GRADE 12 EXAMINATION
NOVEMBER 2015

ADVANCED PROGRAMME MATHEMATICS
CORE MODULE: CALCULUS AND ALGEBRA

MARKING GUIDELINES

Time: 2 hours | 200 marks

These marking guidelines are prepared for use by examiners and sub-examiners, all of whom are required to attend a standardisation meeting to ensure that the guidelines are consistently interpreted and applied in the marking of candidates' scripts.

The IEB will not enter into any discussions or correspondence about any marking guidelines. It is acknowledged that there may be different views about some matters of emphasis or detail in the guidelines. It is also recognised that, without the benefit of attendance at a standardisation meeting, there may be different interpretations of the application of the marking guidelines.

QUESTION 1

1.1 (a)

$e^{\pi x} = 3$
$\therefore \pi x = \ln 3$
$\therefore x = \dfrac{\ln 3}{\pi} = 0,35$ (4)

(b)

$\tan x = 2$
$\therefore x = 1,107 + \pi k$
$\tan x = -2$
$\therefore x = -1,107 + \pi k$
$\therefore x = 1,11 \ \ or \ \ 2,04 \ \ or \ \ 4,25 \ \ or \ \ 5,18$ (6)

(c)

$\log_{\frac{1}{2}} x + \log_2 x^n = -2$
$\therefore -\log_2 x + n\log_2 x = -2$
$\therefore -\log_2 \frac{1}{2} + n\log_2 \frac{1}{2} = -2$
$\therefore 1 - n = -2$
$\therefore n = 3$

OR

$\therefore log_{\frac{1}{2}} \frac{1}{2} + log_2 \left(\frac{1}{2}\right)^n = -2$
$\therefore 1 + log_2 2^{-n} = -2$
$1 - n \ log_2 2 = -2$
$1 - n = -2$
$n = 3$ (4)

1.2 (a) (1)

$P = 70 \times 1 + 22$
$= 92°C$ (2)

(2) Limit as $t \to \infty$
$P = 22°C$ (2)

(b)

$55 = 70 \times 1,2^{-t} + 22$
$t = 4,1245 \min s$

$40 = 70 \times 1,2^{-t} + 22$
$t = 7,449 \min s$

$4,12 \le t \le 7,45$ (7)

[25]

QUESTION 2

Prove true for n = 1:

LHS = $5^2 - 1 = 24$

Which is a multiple of 8

Assume true for n = k: $5^{2k} - 1 = 8p \quad p \in N$

Prove true for n = k + 1:

$5^{2(k+1)} - 1$

$= 5^2.5^{2k} - 1$

$But\ 5^{2k} = 8p + 1\ by\ assumption$

$= 25(8p+1) - 1$

$= 200p + 24$

$= 8(25p+3)$ which is a multiple of 8.

Hence, we have shown that if the expression is divisible by 8 by any one natural value of n then it is also true for the next consecutive value. But it is true for n =1, therefore also true for n =2, 3 , 4 and so on for all natural values of n.

OR

$\therefore$ by the P.M.I. the statement is true for $n \in \mathbb{N}$

[14]

QUESTION 3

3.1 $$\frac{(a+bi)(a+bi)}{a^2+b^2}$$

$$= \frac{a^2 - b^2 + 2abi}{a^2+b^2}$$

$$\therefore real\ part = \frac{a^2-b^2}{a^2+b^2}$$ (7)

3.2 One other solution is $x = 3 + 7i$.

$x - 3 = 7i$

$\therefore x^2 - 6x + 9 = -49$

$\therefore x^2 - 6x + 58 = 0$

By inspection: $(2x-1)(x^2-6x+58) = 2x^3 + px^2 + qx - 58$

p = –13 q = 122 (10)

[17]

QUESTION 4

4.1 $\tan\theta = \sqrt{3}$

$\therefore \theta = \dfrac{\pi}{3}$ *radians*

$r^2 = 1 + 3$ $(Pythag)$

$\therefore r = 2$ (6)

4.2 Area of sector =

$\dfrac{1}{2}r^2\theta$

$= \dfrac{1}{2}(2)^2 . \dfrac{\pi}{3}$

$= 2,09\,units^2$

$\tan\theta = \dfrac{AB}{2}$

$\therefore AB = 3,46\,units$

Area of ΔOAB

$\dfrac{1}{2} \times 2 \times 3,46$

$= 3,46\,units^2$

Shaded area = 3,46 - 2,09 = 1,37 units^2 (9)

[15]

QUESTION 5

5.1 (a) $p(x) = (2+3x)^{-1}$

$\therefore p'(x) = -(2+3x)^{-2}(3)$

$= -\dfrac{3}{(2+3x)^2}$ (4)

(b) $x = \dfrac{1}{2+3y}$

$\therefore 3y = \dfrac{1}{x} - 2$

$\therefore p^{-1}(x) = \dfrac{1-2x}{3x}$ (4)

5.2 (a) $p(q(x))$ (3)

(b) $r(p(x))$ (3)

5.3 $g(x) = \dfrac{x^2+6}{2x-5}$

$\therefore g'(x) = \dfrac{(2x-5)(2x)-(x^2+6)(2)}{(2x-5)^2}$

$\therefore 0 = 4x^2 - 10x - 2x^2 - 12$

$\therefore 0 = x^2 - 5x - 6$

$\therefore x = 6; \quad x = -1$

$y = 6; \quad y = -1$ (8)

[22]

QUESTION 6

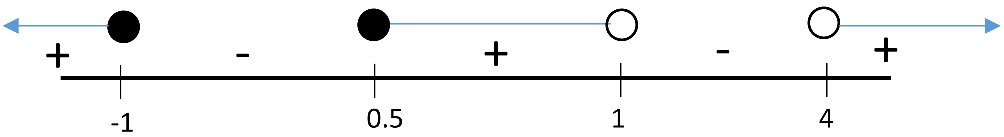
(a) (i) There will be two x-intercepts unless a factor cancels, i.e.

$(x + 1)$ gives p = 5. $(2x – 1)$ gives p = -8,5

(ii) p = 5 and p = -8,5 (see above)

Denominator is a perfect square, i.e. $x = -4$ or $x = 4$.

(b) (i) $f(x)=\dfrac{2x^2+x-1}{x^2-5x+4}=\dfrac{(2x-1)(x+1)}{(x-4)(x-1)}$



$x\geq -1 \;\; or \;\; 0,5\leq x<1 \;\; or \;\; x>4$

(ii)

x-intercepts: $x = –1$; $x = 0.5$
y-intercept: $y = –0.25$
vertical asymptotes: $x = 1$ and $x = 4$
horizontal asymptote: $y = 2$
shape

6.2 (a)

Prove continuity:

$\lim_{x \to 0^-} g(x)$

$= 3$

$\lim_{x \to 0^+} g(x)$

$= |-3| = 3$

$g(3) = |-3| = 3$

$\therefore continuous\ at\ x = 0$

Prove differentiability:

$\lim_{x \to 0^-} g'(x)$

$\lim_{x \to 0^-} (-2x - 1)$

$= -1$

$\lim_{x \to 0^+} g'(x)$

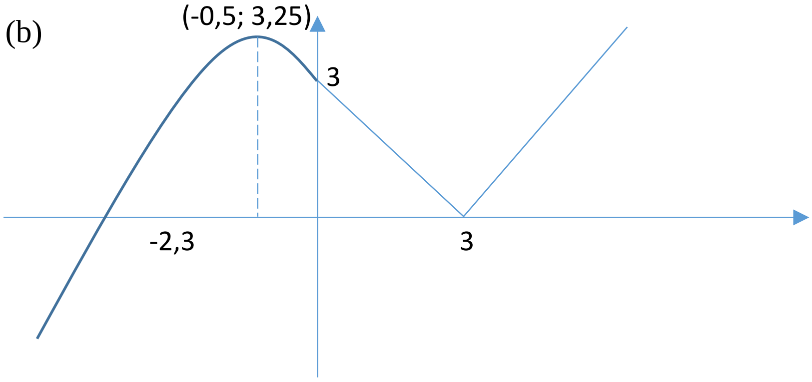
$\lim_{x \to 0^+} -1$

$= -1$

$\therefore differentiable\ at\ x = 0$

(10)

(b)



(8)

[42]

QUESTION 7

7.1 (a)

$f(x) = 3(x-2)^2 - 1 - \frac{4}{x}$

$f(2) = -3$

$f(3) = \frac{2}{3}$

Change of sign implies solution lies in interval, taking into account continuity on the interval $x \in (2;3)$. (3)

(b)

$f'(x) = 6(x-2) + \frac{4}{x^2}$

$x_{r+1} = x_r - \frac{3(x-2)^2 - 1 - \frac{4}{x}}{6(x-2) + \frac{4}{x^2}}$

$x_1 = 2,5$

$x_2 = 3,008241758...$

$x_3 = 2,897330865...$

$x_4 = 2,891354053...$

$x_n = 2,891337 \quad (6dp)$

$= 0,87$ (7)

7.2 (a)

$\frac{d}{dx}\sin y = \frac{d}{dx}x$

$\therefore \cos y.\frac{dy}{dx} = 1$

$\therefore \frac{dy}{dx} = \frac{1}{\cos y}$

$= \frac{1}{\sqrt{1-x^2}}$

(5)

(b)

$\sin y = 0,5$

$y = \frac{\pi}{6} \quad radians$

$\therefore \frac{dy}{dx} = \frac{1}{\cos y}$

$= \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} = 0,87$ (4)

[19]

QUESTION 8

8.1 36 units^2. (2)

8.2 $A_{16} = 36 + \frac{48}{16} + \frac{16}{16^2}$
$= 39,0625$

$\%error = \frac{39,0625 - 36}{36} \times 100$
$= 8,5\%$ (4)

8.3 $\int_0^4 qx^2 + 1\,dx = 36$

$\therefore \left[\frac{qx^3}{3} + x\right]_{x=0}^{x=4} = 36$

$\therefore \frac{64q}{3} + 4 = 36$

$\therefore q = \frac{3}{2}$ (7)

[13]

QUESTION 9

9.1 $\int x(2x-1)^{\frac{2}{3}}\,dx = x.\frac{(2x-1)^{\frac{5}{3}}}{\frac{5}{3}\times 2} - \frac{3}{10}\int (2x-1)^{\frac{5}{3}}.1\,dx$

$= \frac{3x(2x-1)^{\frac{5}{3}}}{10} - \frac{3}{10} \times \frac{(2x-1)^{\frac{8}{3}}}{\frac{8}{3}\times 2}$

$= \frac{3x(2x-1)^{\frac{5}{3}}}{10} - \frac{9(2x-1)^{\frac{8}{3}}}{160} + C$ (8)

9.2 $2\int \frac{1}{2}x^{-\frac{1}{2}}\left(\sqrt{x}+1\right)^3\,dx$

$= 2\frac{\left(\sqrt{x}+1\right)^4}{4} + C$

$= \frac{\left(\sqrt{x}+1\right)^4}{2} + C$ (7)

9.3 (a) $\sec^2 x - 2\sec x \tan x + \tan^2 x$

$= \sec^2 x - 2\sec x \tan x + \sec^2 x - 1$

$= 2\sec^2 x - 2\sec x \tan x - 1$

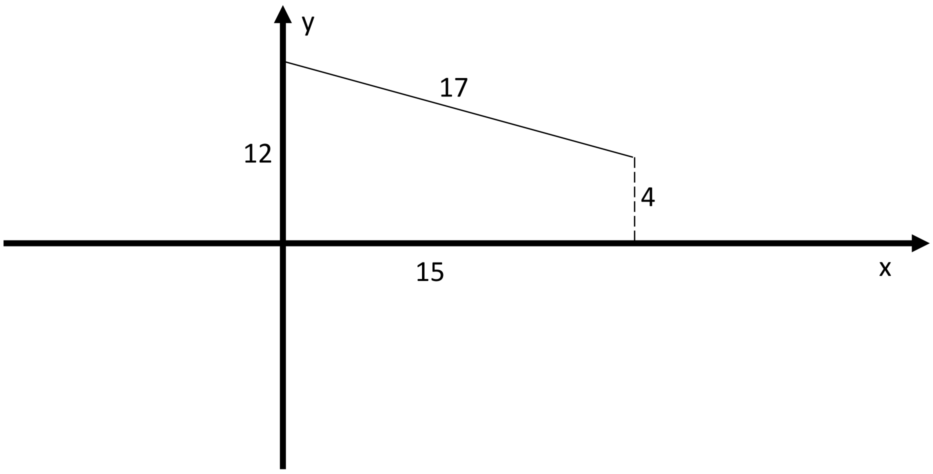
a = 2. b = –1, c = –1 (5)

(b) $\int 2\sec^2 x - 2\sec x \tan x - 1 \, dx$

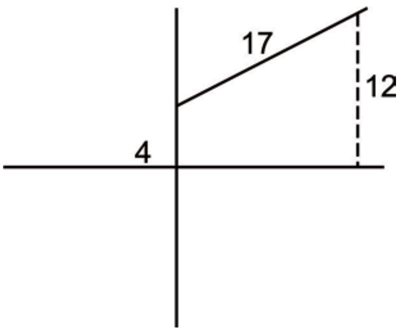
$= 2\tan x - 2\sec x - x + C$ (5)

[25]

QUESTION 10



OR



$$V = \pi \int_0^{15} \left(12 - \frac{8}{15}x\right)^2 \, dx$$

Or any equivalent.

[8]

Total: 200 marks