GRADE 12 EXAMINATION
NOVEMBER 2021

ADVANCED PROGRAMME MATHEMATICS: PAPER I
MODULE 1: CALCULUS AND ALGEBRA

MARKING GUIDELINES

Time: 2 hours — 200 marks

These marking guidelines are prepared for use by examiners and sub-examiners, all of whom are required to attend a standardisation meeting to ensure that the guidelines are consistently interpreted and applied in the marking of candidates' scripts.

The IEB will not enter into any discussions or correspondence about any marking guidelines. It is acknowledged that there may be different views about some matters of emphasis or detail in the guidelines. It is also recognised that, without the benefit of attendance at a standardisation meeting, there may be different interpretations of the application of the marking guidelines.

QUESTION 1

1.1 Solve, for $x \in \mathbb{R}$:

(a) $2e^{x} - 7 + 6e^{-x} = 0$, expressing your answers in exact form using logs

$\therefore 2e^{2x} - 7e^{x} + 6 = 0$

$\therefore (2e^{x} - 3)(e^{x} - 2) = 0$

$\therefore e^{x} = \dfrac{3}{2}$ or $e^{x} = 2$

$\therefore x = \ln\dfrac{3}{2}$ or $x = \ln 2$

(b) $|2x+3| = 5x-2$

$|2x+3| = 5x-2$

$2x+3 = 5x-2$ or $2x+3 = -(5x-2)$

$\therefore 5 = 3x$ or $7x = -1$

$\therefore x = \dfrac{5}{3}$ or $x = -\dfrac{1}{7}$

a check reveals that $x = \dfrac{5}{3}$ only

(c) Solve $\dfrac{(-x^{2}-5)(x^{2}-16)}{|x+3|(x+2)} \geq 0$

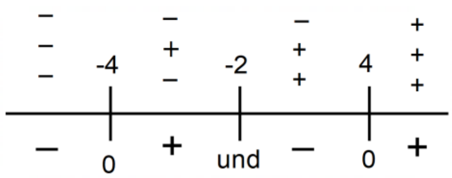
$\dfrac{(-x^{2}-5)(x^{2}-16)}{|x+3|(x+2)} \geq 0$

$\therefore \dfrac{(-x^{2}-5)(x^{2}-16)}{(x+2)} \geq 0$

$\therefore \dfrac{(x^{2}-16)}{(x+2)} \leq 0$

$\therefore \dfrac{(x-4)(x+4)}{(x+2)} \leq 0$

$\therefore x \leq -4$ *or* $-2 < x \leq 4$



1.2 Consider the function $f(x) = x^4 - 3x^3 - 5x^2 + 29x - 30$

(a) Given that $x = 2 - i$ is a root to the equation $f(x) = 0$, write $f\ x$ as a product of two trinomials.

$2 - i$ and $2 + i$ are roots

$\therefore$ one of the trinomials is $x^2 - (2 - i + 2 + i)x + ((2 - i)(2 + i))$

$\therefore$ one of the trinomials is $x^2 - 4x + 5$

$\therefore f(x) = (x^2 - 4x + 5)(x^2 + x - 6)$

(b) Hence, solve $f(x) = 0$ in $\mathbb{C}$.

$(x^2 - 4x + 5)(x^2 + x - 6) = 0$

$\therefore (x^2 - 4x + 5)(x + 3)(x - 2) = 0$

$\therefore x = 2 - i$ or $2 + i$ or -3 or 2

1.3 Kwande defined a new type of complex number called a **Kwande number**.
It has the property that $\text{Im}(z) = 2\text{Re}(z)$
In other words, a **Kwande number**, z is of the form: $z = a + 2ai$

Prove that for all **Kwande numbers** $\frac{z}{z^*} = -\frac{3}{5} + \frac{4}{5}i$

$$\frac{z}{z^*} = \frac{a + 2ai}{a - 2ai}$$

$$= \frac{a + 2ai}{a - 2ai} \times \frac{a + 2ai}{a + 2ai}$$

$$= \frac{a^2 + 4a^2i + 4a^2i^2}{5a^2}$$

$$= \frac{-3a^2 + 4a^2i}{5a^2}$$

$$= -\frac{3}{5} + \frac{4}{5}i$$

QUESTION 2

The number of people, n in a school with population P who have heard a rumour can be modelled by the following function:

$$n = P - Pe^{-0.14t}$$

where t is the time (in days) that have elapsed since the rumour began.

(a) Make t the subject of the formula

$$n = P - Pe^{-0.14t}$$
$$\therefore Pe^{-0.14t} = P - n$$
$$\therefore e^{-0.14t} = \frac{P-n}{P}$$
$$\therefore -0.14t = \ln\left(\frac{P-n}{P}\right)$$
$$\therefore t = \frac{\ln\left(\frac{P-n}{P}\right)}{-0.14}$$

(b) Hence, determine how many days, to the nearest day, it will take for at least 750 people in a school of 1200 to have heard the rumour.

$$\therefore t = \frac{\ln\left(\frac{1200-750}{1200}\right)}{-0.14}$$
$$\therefore t = 7 \text{ days}$$

QUESTION 3

By first principles show that the derivative of $f(x)=\frac{1}{2+3x}$ is $\frac{-3}{(2+3x)^2}$

$$f(x)=\frac{1}{2+3x}$$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{1}{2+3(x+h)}-\frac{1}{2+3x}}{h}$$

$$=\lim_{h\to 0}\frac{\frac{1}{2+3x+3h}-\frac{1}{2+3x}}{h}$$

$$=\lim_{h\to 0}\frac{\frac{(2+3x)-(2+3x+3h)}{(2+3x+3h)(2+3x)}}{h}$$

$$=\lim_{h\to 0}\frac{\frac{2+3x-2-3x-3h}{(2+3x+3h)(2+3x)}}{h}$$

$$=\lim_{h\to 0}\frac{\frac{-3h}{(2+3x+3h)(2+3x)}}{h}$$

$$=\lim_{h\to 0}\frac{-3h}{(2+3x+3h)(2+3x)}\times\frac{1}{h}$$

$$=\lim_{h\to 0}\frac{-3}{(2+3x+3h)(2+3x)}$$

$$=\frac{-3}{(2+3x)^2}$$

QUESTION 4

Prove that $3^{2n+4} - 2^{2n}$ is a multiple of 5 for all $n \in \mathbb{N}$

if $n = 1$ then we have

$3^{2(1)+4} - 2^{2(1)} = 3^6 - 4 = 725$ which is a multiple of 5

so, it is true for $n = 1$

Assume true for $n = k$ *viz*

$3^{2k+4} - 2^{2k} = 5p$ where $p \in \mathbb{N}$ $(*)$

Now in the case where $n = k + 1$ we have

$$3^{2(k+1)+4} - 2^{2(k+1)} = 3^{2k+2+4} - 2^{2k+2}$$
$$= 3^2 \times 3^{2k+4} - 2^2 2^{2k}$$

but from $(*)$ $3^{2k+4} = 5p + 2^{2k}$

so, $3^{2(k+1)+4} - 2^{2(k+1)} = 9\left(5p + 2^{2k}\right) - 4 \times 2^{2k}$ using (*)

$$= 45p + 9 \times 2^{2k} - 4 \times 2^{2k}$$
$$= 5\left(9p + 2^{2k}\right)$$

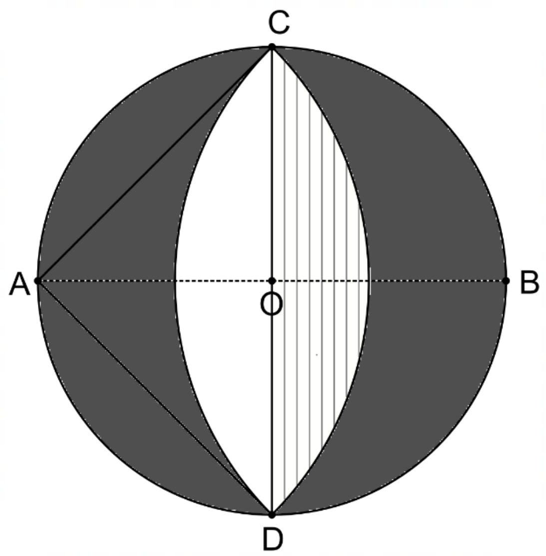
which is clearly a multiple of 5

so, we have proved it true for $n = k + 1$

By the principle of mathematical induction we have proved it true for all $n \in \mathbb{N}$

QUESTION 5

In the diagram below, the circle, centre O, with diameter AB has a radius of 4 cm.
Circular arcs are drawn through C and D with A and B as centres.



Determine the shaded area.

$OA = OC = OD = 4$

$AC = AD = \sqrt{32}$ (pythag)

$\therefore OAC = OAD = 45° = \frac{\pi}{4}$ ($\angle s$ of isos. $\triangle$)

$\therefore CAD = 90° = \frac{\pi}{2}$ (can also use converse of pythag to establish this)

$\therefore$ the area of the striped segment $=$ area sector $ACD -$ area $\triangle ACD$

$$= \frac{1}{2}\left(\sqrt{32}\right)^2 \frac{\pi}{2} - \frac{1}{2}\left(\sqrt{32}\right)\left(\sqrt{32}\right)$$

$$= 8\pi - 16$$

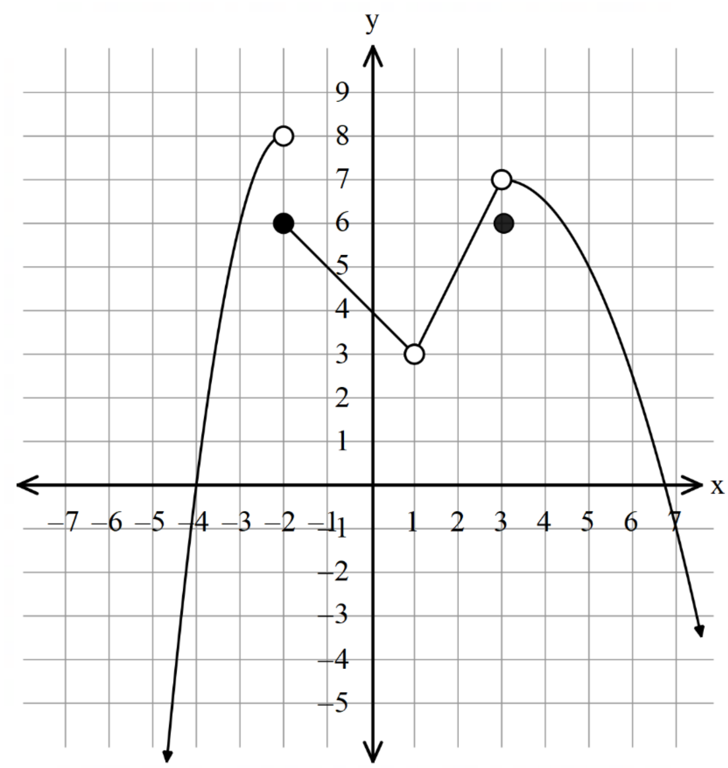
by symmetry the unshaded area $= 16\pi - 32$

Area of circle $= \pi\left(4^2\right) = 16\pi$

So, the shaded area $= 16\pi - \left(16\pi - 32\right) = 32\ cm^2$

QUESTION 6

6.1 Consider the graph of the function *f* shown below.



Answer the following questions paying careful attention to the precision of mathematical notation you use:

(a) Using mathematical notation, justify why *f* is discontinuous at $x=-2$

$\lim\limits_{x\to-2^+} f(x) \neq \lim\limits_{x\to-2^-} f(x)$

so $\lim\limits_{x\to-2} f(x)$ d.n.e.

(b) Using mathematical notation, justify why *f* is discontinuous at $x=1$

$f(1)$ is not defined

(c) Using mathematical notation, justify why *f* is discontinuous at $x=3$.

$\lim\limits_{x\to3} f(x) \neq f(3)$

(d) What is the nature of the discontinuity at $x=-2$?

Jump / non-removable

(e) What is the nature of the discontinuity at $x = 3$?

removable

6.2 Explain why $\lim\limits_{x \to 0} \frac{|x|}{x}$ does not exist.

$\lim\limits_{x \to 0^+} \frac{|x|}{x} = \lim\limits_{x \to 0^+} \frac{x}{x} = \lim\limits_{x \to 0^+} 1 = 1$

but $\lim\limits_{x \to 0^-} \frac{|x|}{x} = \lim\limits_{x \to 0^-} \frac{-x}{x} = \lim\limits_{x \to 0^+} (-1) = -1$

$\therefore \lim\limits_{x \to 0} \frac{|x|}{x}$ does not exist since $\lim\limits_{x \to 0^+} \frac{|x|}{x} \neq \lim\limits_{x \to 0^-} \frac{|x|}{x}$

6.3 Consider the function *g*, defined below:

$$f(x) = \begin{cases} -0.5x^2 + 2x + 3 & x \leq 2 \\ px^2 + qx + 13 & x > 2 \end{cases}$$

Using appropriate mathematical notation, determine the rational values of *p* and *q* if *g* is differentiable at $x = 2$

for differentiability we need continuity

$\lim\limits_{x \to 2^-} f(x) = f(2) = 5$

so, we need $\lim\limits_{x \to 2^+} f(x) = 4p + 2q + 13 = 5$ *or* $4p + 2q = -8$ (1)

$\lim\limits_{x \to 2^-} f'(x) = \lim\limits_{x \to 2^-} (-x + 2) = 0$

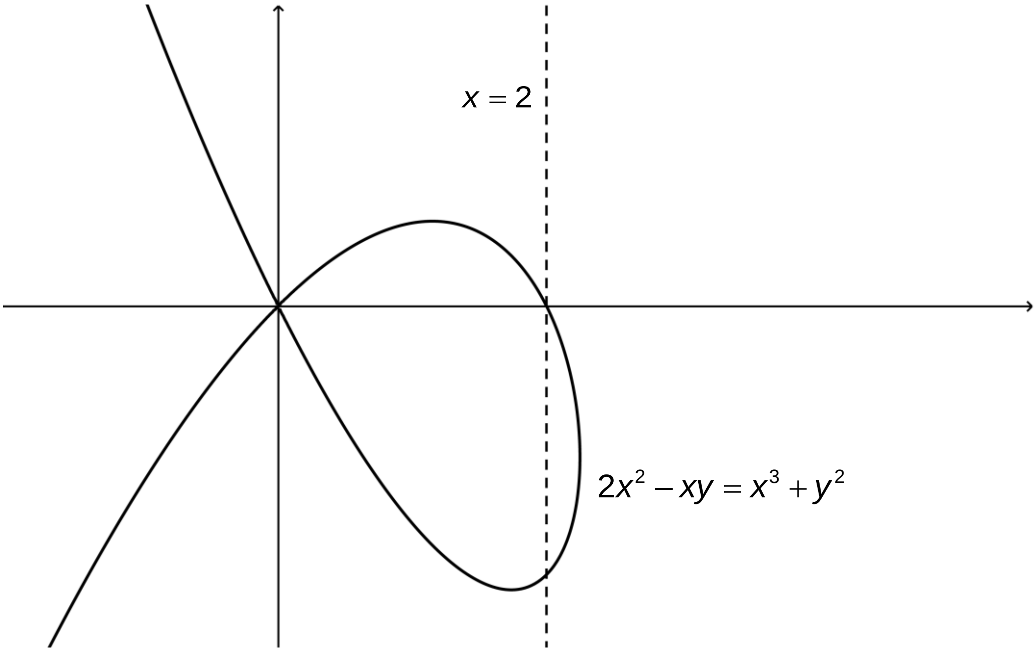
so, we need $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (2px + q) = 4p + q = 0$ (2)

solving (1) and (2) simultaneously:

$q = -8$ *and* $p = 2$

QUESTION 7

The curve $2x^2 - xy = x^3 + y^2$ has two tangents where the *x*-coordinate of the point of contact is 2. Find the equation of the tangent with a positive gradient.



$2x^2 - xy = x^3 + y^2$

when $x = 2$:

$8 - 2y = 8 + y^2$

$\therefore y^2 + 2y = 0$

$\therefore y(y+2) = 0$

$\therefore y = 0 \text{ or } -2$

$\therefore$ *point of contact is* $(2;-2)$

now implicit differentiation yields:

$$4x - \left(y + x\frac{dy}{dx}\right) = 3x^2 + 2y\frac{dy}{dx}$$

$$\therefore 4x - y - x\frac{dy}{dx} = 3x^2 + 2y\frac{dy}{dx}$$

$$\therefore -x\frac{dy}{dx} - 2y\frac{dy}{dx} = 3x^2 + y - 4x$$

$$\therefore \frac{dy}{dx} = \frac{3x^2 + y - 4x}{-x - 2y}$$

$$at\ (2;-2)\ \frac{dy}{dx} = \frac{12 - 2 - 8}{-2 + 4} = \frac{2}{2} = 1$$

$\therefore y + 2 = 1(x - 2)$

$\therefore y = x - 4$

QUESTION 8

8.1 For each of the given functions, determine integral value(s) of *a* if:

(a) $f(x)=\dfrac{ax^2+2x+3}{-\frac{1}{2}x^2+3x+4}$ has an asymptote $y=2$

we need $\dfrac{a}{-\frac{1}{2}}=2$

$\therefore a=-1$

(b) $f(x)=\dfrac{x^2+4}{x^2-4x+a}$ has asymptotes $x=1$ and $x=3$

$(x-1)(x-3)$

$\therefore a=3$

(c) $f(x)=\dfrac{2x^2+2x+3}{x+a}$ has an asymptote $y=2x-4$

$2x^2+2x+3=(x+a)(2x-4)+R$

$\therefore 2ax-4x=2x$

$\therefore 2ax=6x$

$\therefore a=3$

(d) $f(x)=\dfrac{x^2+2x+3}{x^a+3x+4}$ has an asymptote $y=0$

we need $\deg(\text{denominator})>\deg(\text{numerator})$

$\therefore a>2$

8.2 Determine the *x*-coordinate(s) of the stationary point(s) of $f(x)=\dfrac{2x^2-7x+1}{2x+3}$

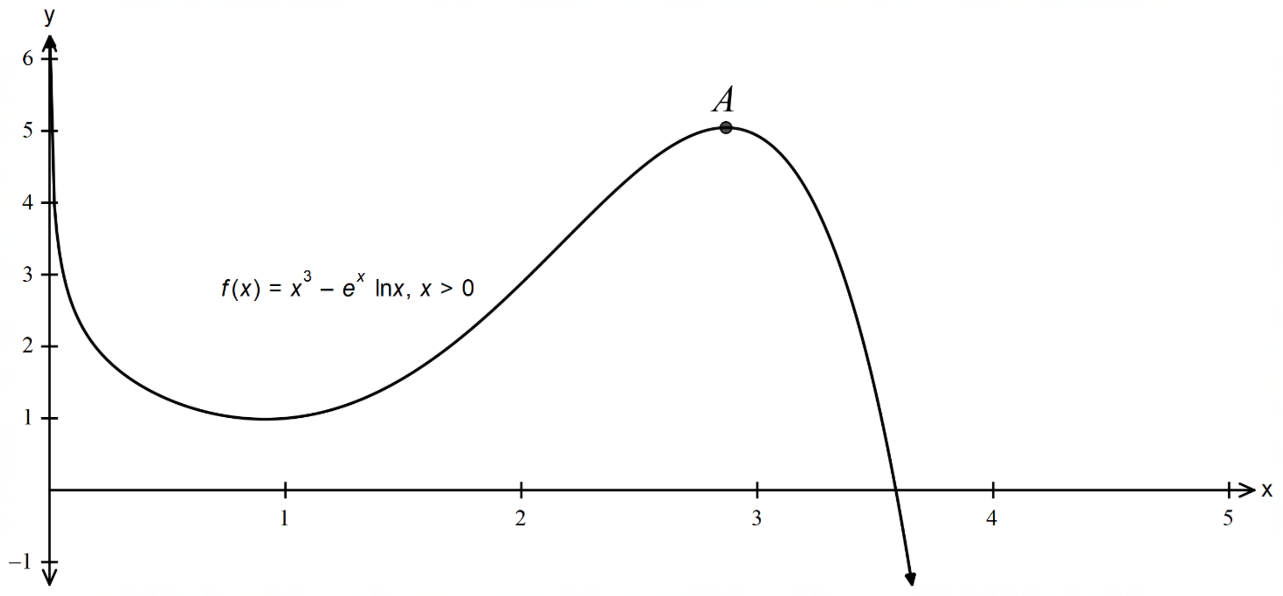
$\therefore f'(x)=\dfrac{(4x-7)(2x+3)-2(2x^2-7x+1)}{(2x+3)^2}=0$

$\therefore 4x^2+12x-23=0$

$\therefore x=1{,}33$ or $x=4{,}33$

QUESTION 9

A portion of the function $f(x) = x^3 - e^x \ln x,\ x > 0$ is shown.



9.1 Show that the equation below would be the one you would have to solve to find the *x*-coordinate of the local maximum at point A.

$$3x^2 - e^x\left(\ln x + x^{-1}\right) = 0$$

At A, $f'(x) = 0$

$\therefore 3x^2 - \left(e^x \ln x + \dfrac{e^x}{x}\right) = 0$

$\therefore 3x^2 - e^x\left(\ln x + \dfrac{1}{x}\right) = 0$

$\therefore 3x^2 - e^x\left(\ln x + x^{-1}\right) = 0$

9.2 Use Newton-Raphson iteration to find, to five decimal places, the x-coordinate of A.

- show the iterative formula you use.
- use $x_0 = 3$ as the first approximation.
- show the value for x_1 to five decimal places.

$$f(x) = 3x^2 - e^x \ln x - e^x x^{-1}$$

$$\therefore f'(x) = 6x - e^x \ln x - \frac{e^x}{x} - \frac{e^x}{x} + \frac{e^x}{x^2}$$

$$\therefore x_{n+1} = x_n - \frac{3x^2 - e^x \ln x - e^x x^{-1}}{6x - e^x \ln x - \frac{2e^x}{x} + \frac{e^x}{x^2}}$$

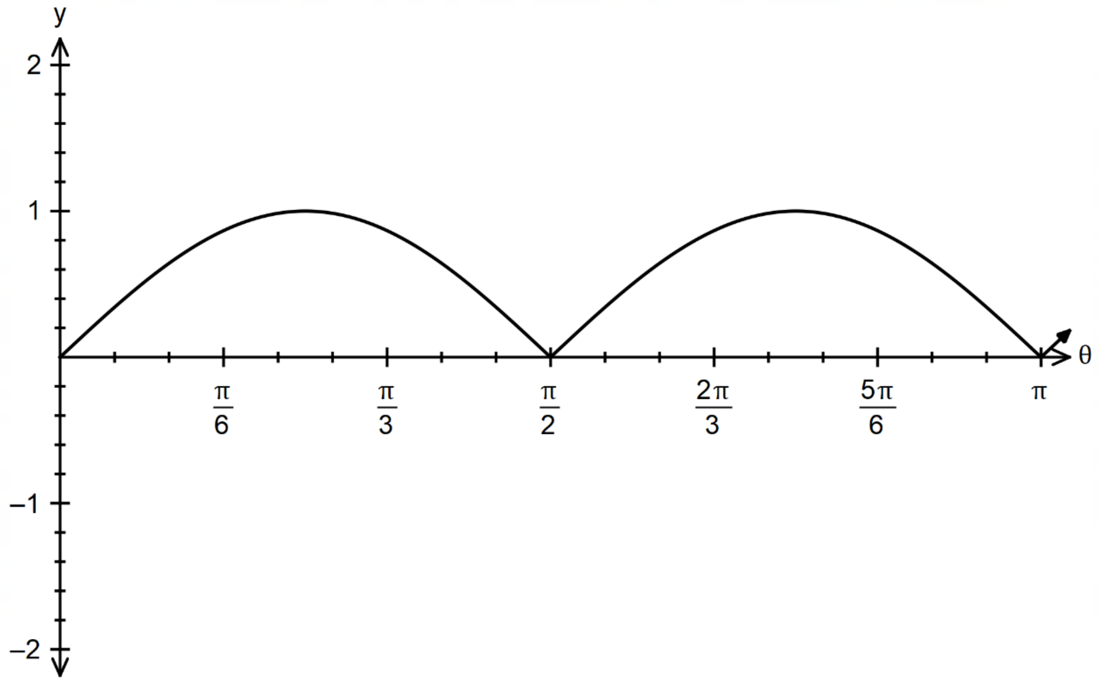
$$\therefore x_1 = 2.88431$$

$$\therefore x = 2.86743$$

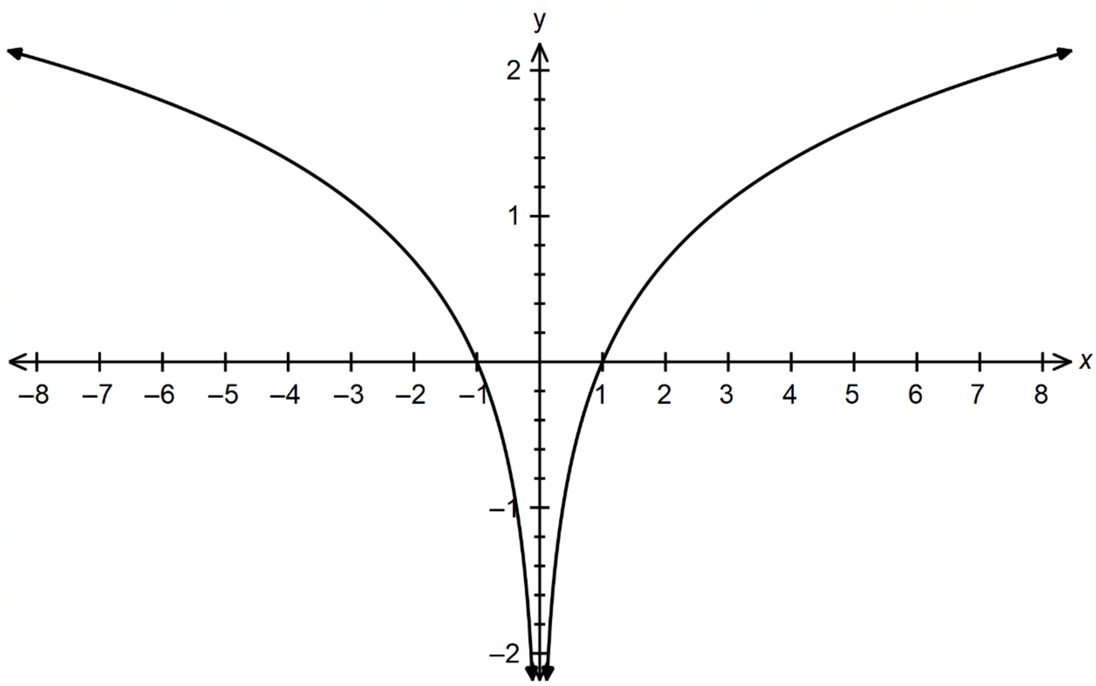
QUESTION 10

10.1 Sketch the following functions, indicating the *x*-intercepts:

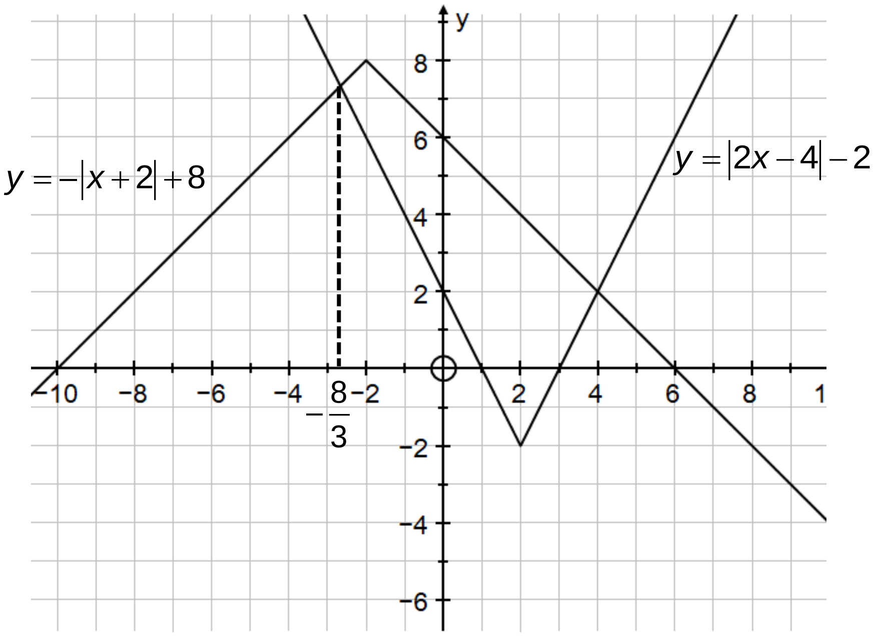
(a) $y = |\sin 2\theta|$ for $\theta \in [0;\pi]$



(b) $y = \ln|x|$



10.2 Use the graphs on the scaled axes below, or otherwise, to solve the given inequalities:



(a) $|2x-4|\le 8$

$|2x-4|-2\le 6$

$\therefore -2\le x\le 6\ \ (\textit{from graph})$

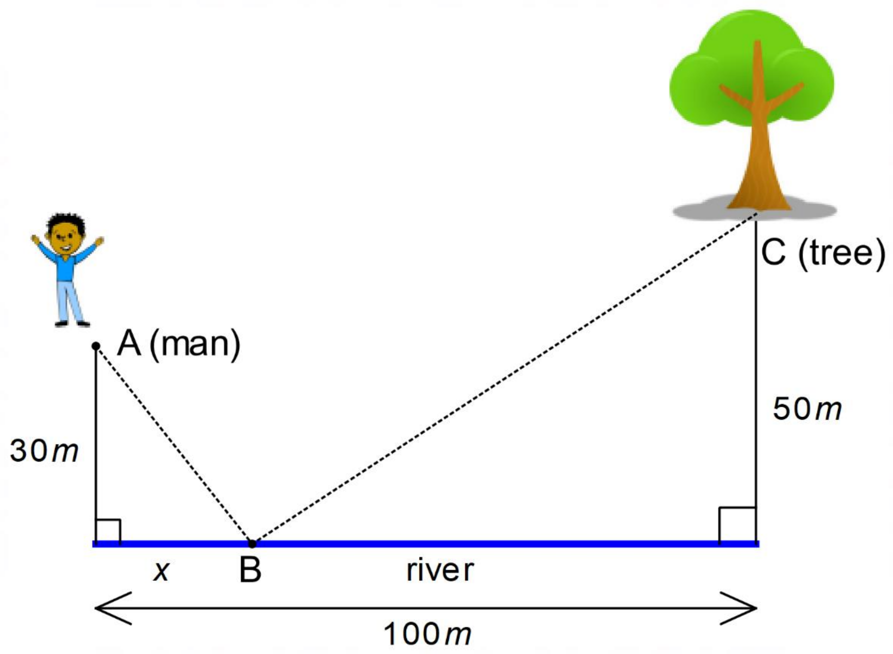
(b) $2|x-2|+|x+2|\le 10$

$\therefore |2x-4|-2\le -|x+2|+8$

$\therefore -\frac{8}{3}\le x\le 4$

QUESTION 11

11.1 A man is standing 30 m away from a straight river. 100 m downstream there is a tree which is 50 m from the riverbank. He wishes to walk to the river to drink and then to the tree to rest in the shade. He walks in straight lines as depicted by the dotted lines in the diagram.



(a) Show that the distance he will walk is given by the expression:

$$d = \sqrt{x^2 + 900} + \sqrt{x^2 - 200x + 12500}$$

distance to river $= \sqrt{x^2 + 30^2} = \sqrt{x^2 + 900}$ (Pythagoras)

distance to tree $= \sqrt{(100 - x)^2 + 50^2} = \sqrt{x^2 - 200x + 12500}$ (Pythagoras)

total distance walked $= \sqrt{x^2 + 900} + \sqrt{x^2 - 200x + 12500}$

(b) Hence, determine the value of *x* which will minimise the distance *d*.

$$d = \sqrt{x^2 + 900} + \sqrt{x^2 - 200x + 12500}$$

$$d = \left(x^2 + 900\right)^{\frac{1}{2}} + \left(x^2 - 200x + 12500\right)^{\frac{1}{2}}$$

$$\therefore \frac{dd}{dx} = \frac{1}{2}\left(x^2 + 900\right)^{-\frac{1}{2}}(2x) + \frac{1}{2}\left(x^2 - 200x + 12500\right)^{-\frac{1}{2}}(2x - 200) = 0$$

$$\therefore \frac{x}{\sqrt{x^2 + 900}} = -\frac{x - 100}{\sqrt{x^2 - 200x + 12500}}$$

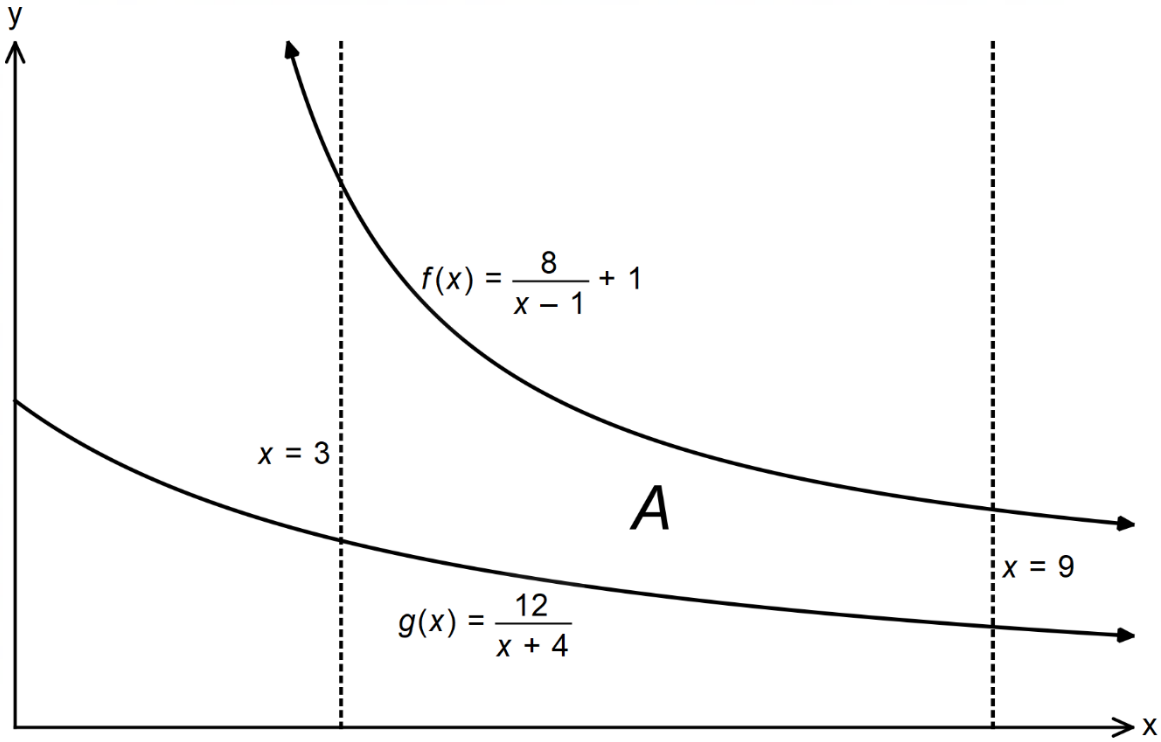
$$\therefore \frac{x^2}{x^2 + 900} = \frac{x^2 - 200x + 10\,000}{x^2 - 200x + 12\,500}$$

$$\therefore x^4 - 200x^3 + 12500x^2 = x^4 - 200x^3 + 10900x^2 - 180000x + 9000000$$

$$\therefore 1600x^2 + 180000x - 9000000 = 0$$

$$\therefore x = 37.5m$$

11.2 Determine the area labelled A below. It is bounded above by the curve $f(x)=\dfrac{8}{x-1}+1$, below by the curve $g(x)=\dfrac{12}{x+4}$, to the left by the line x = 3 and to the right by the line x = 9. You should show the expression involving the integrals which you use to calculate your answer.



$$A=\int_3^9 \frac{8}{x-1}+1-\frac{12}{x+4}dx$$

$$=\left[8\ln|x-1|+x-12\ln|x+4|\right]_3^9$$

$$=8\ln 8+9-12\ln 13-8\ln 2-3+12\ln 7$$

$$=9{,}66\text{ units}^2$$

QUESTION 12

12.1 Determine the following:

(a) $\int \sin 5x \cos 4x \, dx$

$= \int \frac{1}{2}\sin 9x + \frac{1}{2}\sin x \, dx$

$= -\frac{1}{18}\cos 9x - \frac{1}{2}\cos x + c$

(b) $\int xe^x \, dx$

$= xe^x - \int e^x dx$

$= xe^x - e^x + c$

(c) $\int e^{\tan 2x} \sec^2 2x \, dx$

$= \frac{e^{\tan 2x}}{2} + c$

(d) $\int \frac{x^3 - 3}{x^2 - 1} dx$

$x^3 - 3 = x\left(x^2 - 1\right) + x - 3$

$\therefore \frac{x^3 - 3}{x^2 - 1} = x + \frac{x - 3}{(x-1)(x+1)} = x + \frac{A}{x-1} + \frac{B}{x+1} = x - \frac{1}{x-1} + \frac{2}{x+1}$

$\therefore \int \frac{x^3 - 3}{x^2 - 1} dx = \int x - \frac{1}{x-1} + \frac{2}{x+1} dx$

$= \frac{x^2}{2} - \ln|x-1| + 2\ln|x+1| + c$

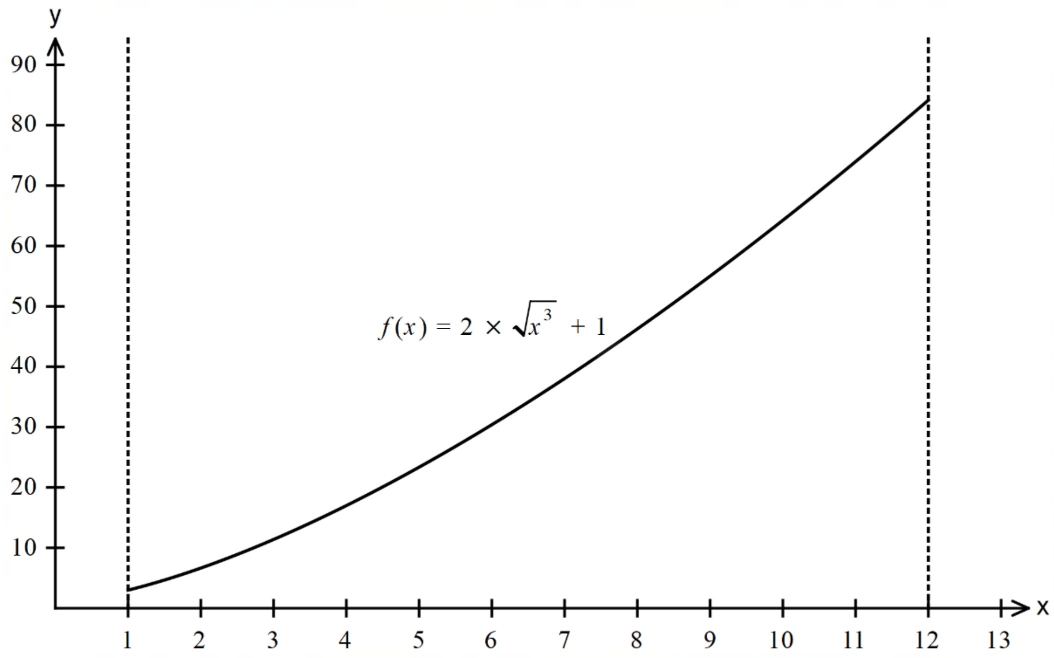
12.2 The arc length of a function $f(x)$ from $x = a$ to $x = b$ is given by the formula:

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx.$$

Given the function $f(x) = 2 \times \sqrt{x^3} + 1$

Use this formula to determine the arc length of the $f\ x$ between $x = 1$ and $x = 12$ as illustrated below.

You must show the integral which you use to calculate your answer.



$f(x) = 2x^{\frac{3}{2}} + 1$

$\therefore f'(x) = 3x^{\frac{1}{2}}$

$\therefore (f'(x))^2 = 9x$

$L = \int_1^{12} (1 + 9x)^{\frac{1}{2}}\, dx$

$= \left[\frac{2}{27}(1 + 9x)^{\frac{3}{2}} \right]_1^{12}$

$= 81{,}95\ units$

Total: 200 marks