INTERNATIONAL SECONDARY CERTIFICATE EXAMINATION
MAY 2023

FURTHER STUDIES MATHEMATICS (STANDARD): PAPER I

MARKING GUIDELINES

Time: 2 hours 200 marks

These marking guidelines are prepared for use by examiners and sub-examiners, all of whom are required to attend a standardisation meeting to ensure that the guidelines are consistently interpreted and applied in the marking of candidates' scripts.

The IEB will not enter into any discussions or correspondence about any marking guidelines. It is acknowledged that there may be different views about some matters of emphasis or detail in the guidelines. It is also recognised that, without the benefit of attendance at a standardisation meeting, there may be different interpretations of the application of the marking guidelines.

QUESTION 1

1.1 Solve for $x \in \mathbb{R}$:

(a) $e^{-x+2} + e^{-x} = 0{,}001$

$e^{-x}\left(e^2 + 1\right) = 0{,}001$

$e^{-x} = \dfrac{0{,}001}{e^2 + 1}$

$-x = \ln\left(\dfrac{0{,}001}{e^2 + 1}\right)$

$x = -\ln\left(\dfrac{0{,}001}{e^2 + 1}\right)$

$x = 9.03$

(b) $|2x - 3| = -3x + 7$

$2x - 3 = -3x + 7$ or $2x - 3 = 3x - 7$

$5x = 10$ or $-x = -4$

$x = 2$ or $x = 4$

but a check reveals that $x = 2$ only

1.2 Given $f(x) = 2x^4 + 11x^3 + 28x^2 + 40x + 24$:

(a) If $g(x) = x^2 + 2x + 4$ is a factor of f then determine the other quadratic factor.

$f(x) = 2x^4 + 11x^3 + 28x^2 + 40x + 24$

$= (x^2 + 2x + 4)(2x^2 + 7x + 6)$

(b) Hence, or otherwise, solve $f(x) = 0$ in $\mathbb{C}$

$x^2 + 2x + 4 = 0$ or $2x^2 + 7x + 6 = 0$

$x = -1 + \sqrt{3}i$ or $-1 - \sqrt{3}i$ or $-\dfrac{3}{2}$ or -2

1.3 Determine p if $\dfrac{4p+i}{p-3i} = 1 + pi$

$$\frac{4p+i}{p-3i} = 1 + pi$$

$\therefore\ 4p + i = (1 + pi)(p - 3i)$

$\therefore\ 4p + i = p + 3p + p^2 i - 3i$

$\therefore\ 4p + i = 4p + (p^2 - 3)i$

$\therefore\ p^2 - 3 = 1$

$\therefore\ p = \pm 2$

ALTERNATE METHOD

$$\frac{4p+i}{p-3i} \times \frac{p+3i}{p+3i} = 1 + pi$$

$$\therefore\ \frac{4p^2 + 13pi - 3}{p^2 + 9} = (1 + pi)$$

$\therefore\ 4p^2 + 13pi - 3 = (1 + pi)(p^2 + 9)$

$\therefore\ (4p^2 - 3) + 13pi = p^2 + 9 + (9p + p^3)i$

$\therefore\ 4p^2 - 3 = p^2 + 9$

$\therefore\ 3p^2 = 12$

$\therefore\ p = \pm 2$

QUESTION 2

Newton's law of cooling says that the rate at which an object cools is roughly proportional to the difference between its temperature and the temperature of its surroundings.

The equation is:

$T = T_s + (T_0 - T_s)e^{-kt}$ where

T is the temperature of the object at time t (in minutes)

T_0 is the initial temperature (ºC)

T_s is the temperature of the surroundings (ºC)

(a) A hard-boiled egg with a temperature of 98 ºC is put into a large sink of water at 18 ºC. After 5 minutes, the egg's temperature is 38 ºC. Determine the value of k to three decimal places.

$$T = T_s + (T_0 - T_s)e^{-kt}$$
$$38 = 18 + (98 - 18)\, e^{-5k}$$
$$0.25 = e^{-5k}$$
$$-5k = \ln 0.25$$
$$k = \frac{\ln 0.25}{-5}$$
$$k = 0.277$$

(b) Using $k = 0{,}277$ determine how long it will take for the egg to reach 20 ºC from the time it was put into the sink. You can assume that the water in the large sink has not been warmed significantly by the egg. Give your answer to the nearest second.

$$T = T_s + (T_0 - T_s)e^{-0.277t}$$
$$20 = 18 + 80e^{-0.277t}$$
$$\frac{1}{40} = e^{-0.277t}$$
$$-0.277t = \ln\frac{1}{40}$$
$$t = \frac{\ln\frac{1}{40}}{-0.277}$$
$$t = 13.317 = 13{:}19 \text{ or } 799 \text{ seconds}$$

QUESTION 3

Use mathematical induction to prove that $n^3 - n + 3$ is divisible by 3 for all $n \in \mathbb{N}$.

$n^3 - n + 3$ is divisible by 3 for all $n \in \mathbb{N}$

when $n = 1$ we have 3 which is divisible by 3 so the statement is true for $n = 1$

Next, we assume true for $n = k$

$k^3 - k + 3 = 3p$ for $p \in \mathbb{N}$

now when $n = k + 1$ we have

$(k + 1)^3 - (k + 1) + 3$

$= k^3 + 3k^2 + 3k + 1 - k - 1 + 3$

$= k^3 - k + 3 + 3k^2 + 3k$

but $k^3 - k + 3 = 3p$ so we have

$3p + 3k^2 + 3k$

$= 3(p + k^2 + k)$

which is clearly divisible by 3

So, we have proved it true for $n = k + 1$

$\therefore$ by the principle of mathematical induction we have proved

that $n^3 - n + 3$ is divisible by 3 for $n \in \mathbb{N}$

QUESTION 4

Determine $f'(x)$ by first principles if $f(x) = \sqrt{5x+3}$

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{5(x+h)+3} - \sqrt{5x+3}}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{5(x+h)+3} - \sqrt{5x+3}}{h} \times \frac{\sqrt{5(x+h)+3} + \sqrt{5x+3}}{\sqrt{5(x+h)+3} + \sqrt{5x+3}}$$

$$f'(x) = \lim_{h \to 0} \frac{5(x+h)+3-5x+3}{h\sqrt{5(x+h)+3} + \sqrt{5x+3}}$$

$$f'(x) = \lim_{h \to 0} \frac{5x+5h+3-5x-3}{h\sqrt{5(x+h)+3} + \sqrt{5x+3}}$$

$$f'(x) = \lim_{h \to 0} \frac{5h}{h\sqrt{5(x+h)+3} + \sqrt{5x+3}}$$

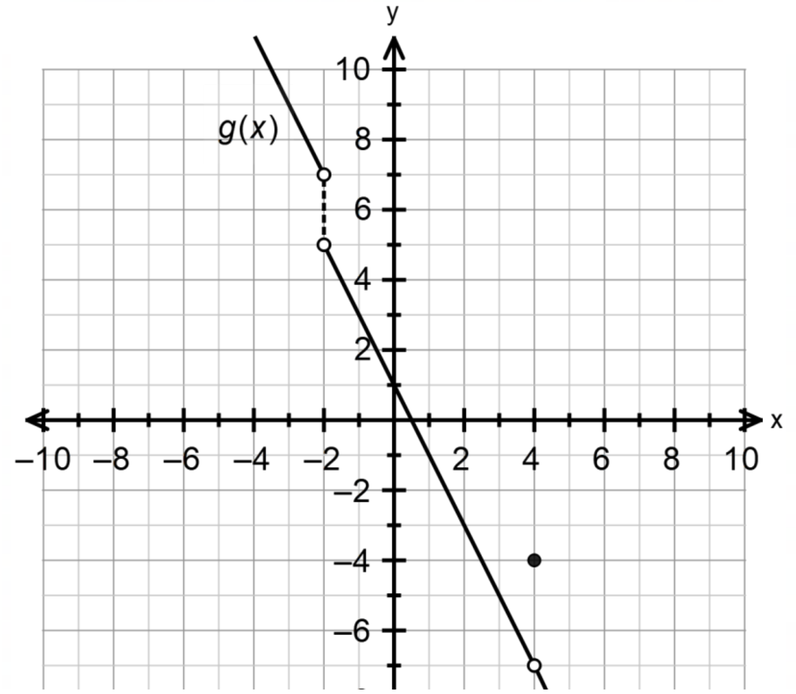
$$f'(x) = \lim_{h \to 0} \frac{5}{\sqrt{5(x+h)+3} + \sqrt{5x+3}}$$

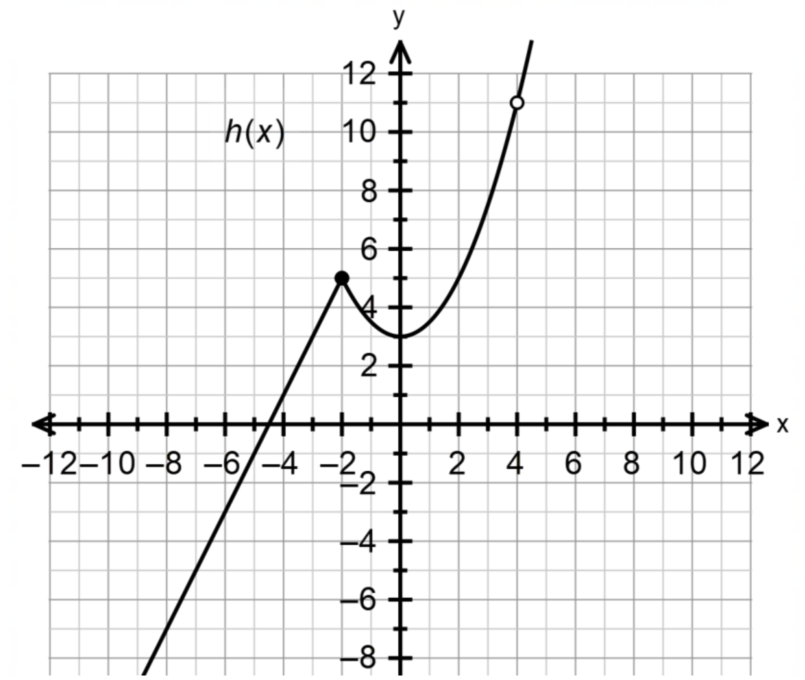
$$f'(x) = \frac{5}{2\sqrt{5x+3}}$$

QUESTION 5

Consider the following four functions and answer the questions that follow:

$$f(x)=\begin{cases}-2x+3 & x<-2\\ 5 & x=-2\\ -4x-1 & -2<x\le 4\\ -x^2-1 & x>4\end{cases}$$





$$j(x)=\begin{cases}x^2+1 & x<-2\\ 5 & x=-2\\ -2x+1 & -2<x\le 4\\ -\frac{1}{4}x^2-3 & x>4\end{cases}$$

(a) State the continuity of each of the functions at $x = -2$. Should any of the functions have a discontinuity then you should justify why and classify it accordingly.

f is discontinuous at $x = -2$ since $f(-2) \neq \lim_{x \to -2} f(x)$

we have a removable discontinuity

g is discontinuous at $x = -2$ since

$\lim_{x \to -2} g(x)$ d.n.e *OR* $\lim_{x \to -2} g(x) \neq \lim_{x \to -2} g(x)$ *OR* $g(-2)$ is not defined

we have a jump or non-removable discontinuity

h is continuous at $x = -2$

j is continuous at $x = -2$

(b) State the differentiability of each of the functions at $x = 4$. Where a function is not differentiable you must justify why not.

f is not differentiable at $x = 4$ since $\lim_{x \to 4^-} f'(x) \neq \lim_{x \to 4^+} f'(x)$ or since f is not continuous at 4.

g is not differentiable at $x = 4$ since it is not continuous at $x = 4$

h is not differentiable at $x = 4$ since h is not continuous at $x = 4$

j is differentiable at $x = 4$

QUESTION 6

6.1 Consider the function $f(x) = \dfrac{x^3 + x^2 - 9x - 9}{x^2 - 2x - 8}$

(a) Determine the equations and nature of all asymptotes.

$f(x) = \dfrac{x^3 + x^2 - 9x - 9}{(x-4)(x+2)}$

vertical asymptotes: $x = 4$ or $x = -2$

$x^3 + x^2 - 9x - 9 = (x^2 - 2x - 8)(x + 3) + 15x + 15$

$\therefore\ f(x) = x + 3 + \dfrac{5x + 15}{x^2 - 2x - 8}$

$\therefore\ y = x + 3$ is an oblique asymptote

(b) Determine the y-intercept and any x-intercepts.

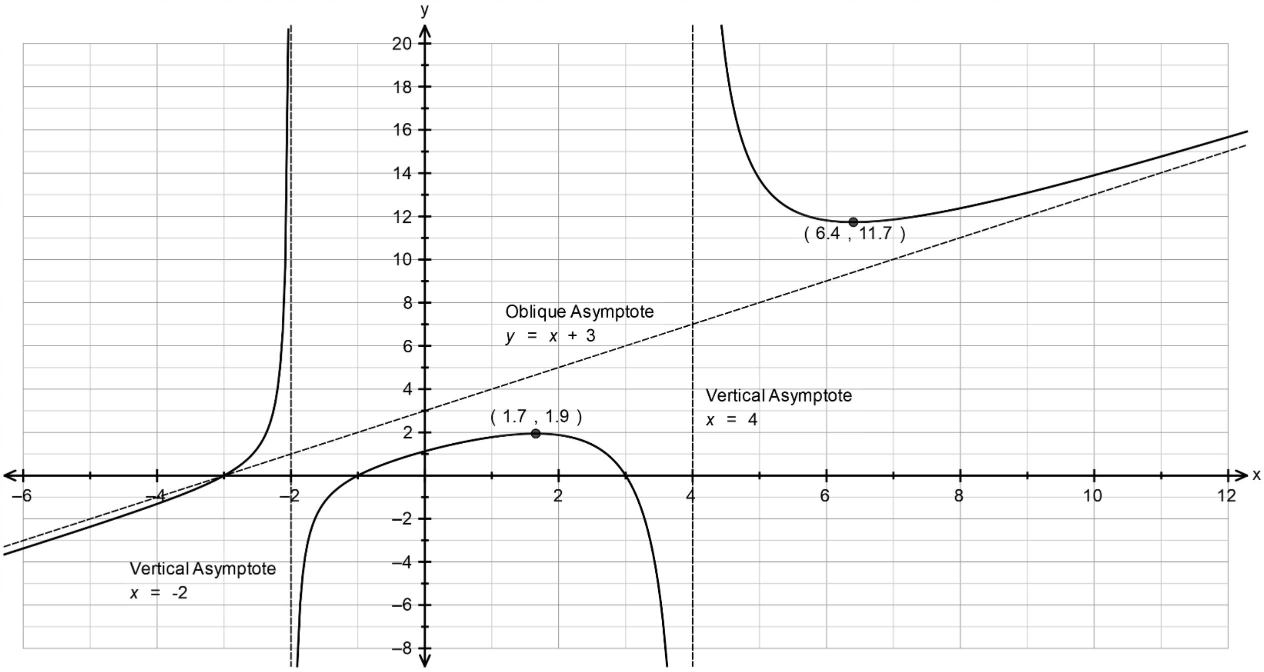
y-intercept – let $x = 0$ so $\left(0;\dfrac{9}{8}\right)$

x-intercepts – let $y = 0$

$x^3 + x^2 - 9x = 0$

$x = -3$ or -1 or 3

(c) Use the above information and the fact that f has stationary points at $(6{,}4\ ;\ 11{,}7)$ and $(1{,}7\ ;\ 1{,}9)$ to draw a sketch of the function on the axes provided. You should draw in and label any asymptotes.



6.2 Justify mathematically why the function $y = \dfrac{x+3}{x^2+4x+1}$ is strictly decreasing.

$$y = \frac{x+3}{x^2+4x+1}$$

$$\frac{dy}{dx} = \frac{1\left(x^2+4x+1\right)-(2x+4)(x+3)}{\left(x^2+4x+1\right)^2}$$

$$\frac{dy}{dx} = \frac{-x^2-6x-11}{\left(x^2+4x+1\right)^2}$$

$$\frac{dy}{dx} = \frac{-\left[x^2+6x+11\right]}{\left(x^2+4x+1\right)^2}$$

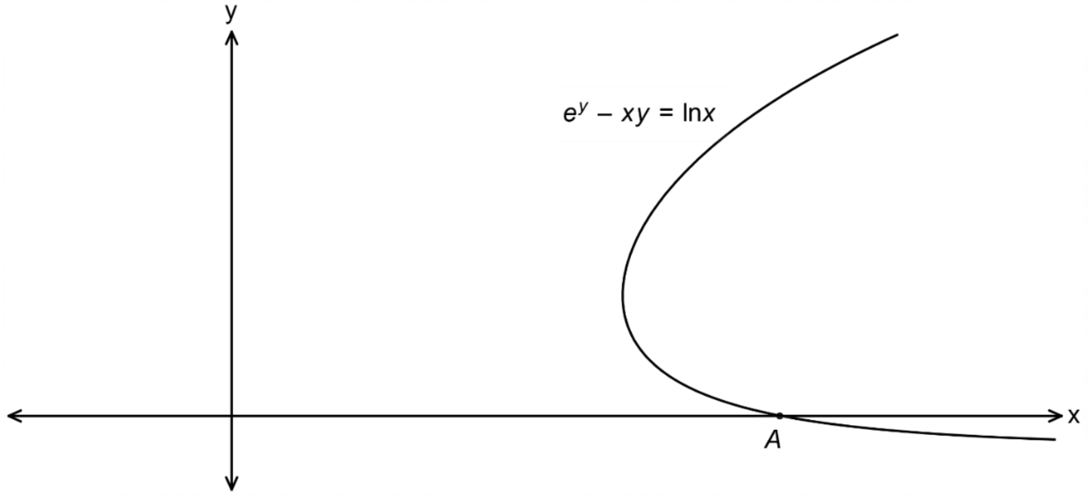
$$\frac{dy}{dx} = \frac{-\left[(x+3)^2+2\right]}{\left(x^2+4x+1\right)^2}$$

$$\frac{dy}{dx} = \frac{-\left[(x+3)^2-2\right]}{\left(x^2+4x+1\right)^2}$$

since numerator is ≤ –2 and denominator > 0
the gradient is < 0 so the function is strictly decreasing

QUESTION 7

A portion of the graph of an implicitly defined relation $e^y - xy = \ln x$ is shown below.



(a) Determine the coordinates of point A, the x-intercept.

$$e^y - xy = \ln x$$

for x-intercept we let $y = 0$

$$e^0 - 0 = \ln x$$

$$1 = \ln x$$

$$x = e$$

so, $A(e;0)$

(b) Determine the equation of the tangent to the curve at the x-intercept. Should you be unsuccessful in finding the x-intercept in part (a) you can make up a value to use. Round your answers to three decimal places.

$$e^y - xy = \ln x$$

$$e^y\left(\frac{dy}{dx}\right) - y - x\frac{dy}{dx} = \frac{1}{x}$$

$$\therefore \frac{dy}{dx}\left(e^y - x\right) = \frac{1}{x} + y$$

$$\therefore \frac{dy}{dx} = \frac{\frac{1}{x} + y}{e^y - x}$$

at $(e;0)$ $\frac{dy}{dx} = \frac{\frac{1}{e}}{1-e} = -0.214$

$$\therefore y = -0.214(x - e)$$

$$\therefore y = -0.214x + 0.582$$ Final line need not be shown

QUESTION 8

Given the function $f(x) = \sin^2 x \tan x - \sec x$.

(a) Given that the function is continuous on the interval $x \in [0\ ;\ 4]$ prove that there must be at least one solution to $f(x) = 0$ on the interval [0 ; 4].

$f(x) = \sin^2 x \tan x - \sec x$

$f(0) = \sin^2(0)\tan(0) - \sec(0) = -1$

$f(4) = \sin^2(4)\tan(4) - \sec(4) = 2{,}193$

since $f(0)f(4) < 0$ there must be at least one root between 0 and 4

(b) Use the Newton-Raphson method to determine a solution to $f(x) = 0$ to 5 decimal places.

You should:

- Show the iterative formula you use
- Use an initial approximation of $x_0 = 1$
- Also, give x_1 to 3 decimal places

$f(x) = \sin^2 x \tan x - \sec x$

$f'(x) = 2\sin x\cos x\tan x + \sec^2 x\sin^2 x - \sec x\tan x$

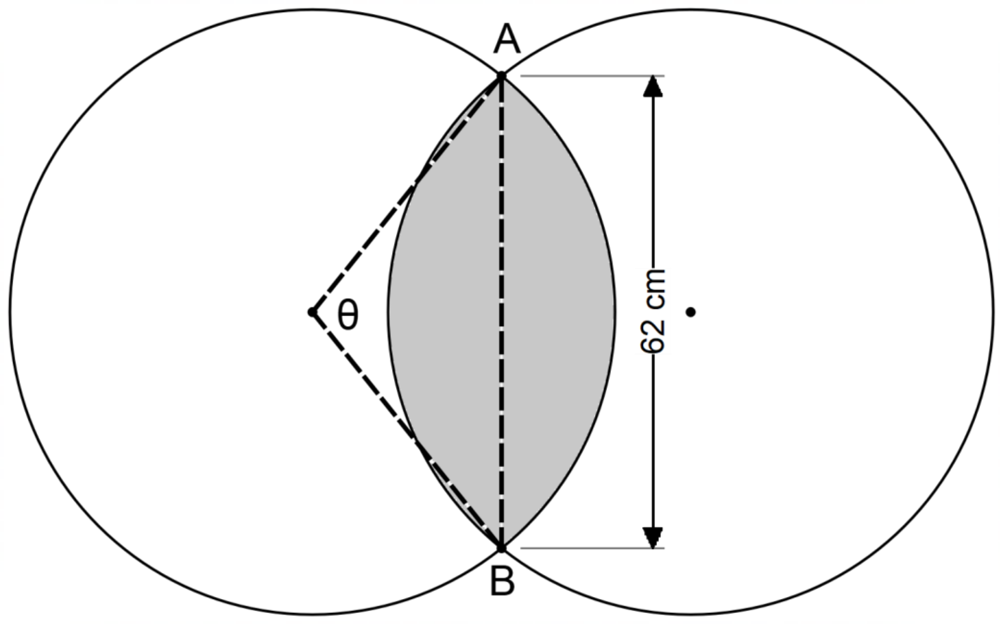
$$x_{n+1} = x_n - \frac{\sin^2 x_n \tan x_n - \sec x_n}{2\sin x_n \cos x_n \tan x_n + \sec^2 x_n \sin^2 x_n - \sec x_n \tan x_n}$$

$x_1 = 1{,}77988$

$x = 1{,}570797$

QUESTION 9

In the diagram below, the two circles are the same size, each with a radius of 40 cm. The distance between A and B, the points at which they intersect is 62 cm. Calculate the shaded area.



$$\cos\theta = \frac{40^2 + 40^2 - 62^2}{2(40)(40)}$$

$$\theta = \cos^{-1}\left(\frac{40^2 + 40^2 - 62^2}{2(40)(40)}\right)$$

$$\theta = 1.77343$$

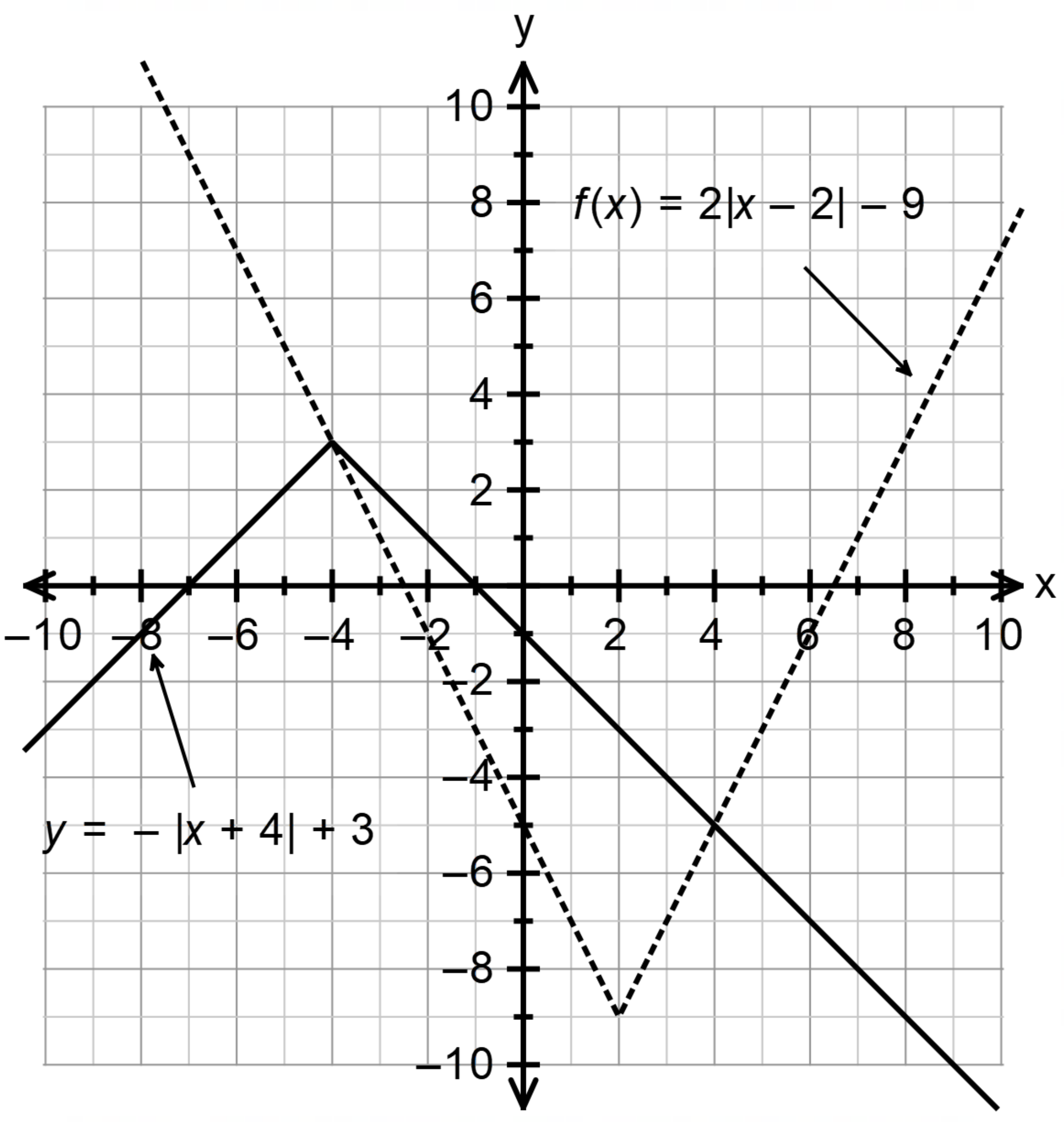
Area of segment on right hand side of *AB* = $\frac{1}{2} \times 40^2(1.773 - \sin(1.773))$

Area of segment on right hand side of *AB* = 634.7

so, total shaded area is double this or 1 269,4 cm^2

QUESTION 10

The function $f(x) = 2|x - 2| - 9$ is drawn below.



(a) On the same set of axes draw the function $g(x) = -|x + 4| + 3$.

(b) Hence, or otherwise, solve $|x - 2| + \frac{|x + 4|}{2} \leq 6$

$2|x - 2| + |x + 4| \leq 12$

$2|x - 2| - 9 + |x + 4| \leq 3$

$2|x - 2| - 9 \leq -|x + 4| + 3$

$-4 \leq x \leq 4$

QUESTION 11

(a) Prove that $\frac{\cos\theta}{(1-\cos\theta)(1+\cos\theta)} = \text{cosec}\theta\cot\theta$

$LHS = \frac{\cos\theta}{1-\cos^2\theta}$

$= \frac{\cos\theta}{\sin^2\theta}$

$= \frac{1}{\sin\theta} \times \frac{\cos\theta}{\sin\theta}$

$= \text{cosec}\theta\cot\theta$

$= RHS$

(b) Using the result from (a), or otherwise, determine $\int \frac{\cos 3\theta}{(1-\cos 3\theta)(1+\cos 3\theta)}\, d\theta$

$\int \frac{\cos 3\theta}{(1-\cos 3\theta)(1+\cos 3\theta)}\, d\theta$

$= \int \text{cosec}3\theta\cot 3\theta\, d\theta$

$= -\frac{\text{cosec}3\theta}{3} + c$

QUESTION 12

(a) Use differentiation to confirm that = $\int \ln x \, dx = x\ln x - x + c$

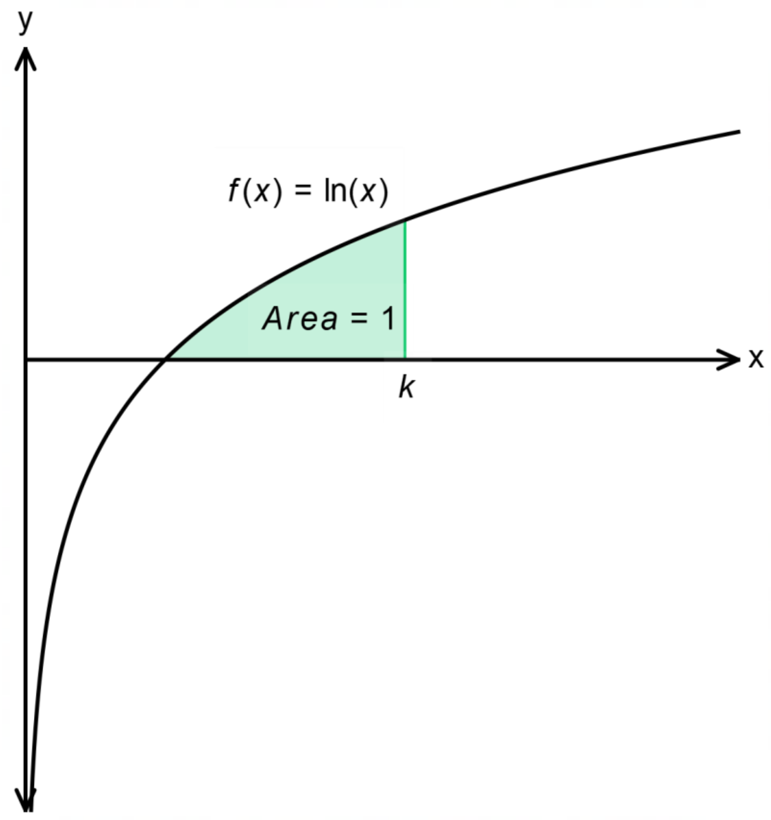
$\frac{d}{dx}\left(x\ln x - x + c\right)$

$= 1 \ln x + x\left(\frac{1}{x}\right) - 1$

$= \ln x + 1 - 1$

$= \ln x$ as required

(b) Hence or otherwise determine k if the shaded area below is equal to 1.



x-intercept – (1;0)

$$\int_1^k \ln x\,dx = 1$$

$$\therefore \left[x\ln x - x\right]_1^k = 1$$

$$\therefore\ k\ln k - k - (1\ \ln 1 - 1) = 1$$

$$\therefore\ k\ln k - k + 1 = 1$$

$$\therefore\ k\ln k - \text{k} = 0$$

$$\therefore\ k(\ln k - 1) = 0$$

$$\therefore\ \ln k = 1 \text{ since } k \neq 0$$

$$\therefore\ k = e$$

QUESTION 13

13.1 Determine the following integrals:

(a) $\int \frac{x-1}{x^2-1}dx$

$\int \frac{x-1}{x^2-1}dx$

$= \int \frac{1}{x+1}dx$

$= \ln |x + 1| + c$

(b) $\int (x+2)(x+5)^{11}dx$

by parts: $f(x) = x + 2$ so $f'(x) = 1$ and $g'(x) = (x + 5)^{11}$ so $g(x) = \frac{(x+5)^{12}}{12}$

$= \frac{(x+2)(x+5)^{12}}{12} - \frac{1}{12}\int (x+5)^{12}dx$

$= \frac{(x+2)(x+5)^{12}}{12} - \frac{1}{12}\left(\frac{(x+5)^{13}}{13}\right) + c$

$= \frac{(x+2)(x+5)^{12}}{12} - \frac{(x+5)^{13}}{156} + c$

OR

$\int (x+2)(x+5)^{11}dx$

by substitution: let $x + 5 = u$ then $x + 2 = u - 3$ and $\frac{du}{dx} = 1$ so $du = dx$

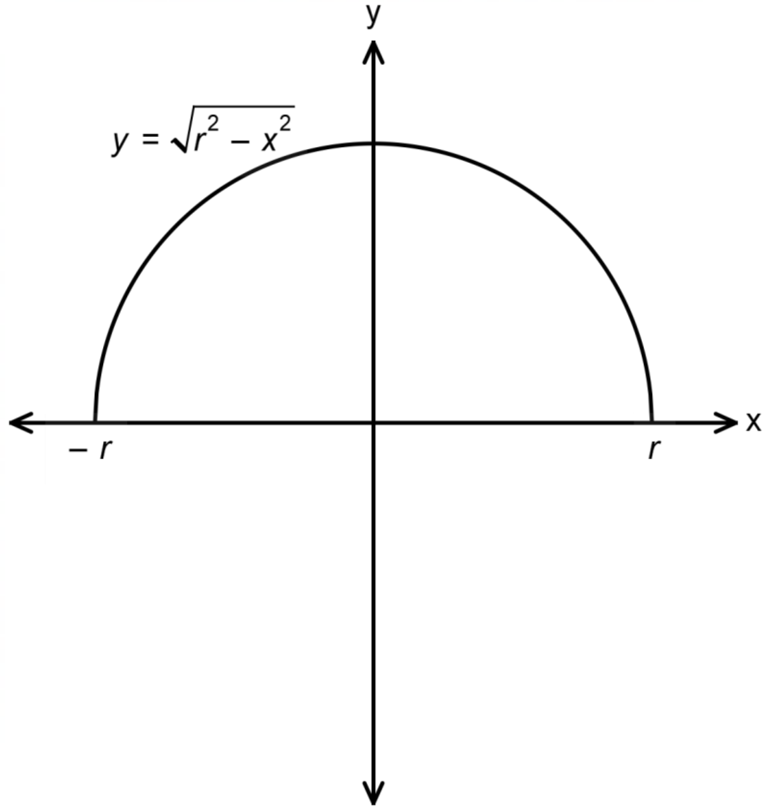
$= \int (u-3)u^{11}\,du$

$= \int u^{12} - 3u^{11}\,du$

$= \frac{u^{13}}{13} - \frac{3u^{12}}{12} + c$

$= \frac{(x+5)^{13}}{13} - \frac{(x+5)^{12}}{4} + c$

13.2 Use integration to derive the formula for the volume of a sphere using the sketch below. You should show all working details.



$$V = \pi \int_{-r}^{r} \left(\sqrt{r^2 - x^2}\right)^2 dx$$

$$= \pi \int_{-r}^{r} r^2 - x^2 dx$$

$$= \pi \left[r^2 x - \frac{x^3}{3} \right]^r$$

$$= \pi \left[r^3 x - \frac{x^3}{3} - \left(-r^3 + \frac{r^3}{3} \right) \right]$$

$$= \pi \left[2r^3 - \frac{2r^3}{3} \right]$$

$$= \frac{4}{3} \pi r^3$$

Total: 200 marks