GRADE 12 EXAMINATION
NOVEMBER 2020

ADVANCED PROGRAMME MATHEMATICS: PAPER I
MODULE 1: CALCULUS AND ALGEBRA

MARKING GUIDELINES

Time: 2 hours 200 marks

These marking guidelines are prepared for use by examiners and sub-examiners, all of whom are required to attend a standardisation meeting to ensure that the guidelines are consistently interpreted and applied in the marking of candidates' scripts.

The IEB will not enter into any discussions or correspondence about any marking guidelines. It is acknowledged that there may be different views about some matters of emphasis or detail in the guidelines. It is also recognised that, without the benefit of attendance at a standardisation meeting, there may be different interpretations of the application of the marking guidelines.

QUESTION 1

1.1 Solve for $x \in \mathbb{R}$ without using a calculator and showing all working:

(a) $2\left|e^x - 5\right| + 3 = 11$

$\left|e^x - 5\right| = 4$ ✓a

$\therefore e^x - 5 = 4$ or $e^x - 5 = -4$ ✓m – 2 cases ✓ca

$\therefore e^x = 9$ or $e^x = 1$ ✓ca ✓ca

$\therefore x = \ln 9$ or $x = 0$ ✓m – logs ✓ca

(7)

(b) $\ln x = 3$

$\therefore x = e^3$ ✓m – exponent form

$\therefore x = 20{,}1$ ✓a (2)

1.2 Determine a and b if $\dfrac{a+bi}{5-i} = \dfrac{1}{2} + \dfrac{1}{2}i$.

$\dfrac{a+bi}{5-i}$

$= \dfrac{a+bi}{5-i} \times \dfrac{5+i}{5+i}$ ✓m ✓a

$= \dfrac{5a + bi^2 + ai + 5bi}{26}$ ✓a

$= \dfrac{5a-b}{26} + \dfrac{a+5b}{26}i = \dfrac{13}{26} + \dfrac{13}{26}i$ ✓ca ✓m – separating

Equating real and imaginary parts: ✓m – equating

$5a - b = 13 \ (1)$ and $a + 5b = 13 \ (2)$

multiplying (2) by 5: ✓m – solving simultaneously

$5a + 25b = 65 \ (3)$

subtracting (1) from (3) ✓a

$26b = 52$

$\therefore b = 2$ and $a = 3$ ✓ca ✓ca (10)

1.3 Determine, in standard form, a quartic (degree 4) equation with rational coefficients where two of the roots are equal to $2+i$ and $1-\sqrt{3}$.

We know that $2-i$ and $1+\sqrt{3}$ are also roots since ✓m – conjugates complex and irrational roots occur in conjugate pairs ✓a

$\left(x-(2+i)\right)\left(x-(2-i)\right)\left(x-\left(1-\sqrt{3}\right)\right)\left(x-\left(1+\sqrt{3}\right)\right) = 0$ ✓m –factors ✓a

$\therefore \left(x^2 - 4x + 5\right)\left(x^2 - 2x - 2\right) = 0$ ✓ca ✓ca

$\therefore x^4 - 6x^3 + 11x^2 - 2x - 10 = 0$ ✓ca ✓ca (8)

[27]

QUESTION 2

For a given annual interest rate, the yield is improved by compounding the interest more frequently. However, a limit exists. If interest is compounded continuously then the following formula applies:

$A = Pe^{r\,t}$

Where:

- P is the principle invested
- A is the accumulated amount
- r is the annual interest rate expressed as a percentage
- t is the time in years

2.1 By first making t the subject of the formula, determine how long it will take the money invested to triple in value if interest is 10% per annum. Express your answer to the nearest year.

$\frac{A}{P} = e^{rt}$ ✓a

$\therefore rt = \ln\frac{A}{P}$ ✓m – taking logs ✓a

$\therefore t = \frac{\ln\frac{A}{P}}{r}$ ✓a

$\therefore t = \frac{\ln 3}{0{,}1}$ ✓m – substitution

$\therefore t = 11$ years ✓ca (6)

2.2 By first making r the subject of the formula, determine the annual interest rate (expressed as a percentage to 2 decimal places) that will increase R500 to a total of R900 in 3 years.

$\therefore rt = \ln\frac{A}{P}$ ✓m – from above

$\therefore r = \frac{\ln\frac{A}{P}}{t}$ ✓a

$\therefore r = \frac{\ln\frac{900}{500}}{3}$ ✓m – substitution

$\therefore r = 19{,}59\%$ ✓ca (4)

[10]

QUESTION 3

Use mathematical induction to prove that $n^3 + 2n$ is divisible by 3 for $n \in \mathbb{N}$.

When $n = 1$ we have $1^3 + 2(1) = 3$ which is divisible by 3 ✓m – proving true for $n = 1$
so it is true for $n = 1$ ✓a
Assume true for $n = k$ ✓m – assume true
viz. $k^3 + 2k = 3p$ for $p \in \mathbb{N}$ ✓a ✓a for p element of N

now $(k+1)^3 + 2(k+1) = k^3 + 3k^2 + 3k + 1 + 2k + 2$ ✓m for $k + 1$ ✓a

$= (k^3 + 2k) + 3k^2 + 3k + 3$ ✓a

$= 3p + 3(k^2 + k + 1)$ ✓m – using assumption ✓a

which is clearly divisible by 3
so, we have proved it true for $n = k + 1$ ✓a ✓a
$\therefore$ by the principle of mathematical induction, we have proved it true for $n \in \mathbb{N}$

[12]

QUESTION 4

Determine $f'(x)$ by first principles if $f(x) = \sqrt{1-x}$.

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{1-(x+h)} - \sqrt{1-x}}{h}$$ ✓m ✓a

$$= \lim_{h \to 0} \frac{\sqrt{1-(x+h)} - \sqrt{1-x}}{h} \times \frac{\sqrt{1-(x+h)} + \sqrt{1-x}}{\sqrt{1-(x+h)} + \sqrt{1-x}}$$ ✓m – rationalising ✓a

$$= \lim_{h \to 0} \frac{1-(x+h)-(1-x)}{h\left(\sqrt{1-(x+h)} + \sqrt{1-x}\right)}$$ ✓a ✓a

$$= \lim_{h \to 0} \frac{-h}{h\left(\sqrt{1-(x+h)} + \sqrt{1-x}\right)}$$ ✓ca

$$= \lim_{h \to 0} \frac{-1}{\sqrt{1-(x+h)} + \sqrt{1-x}}$$ ✓ca

$$= \frac{-1}{2\sqrt{1-x}}$$ ✓ca ✓ notation throughout

[10]

QUESTION 5

5.1 Consider the function $f(x)=\dfrac{2x^2+2x-3}{x^2-5x-6}$.

(a) Give the equations and nature of all asymptotes.

$$f(x)=\frac{2x^2+2x-3}{(x-6)(x+1)}$$ ✓m factoring

✓a

so, vertical asymptotes of $x=6$ and $x=-1$ ✓a ✓a

✓a horizontal asymptote of $y=2$ ✓a (6)

(b) Prove that the function is strictly decreasing.

$$f(x)=\frac{2x^2+2x-3}{x^2-5x-6}$$ ✓a ✓a

$$\therefore f'(x)=\frac{(4x+2)(x^2-5x-6)-(2x-5)(2x^2+2x-3)}{(x^2-5x-6)^2}$$ ✓m – quotient rule ✓a

$$\therefore f'(x)=\frac{4x^3-18x^2-34x-12-(4x^3-6x^2-16x+15)}{(x^2-5x-6)^2}$$ ✓a

$$\therefore f'(x)=\frac{-12\left[x^2+\frac{3}{2}x+\frac{27}{12}\right]}{(x^2-5x-6)^2}$$ ✓m completing square

$$\therefore f'(x)=\frac{-12\left[\left(x+\frac{3}{4}\right)^2-\frac{9}{16}+\frac{27}{12}\right]}{(x^2-5x-6)^2}$$ ✓a

$$\therefore f'(x)=\frac{-12\left[\left(x+\frac{3}{4}\right)^2+\frac{27}{16}\right]}{(x^2-5x-6)^2}$$ ✓a

$$\therefore f'(x)=\frac{-12\left(x+\frac{3}{4}\right)^2-\frac{81}{4}}{(x^2-5x-6)^2}$$ ✓a

which is always negative so the function is strictly decreasing ✓a ✓a (11)

5.2 Give the equation of a rational function which has:

- an oblique asymptote of $y = 2x + 1$
- a vertical asymptote of $x = -2$
- no *x*-intercepts

✓m ✓a ✓a ✓a

$(x+2)(2x+1) + \text{remainder}$

$$y = \frac{2x^2 + 5x + 13}{x+2}$$ ✓a ✓a ✓a ✓a ✓a ✓a

NOTE:
Answer is not unique c must be bigger than 25/8
marks awarded for thinking rather than working

(10)

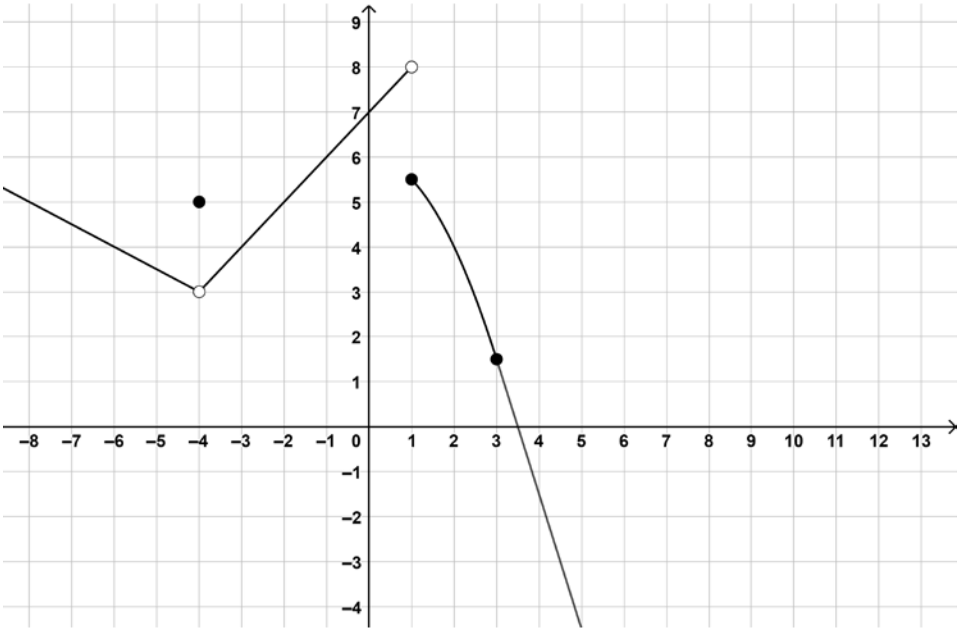
[27]

QUESTION 6

Consider the function, *f*, defined as follows:

$$f(x)=\begin{cases} -0{,}5x+1 & x<-4 \\ 5 & x=-4 \\ x+7 & -4<x<1 \\ -0{,}5x^2+6 & 1\le x\le 3 \\ ax+b & x\ge 3 \end{cases}$$

f is depicted in the graph below:



6.1 Identify, by means of their *x*-coordinates, any points of discontinuity. You should also classify the discontinuity and justify your classifications mathematically. Pay careful attention to notation.

There is a removable discontinuity ✓a at $x=-4$ ✓a
since $\lim_{x\to -4} f(x)\neq f(-4)$ ✓a

There is a non-removable/jump ✓a discontinuity at $x=1$ ✓a
since $\lim_{x\to 1} f(x)$ *d.n.e* ✓a (6)

6.2 Determine *a* and *b* if *f* is differentiable at $x = 3$.

If *f* is differentiable at 3 it needs to be continuous at 3. ✓a

$\lim_{x\to 3^-} f(x) = 1{,}5$ ✓a

so $\lim_{x\to 3^+} (ax + b) = 1{,}5$ ✓a

$\therefore 3a + b = 1{,}5$ ✓a

but we also know that $\lim_{x\to 3^-} f'(x) = \lim_{x\to 3^+} f'(x)$ ✓a

so $\lim_{x\to 3^-} (-x) = \lim_{x\to 3^+} a$ ✓a

$\therefore -3 = a$ ✓ca

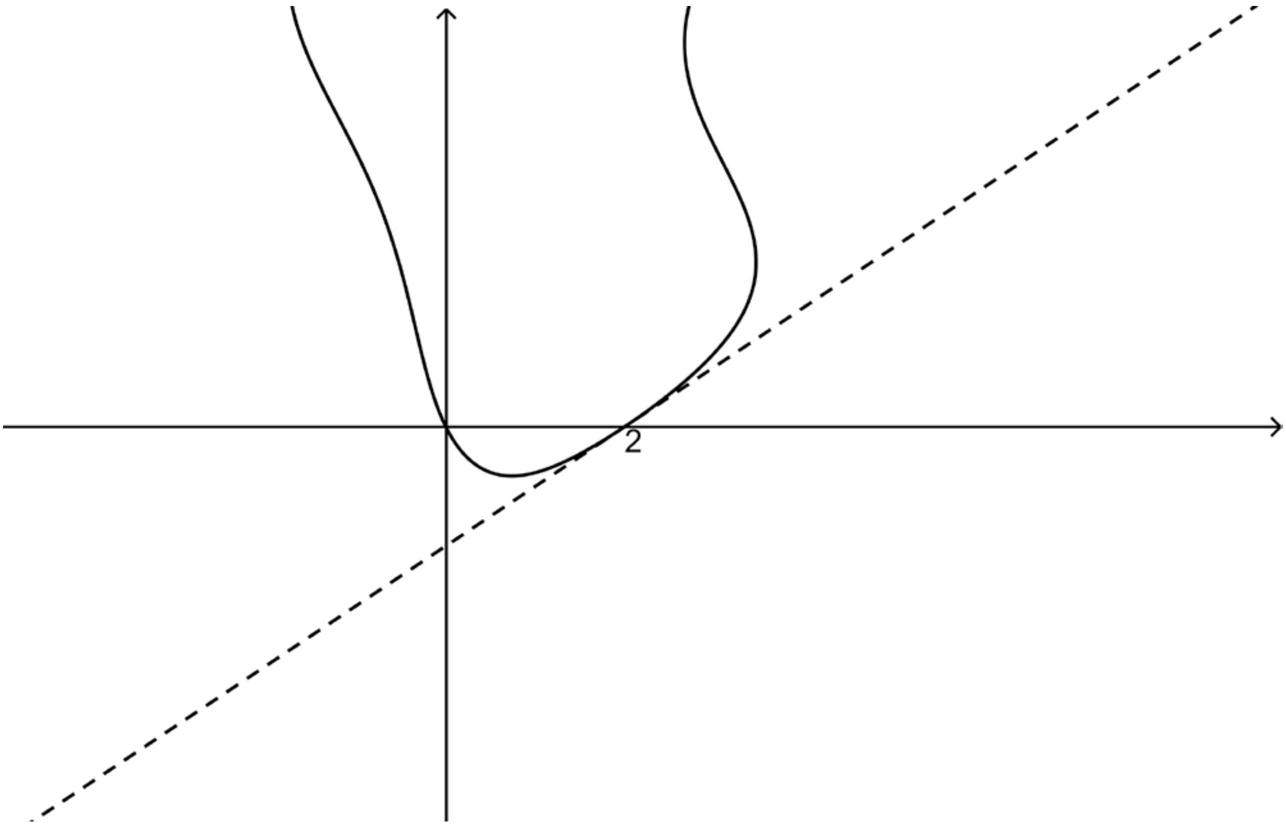
$\therefore b = 10{,}5$ ✓ca (8)

[14]

QUESTION 7

A portion of the implicitly defined curve $x^2 - x\sin y = y + 2x$ is shown below.

Determine the equation of the tangent to the curve at the point $(2;0)$.



$$2x - (1\sin y + xy'\cos y) = y' + 2$$ ✓a ✓a ✓a ✓m – implicit differentiation

$$\therefore 2x - \sin y - xy'\cos y - y' = 2$$

$$\therefore y'(-x\cos y - 1) = 2 - 2x + \sin y$$ ✓m – factoring y' out

$$\therefore y' = \frac{2 - 2x + \sin y}{-x\cos y - 1}$$ ✓ca

when $x = 2$ and $y = 0$ ✓m – substitution

$$y' = \frac{2 - 4 + 0}{-2 - 1} = \frac{2}{3}$$ ✓ca

$$\therefore y - 0 = \frac{2}{3}(x - 2)$$ ✓m – straight-line formula

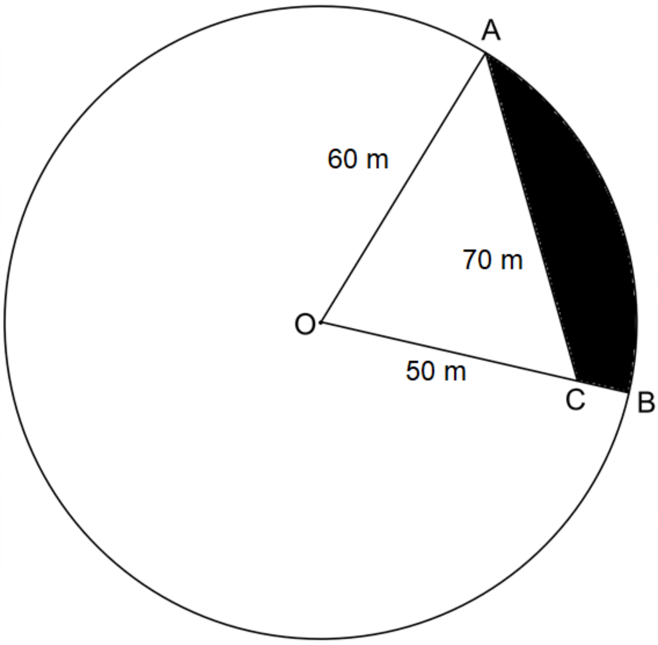
$$\therefore y = \frac{2}{3}x - \frac{4}{3}$$ ✓ca

[10]

QUESTION 8

[Source: <https://www.northwesthydro.com.au/blog/solar-pumping-for-centre-pivot-irrigation/>]

The sketch below is of a circular field with a centre pivot irrigation system on it. O is the centre and OCB is a straight line. OA is 60 m and OC is 50 m. AC is 70 m.



An aerial photo of the field has shown that the shaded area is infected with weeds. What percentage of the field is infected?

let $B\hat{O}A = \theta$ then $\cos\theta = \frac{50^2 + 60^2 - 70^2}{2(50)(60)} = \frac{1}{5}$ ✓m ✓a – cos rule for theta

$\therefore \theta = \cos^{-1}\frac{1}{5} = 1{,}369$ ✓ca

now shaded area $=$ sector $AOB - \Delta AOC$ ✓ca

$= \frac{1}{2}60^2(1{,}369) - \frac{1}{2}(60)(50)\sin(1{,}369)$ ✓m ✓a ✓m ✓a

$= 995{,}295$ ✓ca

As a percentage of the total: ✓m

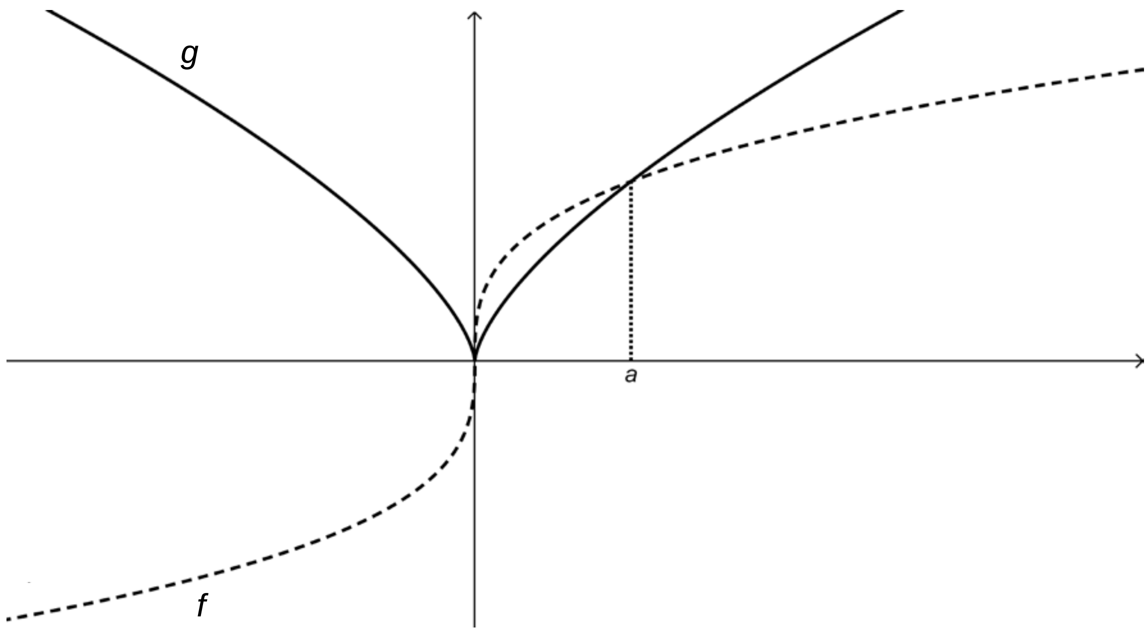
$\frac{995{,}295}{\pi \times 60^2} \times 100 = 8{,}8\%$ ✓a ✓ca

[12]

QUESTION 9

9.1 Consider the two functions below:

f is an **odd** function since $f(-x) = -f(x)$ while *g* is an **even** function since $g(-x) = g(x)$. To help you distinguish them, *f* has been drawn with a dotted line and *g* with a solid line. *f* and *g* intersect at x = a.



If it is further given that $\int_0^a f(x)\,dx = 0{,}75$ and $\int_0^a g(x)\,dx = 0{,}6$ then determine the following:

(a) $\int_0^a f(x) - g(x)\,dx$

$= \int_0^a f(x)\,dx - \int_0^a g(x)\,dx$ ✓m – subtracting
$= 0{,}75 - 0{,}6$
$= 0{,}15$ ✓ca (2)

(b) $\int_{-a}^{0} f(x) + g(x)\, dx$

$= \int_{-a}^{0} f(x)\, dx + \int_{-a}^{0} g(x)\, dx$ ✓m – splitting
✓a
$= -0{,}75 + 0{,}6$
$= -0{,}15$ ✓a

(3)

(c) $\int_{-a}^{a} 2f(x) + 3g(x)\, dx$

$= 2\int_{-a}^{a} f(x)\, dx + 3\int_{-a}^{a} g(x)\, dx$ ✓m – splitting ✓a
$= 2(0) + 3(2 \times 0{,}6)$ ✓a
$= 3{,}6$ ✓ca

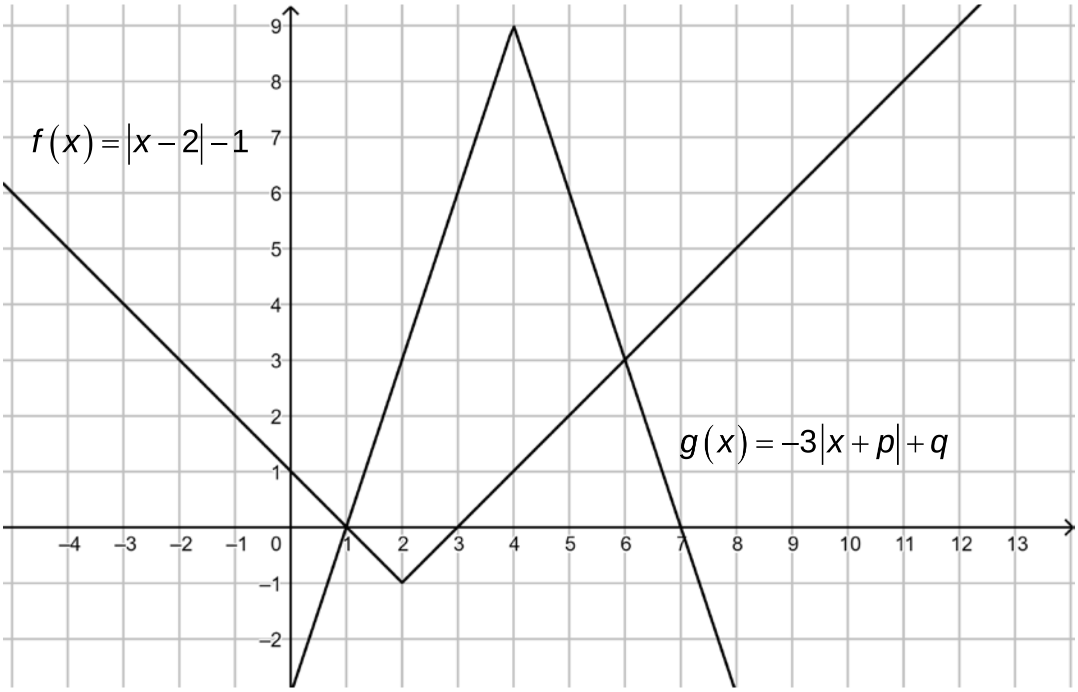
(4)

(d) $\int_{-a}^{a} f(|x|)\, dx$

$= 2\int_{0}^{a} f(x)\, dx$ ✓m – symmetry
$= 2 \times 0{,}75$ ✓a
$= 1{,}5$ ✓ca

(3)

9.2 Consider the functions $f(x) = |x-2| - 1$ and $g(x) = -3|x+p| + q$ drawn below:



(a) Determine the values of *p* and *q.*

$p = -4$ ✓a ✓a and $q = 9$ ✓a ✓a (4)

(b) Using the graphs, or otherwise, solve: $|x-2| + 3|x-4| > 10$.

$|x-2| + 3|x-4| > 10$

$\therefore |x-2| > -3|x-4| + 10$ ✓m – manipulating

$\therefore |x-2| - 1 > -3|x-4| + 9$ ✓a ✓a

but this is just where $f > g$ ✓m – using graphs

so, $x < 1$ ✓a or $x > 6$ ✓a

(6)

(c) Determine $\int_1^7 g(x)\,dx$

By areas $\int_1^7 g(x)\,dx = \int_1^3 g(x)\,dx + \int_3^7 g(x)\,dx$ ✓m – splitting ✓a

$= -1 + 8$ (using areas) ✓m – using areas

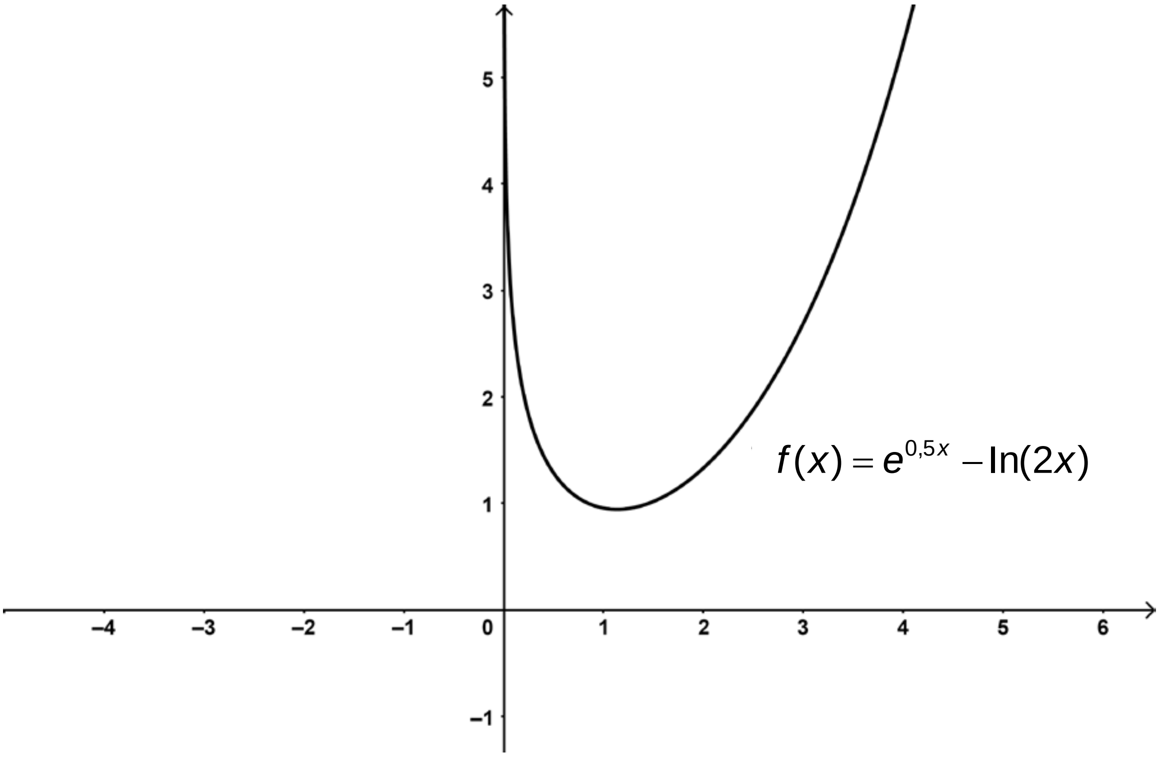
$= 7$ ✓ca

(4)

[26]

QUESTION 10

Use Newton-Raphson iteration to find the turning point of the given function.



You should:

- Show the iterative formula you use.
- Use an initial approximation of $x = 2$.
- Show your first approximation to 5 decimal places.

$f(x) = e^{0,5x} - \ln(2x)$

$so\ f'(x) = \frac{1}{2}e^{0,5x} - \frac{2}{2x}$ ✓m – differentiating

we want to solve $\frac{1}{2}e^{0,5x} - \frac{1}{x} = 0$ ✓m – equating to 0 ✓a

so this is our $f(x)$

then $f'(x) = \frac{1}{4}e^{0,5x} + \frac{1}{x^2}$ ✓a ✓a ✓m

so $x_{n+1} = x_n - \dfrac{\frac{1}{2}e^{0,5x_n} - \frac{1}{x_n}}{\frac{1}{4}e^{0,5x_n} + \frac{1}{x_n^{\,2}}}$ ✓m – Newton formula ✓a

$\therefore x_1 = 1{,}07577$ ✓a ✓a

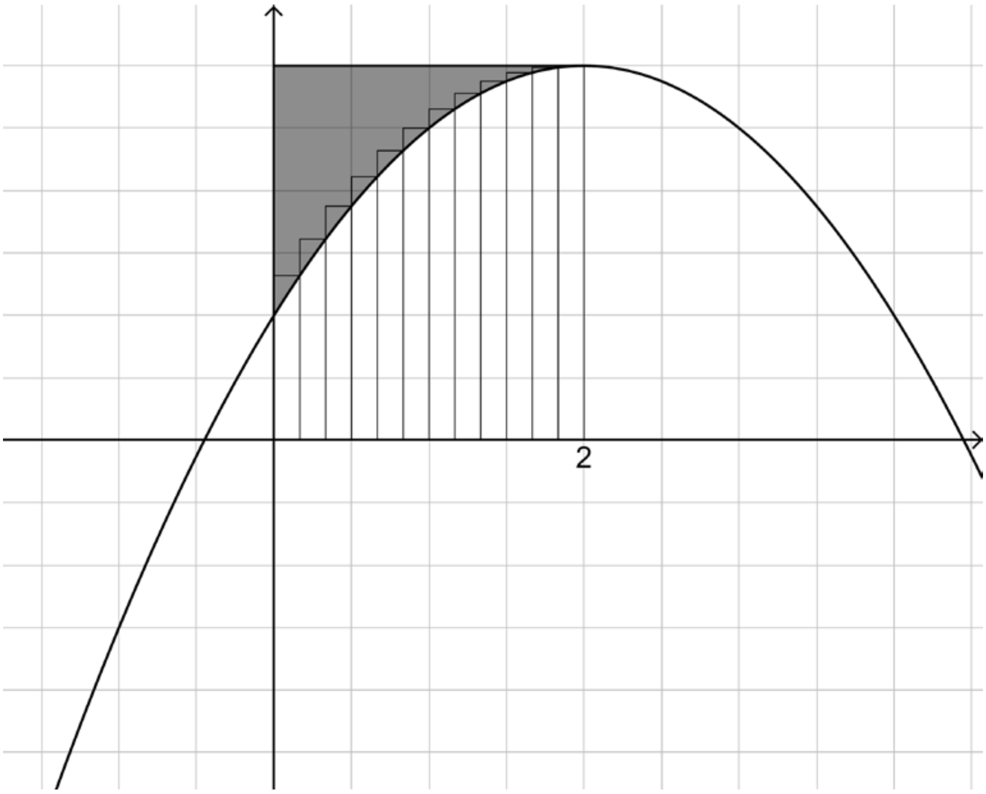
$\therefore x = 1{,}13429$ ✓ca ✓ca

[12]

QUESTION 11

11.1 When the area bounded by the curve *f*, the *x*-axis and the lines $x = 0$ and $x = 2$ is partitioned into *n* rectangles the area is given by:

$$A = -\frac{8}{3} - \frac{4}{3n^2} + 12 + \frac{4}{n}$$



If it is further given that $f(2) = 6$ then determine the shaded area, correct to 2 decimal places.

Area between curve and *x*-axis $= \lim_{n\to\infty}\left(-\frac{8}{3} - \frac{4}{3n^2} + 12 + \frac{4}{n}\right) = \frac{28}{3}$ ✓m – limit ✓a

so, shaded area $= 2\times 6 - \frac{28}{3}$ ✓m – calculating rectangle ✓m – subtracting

$= 12 - \frac{28}{3}$

$= \frac{36}{3} - \frac{28}{3}$ ✓a

$= \frac{8}{3}$ units^2 ✓ca (6)

11.2 Determine:

(a) $\int x\left(3x^2+7\right)^3 dx$

$= \dfrac{\left(3x^2+7\right)^4}{24} + c$ ✓a ✓a ✓a ✓a

✓✓ 2 – for method (substitution)
Need not be shown

(6)

(b) $\int e^{2x} x\, dx$

let $f(x) = x$ ✓a and $g'(x) = e^{2x}$ ✓a ✓m – parts

then $f'(x) = 1$ and $g(x) = \dfrac{1}{2}e^{2x}$ ✓ca ✓ca

$\int e^{2x} x\, dx = \dfrac{1}{2}xe^{2x} - \dfrac{1}{2}\int e^{2x} dx$ ✓ca ✓ca

$= \dfrac{1}{2}xe^{2x} - \dfrac{1}{4}e^{2x} + c$ ✓ca

(8)

(c) $\int \dfrac{3x-5}{x^2-2x-3}\, dx$

$\dfrac{3x-5}{x^2-2x-3} = \dfrac{3x-5}{(x-3)(x+1)} = \dfrac{1}{x-3} + \dfrac{2}{x+1}$ (cover-up method) ✓a ✓a ✓a ✓m – partial fractions

$\therefore \int \dfrac{3x-5}{x^2-2x-3} dx = \int \dfrac{1}{x-3} dx + \int \dfrac{2}{x+1} dx$ ✓m – splitting up ✓a

$= \ln|x-3| + 2\ln|x+1| + c$

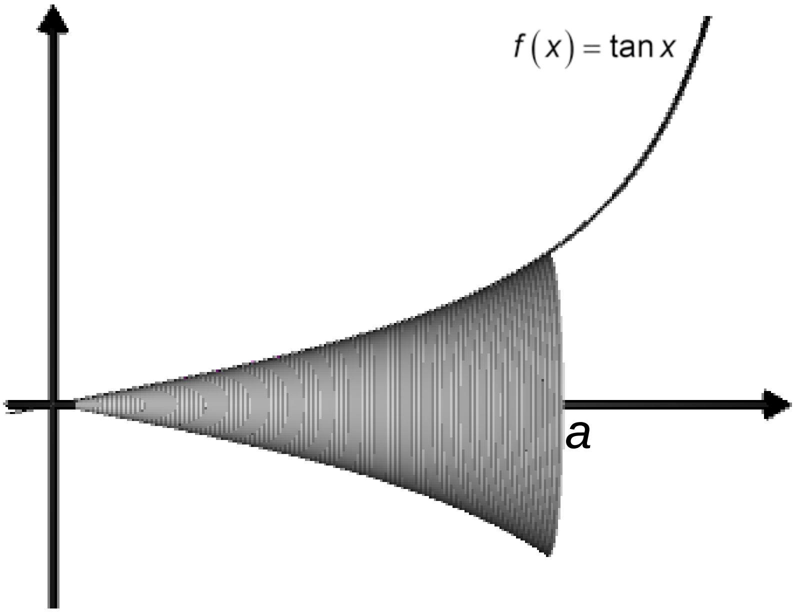
✓ca ✓ca
✓ca ✓ca

(10)

[30]

QUESTION 12

The area bounded by the curve $f(x) = \tan x$, the x-axis, the line $x = 0$ and the line $x = a$, $a < \frac{\pi}{2}$ is rotated about the x-axis.



Give an expression for the volume in terms of *a*.

volume $= \pi\int_0^a (\tan(x))^2\,dx$ ✓m – volume formula ✓a

$= \pi\int_0^a \tan^2 x\,dx$

$= \pi\int_0^a \sec^2 x - 1\,dx$ ✓m – using identity ✓a

✓a ✓a

$= \pi[\tan x - x]_0^a$

✓ca ✓ca

$= \pi[\tan a - a] - \pi[\tan 0 - 0]$

$= \pi[\tan a - a]$ ✓ca ✓ca

[10]

Total: 200 marks