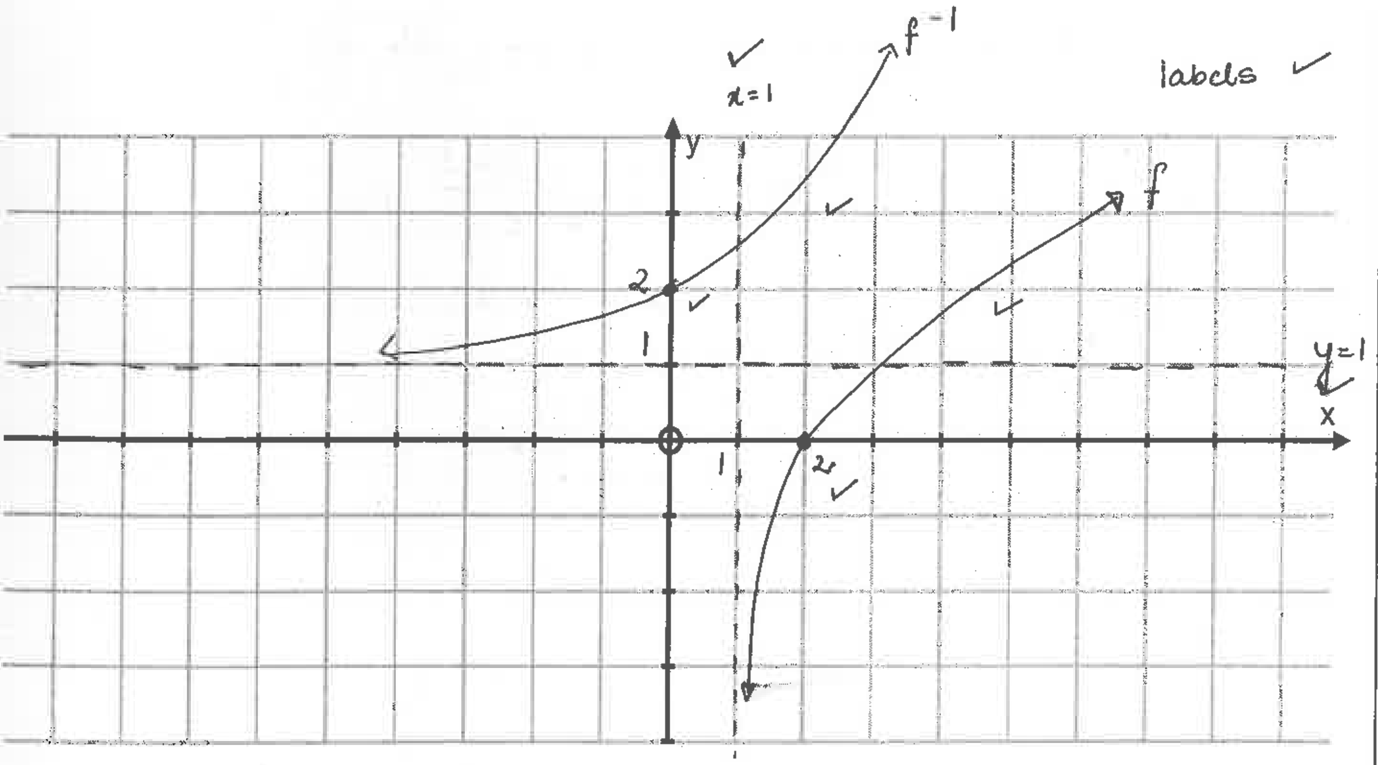
$$e^{\frac{x}{2}} = y - 1 \checkmark$$

$$e^{x/2} + 1 = y$$

$$\therefore f^{-1}(x) = e^{x/2} + 1 \checkmark$$

ii) Sketch the graphs of f(x) and $f^{-1}(x)$ on the same set of axes below. Clearly label all intersections with the axes and any asymptotes.

(7)



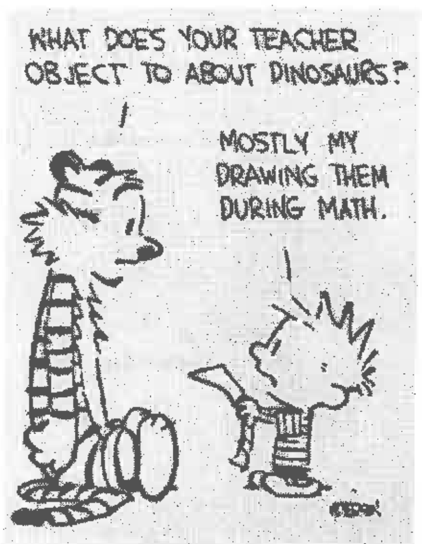
QUESTION 2 [6 marks]

Determine g(f(x)) if $f(g(x)) = \frac{1}{x+3} + x^2 + 6x + 9$ and $g(g(x)) = x + 6$ (6)

$f(x) = \frac{1}{x} + x^2$ ✓✓ $\qquad g(x) = x + 3$ ✓✓

$g(f(x)) = g\left(\frac{1}{x} + x^2\right)$

$= \frac{1}{x} + x^2 + 3$ ✓✓



QUESTION 3 **[16 marks]**

a) Prove by Mathematical Induction that:

$\sum_{r=1}^{n} r(3r-1) = n^3 + n^2$ for all Natural numbers n. (12)

$$2 + 10 + \ldots + n(3n-1) = n^3 + n^2$$

1. Prove true for $n=1$

LHS: 2 ✓ RHS: $1^3 + 1^2$

$= 2$ ✓

LHS = RHS

∴ true for $n=1$

2. Assume true for $n=k$

$$2 + 10 + \ldots + k(3k-1) = k^3 + k^2 \checkmark\checkmark$$

3. Prove true for $n=k+1$

LHS: $\underbrace{2 + 10 + \ldots + k(3k-1)}_{k^3+k^2} + (k+1)(3(k+1)-1)$ ✓

$k^3 + k^2$ ✓ $+ (k+1)(3k+2)$

$k^2(k+1)$ ✓ $+ (k+1)(3k+2)$

$(k+1)$ ✓ $(k^2 + 3k + 2)$

$(k+1)(k+2)(k+1)$ ✓ or $k^3 + 4k^2 + 5k + 2$

RHS: $(k+1)^3 + (k+1)^2$ ✓

$= (k+1)^2(k+1+1)$

$= (k+1)^2(k+2)$ ✓

$=$ LHS.

4. Conclusion ✓ ...

b) Hence, determine the value of:

$2 + 10 + 24 + \ldots + 660$ (4)

$r(3r-1) = 660$ ✓

$3r^2 - r - 660 = 0$

$(3r+44)(r-15) = 0$

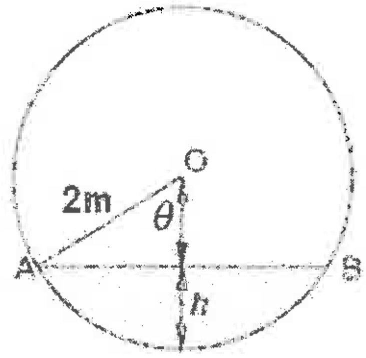
$\therefore r = 15$ ✓

$\therefore \text{Sum} = 15^3 + 15^2$ ✓

$= 3600$ ✓

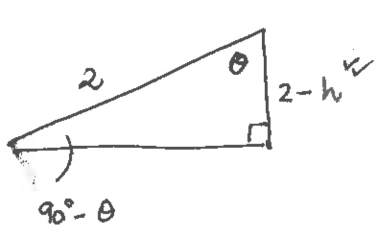
QUESTION 4 [14 marks]

Petrol stations only run out of petrol when there is a major price increase. Generally, they manage their petrol levels in the tanks to ensure that they have sufficient stock. The petrol is stored in an underground cylindrical tank and the amount left in the tank is checked using a dipstick. Our problem is to calibrate the dipstick.



It is easy to measure the height, h, of the petrol left in the tank using a dipstick. However, the volume is proportional to the cross-sectional area.

a) Determine $\cos\theta$ in terms of h. (4)



$\cos\theta = \dfrac{2-h}{2}$ ✓✓

b) Show that the cross sectional area of petrol in the tank is determined by:

$$A = 4\theta - 2\sin(2\theta)$$ (6)

Area of sector $= \frac{1}{2} r^2 \theta$

$= \frac{1}{2} (2)^2 \theta$ ✓

$= 2\theta$ ✓ $\times 2 = 4\theta$ ✓

Area $\Delta = \frac{1}{2} \cdot 2 \cdot 2 \cdot \sin 2\theta$ ✓

$= 2 \sin 2\theta$ ✓

$\therefore A = 4\theta - 2\sin 2\theta$ ✓

c) What fraction of the tank is full when $\theta = \frac{\pi}{4}$? (4)

$$A = 4\left(\frac{\pi}{4}\right) - 2\sin 2\left(\frac{\pi}{4}\right)$$

$$= \pi - 2\sin\frac{\pi}{2}$$

$$= \pi - 2$$

$$A = \pi r^2 = 4\pi$$

$$\therefore \text{fraction} = \frac{\pi - 2}{4\pi}$$

$= 0{,}09$ of the tank.

QUESTION 5 **[8 marks]**

The function f(x) is defined as follows:

$$f(x) = \begin{cases} \frac{a}{x} & \text{if } x \geq 1 \\ b - 2x & \text{if } x < 1 \end{cases}$$

Determine the value(s) of a and b if f(x) is differentiable at x = 1. (8)

Continuity: $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$

$$b - 2 = \frac{a}{1}$$

$$b - 2 = a$$

differentiability: $\lim_{x \to 1^-} f'(x) = \lim_{x \to 1^+} f'(x)$

$$\lim_{x \to 1^-} -2 = \lim_{x \to 1^+} -\frac{a}{x^2}$$

$$-2 = -a$$

$$2 = a$$

$$\therefore b = 4$$

QUESTION 6 [33 marks]

a) i) Prove that $\frac{1}{\sec A - 1} + \frac{1}{\sec A + 1} = 2\cos A.\operatorname{cosec}^2 A$ (6)

LHS:

$$\frac{\sec A + 1 + \sec A - 1}{(\sec A - 1)(\sec A + 1)}$$

$$= \frac{2\sec A}{\sec^2 A - 1}$$

$$= \frac{2\sec A}{\tan^2 A}$$

$$= \frac{2}{\cos A} \times \frac{\cos^2 A}{\sin^2 A}$$

$$= \frac{2\cos A}{\sin^2 A}$$

RHS: $\frac{2\cos A}{\sin^2 A}$

ii) Hence, determine $\lim_{A\to 0}\left(\frac{1}{\sec A - 1} + \frac{1}{\sec A + 1}\right)A^2$ (5)

$$\lim_{A\to 0} 2\cos A \cdot \operatorname{cosec}^2 A \cdot A^2$$

$$= \lim_{A\to 0} 2\cos A \cdot \frac{A^2}{\sin^2 A}$$

$$= 2\lim_{A\to 0}\cos A \cdot \lim_{A\to 0}\frac{A^2}{\sin^2 A}$$

$$= 2 \cdot 1 \cdot 1$$

$$= 2$$

b) Given: $y = \sqrt{4x^2 + 1}$

i) Show that $\dfrac{dy}{dx} = \dfrac{4x}{y}$ (6)

$$y = (4x^2 + 1)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}\checkmark (4x^2+1)^{-\frac{1}{2}} \checkmark \cdot 8x \checkmark$$

$$= \frac{4x \checkmark}{\sqrt{4x^2+1}} \checkmark$$

$$= \frac{4x}{y} \checkmark$$

ii) Hence, or otherwise, show that $\dfrac{d^2y}{dx^2} = \dfrac{4}{y} - \dfrac{16x^2}{y^3}$ (6)

$$\frac{d^2y}{dx^2} = \frac{4 \cdot y \checkmark - 4x \cdot \frac{dy}{dx} \checkmark}{y^2 \checkmark}$$

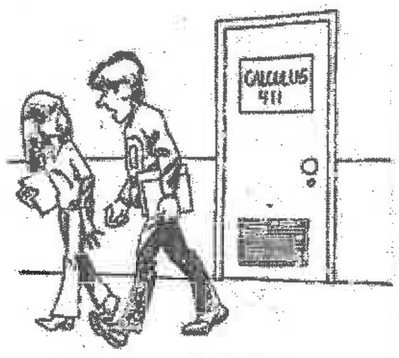
✓ in quotient rule

$$= \frac{4y}{y^2} - \frac{4x \cdot \frac{4x}{y}}{y^2} \checkmark$$

substitution

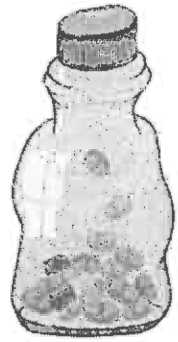
✓ split denominator

$$= \frac{4}{y} - \frac{16x^2}{y^3}$$



"I think I understood the part about bringing two sharp pencils to class every day."

c) If a drug is given to a patient and the percentage of the concentration of the drug in the blood stream t hours later is given by:



$$k(t) = \frac{5t}{t^2+1}$$

Determine when the concentration of the drug is increasing. (10)

$$K'(t) > 0$$

$$\frac{5(t^2+1) - 5t(2t)}{(t^2+1)^2} > 0 \quad \text{quotient rule}$$

$$\frac{5t^2+5-10t^2}{(t^2+1)^2} > 0$$

$$\frac{-5t^2+5}{(t^2+1)^2} > 0$$

$$\frac{-5(t^2-1)}{(t^2+1)^2} > 0$$

$$\frac{-5(t+1)(t-1)}{(t^2+1)^2} > 0$$

CV's ± 1

$$\begin{array}{ccccc} & & + & & - \\ \hline & 0 & & 1 & \end{array}$$

$0 < t < 1$ hours.

QUESTION 7 [28 marks]

a) It is given that: $g(x) = \dfrac{3x^2 + x - 2}{x^2 + px - 2}$

i) For which value(s) of p will the graph of g have:

(1) one x-intercept. (5)

$$g(x) = \frac{3x^2 + x - 2}{x^2 + px - 2} = \frac{(3x-2)(x+1)}{x^2 + px - 2}$$

So, $x^2 + px - 2 = (3x-2)(\frac{1}{3}x + 1)$ $\quad p = 7/3$

or, $x^2 + px - 2 = (x+1)(x-2)$ $\quad p = -1$

(2) an oblique asymptote. (2)

None, degree numerator is not one greater than degree of denominator.

ii) If p = - 1,

(1) show that g is a hyperbola with a removable discontinuity at x = - 1. (3)

$$g(x) = \frac{3x^2 + x - 2}{x^2 - x - 2} = \frac{(3x-2)\cancel{(x+1)}}{(x-2)\cancel{(x+1)}} = \frac{3x-2}{x-2}$$

Shows a removable discontinuity at $x = -1$

(2) hence, determine the equations of the asymptotes of g. (3)

$$\begin{array}{r|l} & 3 \\ \hline x-2 & 3x - 2 \\ & -(3x - 6) \\ \hline & \quad 4 \end{array}$$

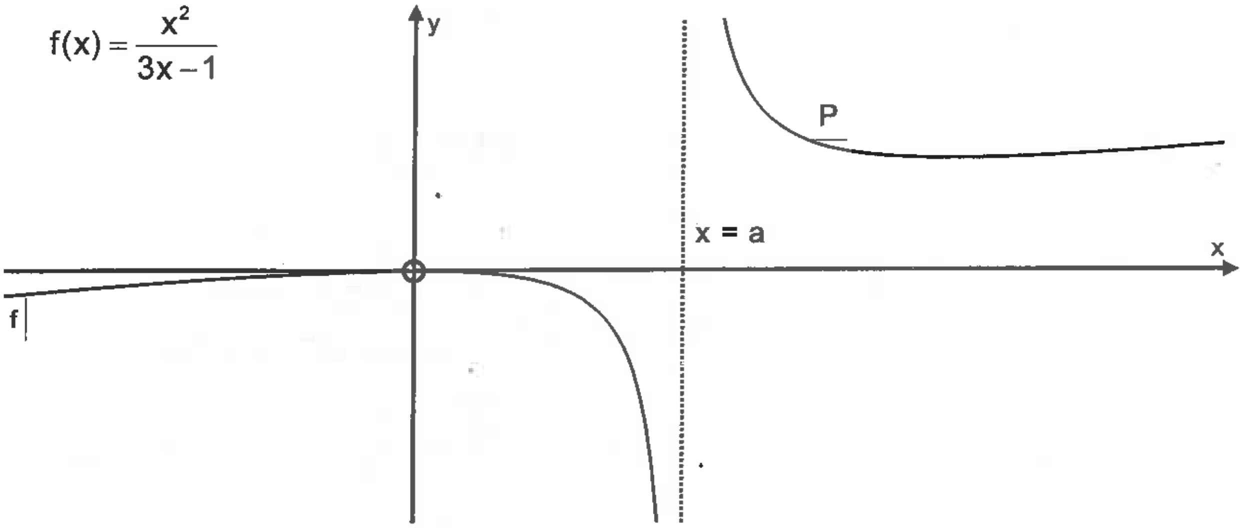
$\therefore g(x) = 3 + \dfrac{4}{x-2}$ — hyperbola.

horizontal asymptote is $y = 3$

Vertical asymptote is $x = 2$

b) The sketch shows part of the curve defined by:

$$f(x) = \frac{x^2}{3x-1}$$



i) Write down the value of a. (1)

$a = \frac{1}{3}$

ii) If P is the turning point of f, show that the x co-ordinate of P is $\frac{2}{3}$. (6)

$f'(x) = 0$ ✓

✓m

$$\frac{2x(3x-1) - x^2 . 3}{(3x-1)^2} = 0$$

$6x^2 - 2x - 3x^2 = 0$ ✓

$3x^2 - 2x = 0$ ✓

$x(3x-2) = 0$ ✓

$x = 0$ or $x = 2/3$ ✓

iii) If $f(x) = k$ has no real solutions, write down the possible values of k. (4)

$(0;0)$ ✓ $P(2/3 ; 4/9)$ ✓

$0 < k < 4/9$ ✓ ✓

iv) Determine the equation of the oblique asymptote of f. (4)

$$\begin{array}{r} \tfrac{1}{3}x + \tfrac{1}{9} \\ 3x-1 \,\overline{)\, x^2 \qquad\qquad} \\ -\left(x^2 - \tfrac{1}{3}x\right) \\ \hline \tfrac{1}{3}x \\ -\left(\tfrac{1}{3}x - \tfrac{1}{9}\right) \\ \hline \tfrac{1}{9} \end{array}$$

$$y = \frac{1}{3}x + \frac{1}{9}$$

QUESTION 8 [30 marks]

a) Determine the following indefinite integrals:

i) $\int 2\sin 5x.\cos 3x\,dx$ (4)

$$= 2\int \frac{1}{2}\left[\sin 8x + \sin 2x\right] dx$$

$$= \int \sin 8x\,dx + \int \sin 2x\,dx$$

$$= -\frac{\cos 8x}{8} - \frac{\cos 2x}{2} + c$$

ii) $\int x^2 \sec^2(2x^3)\, dx$ (8)

let $u = 2x^3$

$\frac{du}{dx} = 6x^2$

$\frac{1}{6} du = x^2\, dx$

$\int \frac{1}{6} \sec^2 u \cdot du$

$= \frac{1}{6} \int \sec^2 u \, du$

$= \frac{1}{6} \tan u + c$

$= \frac{1}{6} \tan(2x^3) + c$

b) Calculate a if: (8)

$$\int_0^a \frac{2}{\sqrt{x+4}} dx = 4$$

$\left[4\sqrt{x+4} \right]_0^a = 4$

$4\sqrt{a+4} - 4\sqrt{4} = 4$

$\sqrt{a+4} = 3$

$a + 4 = 9$

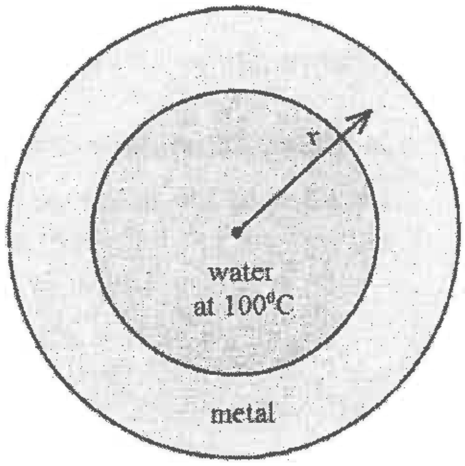
$a = 5$

Valid

c) A metal tube has a cross-section as shown alongside.

The outer radius is 4 cm and the inner radius is 2 cm.

Within the tube, water is maintained at a temperature of 100 ° C.



Tubular Cross-Section

Within the metal the temperature drops from inside to outside according to

$\frac{dT}{dx} = -\frac{10}{x^2}$, where x is the distance from the central axis and $2 \leq x \leq 4$.

Determine the temperature of the outer surface of the tube. (10)

$$T = \int -\frac{10}{x^2}\, dx \checkmark$$

$$= \int -10x^{-2}\, dx$$

$$= -\frac{10x^{-1}}{-1} + c \checkmark\checkmark$$

$$= \frac{10}{x} + c \checkmark$$

At $T(2) = 100 \checkmark$

$\frac{10}{2} + c = 100 \checkmark$

$c = 95°C \checkmark$

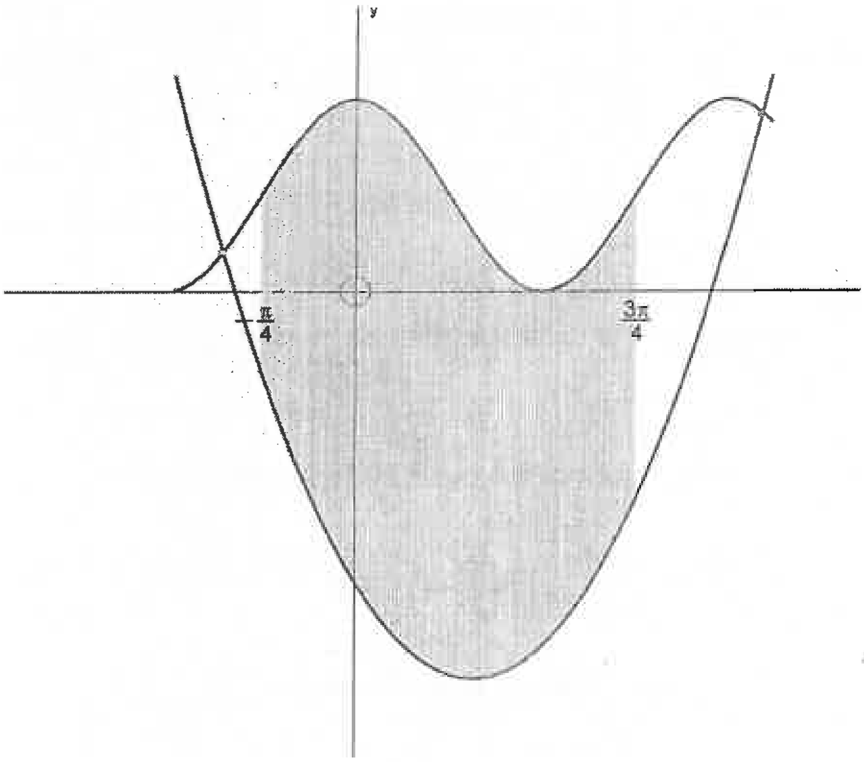
$\therefore T = \frac{10}{x} + 95°C \checkmark$

$T(4) \overset{\checkmark}{=} \frac{10}{4} + 95°C$

$= 97{,}5°C \checkmark$

QUESTION 9 [22 marks]

In the diagram below, the shaded region is defined on the interval $\left[-\frac{\pi}{4};\frac{3\pi}{4}\right]$ and is bounded by the functions $h(x) = \cos 2x + 1$ and $g(x) = x^2 - 2x - 3$. The functions do not intersect at any point on this interval.



a) Show that the maximum distance between the two graphs on this interval can be found by solving the equation $\sin 2x = 1 - x$. (8)

$$\text{Distance} = \cos 2x + 1 - (x^2 - 2x - 3)$$

$$D = \cos 2x + 1 - x^2 + 2x + 3$$

$$\text{Max}: \frac{dD}{dx} = 0$$

$$-\sin 2x \cdot 2 - 2x + 2 = 0$$

$$-2x + 2 = 2\sin 2x$$

$$-x + 1 = \sin 2x$$

$$1 - x = \sin 2x$$

b) Use Newton's method to write down a recursive equation that can be used to solve the equation in QUESTION 9a.
Hence, taking $x_0 = 0{,}5$ as an initial value, determine the answer correct to FIVE decimal places. (8)

$$f(x) = \sin 2x - 1 + x \quad ✓$$

$$f'(x) = 2\cos 2x + 1 \quad ✓✓$$

$$a_{n+1} = a_n - \frac{\sin 2a_n - 1 + a_n}{2\cos 2a_n + 1} \quad ✓✓$$

$x = 0{,}5$

$x_1 = 0{,}335\ 87\ 896\ 38$

$x_2 = 0{,}35\ 21\ 56\ 0239$

$x_3 = 0{,}35\ 22\ 88\ 4475$ ✓✓

$x_4 = 0{,}35\ 22\ 88\ 4565$

$\therefore x = 0{,}35\ 22\ 9$ ✓

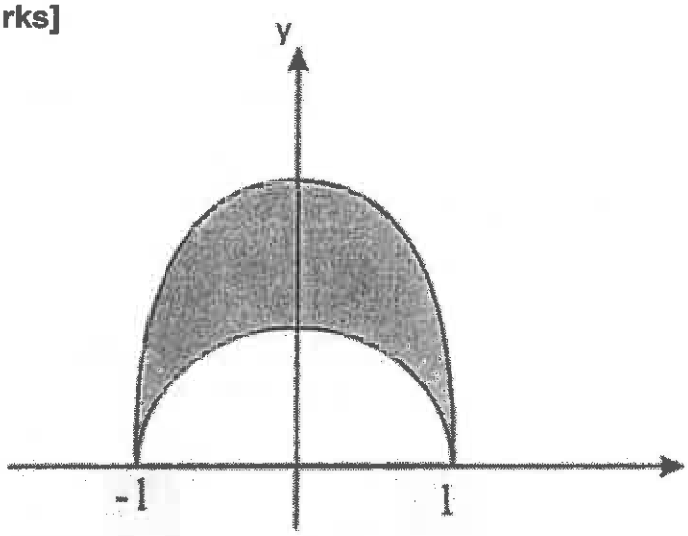
c) Now find the area of the shaded region. (6)

$$A = \int_{-\pi/4}^{3\pi/4} \cos 2x + 1 \; dx \quad ✓✓ \quad - \int x^2 - 2x - 3 \; dx \quad ✓✓$$

$$= \left[\frac{\sin 2x}{2} + x - \frac{x^3}{3} + x^2 + 3x \right]_{-\pi/4}^{3\pi/4}$$

$$= 12{,}97 \text{ units}^2 \quad ✓✓$$

QUESTION 10 [8 marks]



The area enclosed between the parts of two curves $x^2 + y^2 = 1$ and $4x^2 + y^2 = 4$ is rotated by 2π radians about the x-axis.

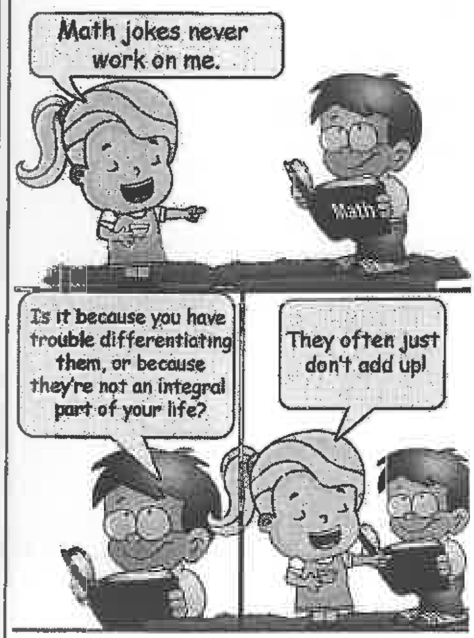
Determine the volume of the solid formed.
(Leave your answer in terms of π) (8)

$$y^2 = 1 - x^2 \qquad y^2 = 4 - 4x^2$$

$$V = \pi \int_{-1}^{1} 4 - 4x^2 \, dx - \pi \int_{-1}^{1} 1 - x^2 \, dx$$

$$= \pi \left[4x - \frac{4x^3}{3} \right]_{-1}^{1} - \pi \left[x - \frac{x^3}{3} \right]_{-1}^{1}$$

$$= 4\pi$$



TOTAL: 200