INTERNATIONAL SECONDARY CERTIFICATE EXAMINATION
MAY 2023

FURTHER STUDIES MATHEMATICS (EXTENDED): PAPER II

MARKING GUIDELINES

Time: 1 hour 100 marks

These marking guidelines are prepared for use by examiners and sub-examiners, all of whom are required to attend a standardisation meeting to ensure that the guidelines are consistently interpreted and applied in the marking of candidates' scripts.

The IEB will not enter into any discussions or correspondence about any marking guidelines. It is acknowledged that there may be different views about some matters of emphasis or detail in the guidelines. It is also recognised that, without the benefit of attendance at a standardisation meeting, there may be different interpretations of the application of the marking guidelines.

MODULE 2 STATISTICS

QUESTION 1

1.1 $X \sim B(24;0,1)$

1.2 $P(X \geq 2) = 1 - [P(X = 0) + P(X = 1)]$

$= 1 - \left[(0,9)^{24} + \binom{24}{1}(0,1)(0,9)^{23} \right]$

$= 0,7075$

1.3 $X \sim B(15;0,7075)$

$P(X = 9) = \binom{15}{9}(0,7075)^{9}(0,2925)^{6}$

$= 0,1392$

1.4 (a) $X \sim B(150;0,1) \rightarrow X \sim N\left(15;\sqrt{13.5}^{2}\right)$

$P(X \leq 20) \rightarrow P(X < 20,5)$

$= P\left(Z < \frac{20,5 - 15}{\sqrt{13.5}}\right)$

$= P(Z < 1,5)$

$= 0,5 + 0,4332$

$= 0,9332$

(calc answer 0,9328)

(b) $np > 5$ and $nq > 5$

QUESTION 2

2.1 $f(x) \geq 0$ but $A = \frac{1}{2}(0,1)(10) = 0,5$

$A \neq 1$ hence not a probability density function.

2.2 For some values of x $f(x) < 0$ hence not a probability density function.

2.3 $f(x) \geq 0$ and $A = \frac{1}{2}(0,1 + 0,15)(8) = 1$

hence a probability density function.

2.4 $f(x) \geq 0$ and $A = \frac{1}{2}\pi\left(\sqrt{\frac{2}{\pi}}\right)^2 = 1$

hence a probability density function.

QUESTION 3

3.1 (a) $X \sim N(2; 0,25^2)$

$\mu = \frac{39 + 63}{2} = 51$

$P(X > 63) = 0,25$

$0,67 = \frac{63 - 51}{\sigma}$

$\sigma = 17,9104$

OR Alternative:

$P(X > 39) = 0,75$

$-0,67 = \frac{39 - 51}{\sigma}$

$\sigma = 17,9104$

(b) $(0,25)(0,75) \times 2 = 0,375$

3.2 (a) $\bar{x} = \frac{620}{20} = 31$

A 98% CI for μ is:

$31 \pm 2.33\left(\frac{3}{\sqrt{20}}\right)$

$(29,437; 32,563)$

(b) This claim should be accepted as the stated mean falls within the confidence interval.

QUESTION 4

4.1 $\dfrac{\bar{x}-400}{\dfrac{60}{\sqrt{25}}} > 1{,}645$

$\therefore \bar{x} > 419{,}74$

4.2 $Z = \dfrac{382-400}{\dfrac{60}{\sqrt{25}}}$

$Z = -1{,}5$

$\therefore P(Z < -1{,}5) = 0{,}5 - 0{,}4332$

$= 0{,}0668$

Hence a 6,68% level of significance

4.3 $Z = \dfrac{379-400}{\dfrac{60}{\sqrt{25}}}$

$Z = -1{,}75$

Since $Z > -1{,}96$ fail to reject H_0 at 5% l.o.s. and suggest insufficient evidence to support the claim.

QUESTION 5

5.1 $a + b = 2(c + 0{,}25)$ .....(1)

$a + b = 1 - (c + 0{,}25)$ ....(2)

$\therefore 2c + 0{,}5 = 0{,}75 - c$

$3c = 0{,}25$

$c = \dfrac{1}{12}$

$P(2 \cap 4) + P(4 \cap 2) + P(3 \cap 3)$

$P(X = 2) \times P(X = 4) \times 2 + P(X = 3) \times P(X = 3)$

$\left(\dfrac{1}{12}\right)(0{,}15) \times 2 + (0{,}1)^2$

$= \dfrac{7}{200}$

5.2 (a) $\dfrac{7!}{4!3!} \times \dfrac{7!}{2!2!3!} \times 2 = 14\,700$

(b) $\dfrac{\binom{3}{1}\binom{4}{1}\binom{7}{2}}{\binom{14}{4}} = \dfrac{36}{143} = 0{,}2517$

Total for Module 2: 100 marks

MODULE 3 FINANCE AND MODELLING

QUESTION 1

1.1 (a) $B_{30} = \dfrac{5\,500\left[\left(1+\dfrac{0{,}078}{12}\right)^{31}-1\right]}{\dfrac{0{,}078}{12}}$

$= 188\,217{,}52$

(b) $B_{36} = \dfrac{7\,000\left[\left(1+\dfrac{0{,}078}{12}\right)^{6}-1\right]}{\dfrac{0{,}078}{12}} + 188\,217{,}52\left(1+\dfrac{0{,}078}{12}\right)^{6}$

$238\,366{,}77$

1.2 $B_{48} = 238\,366{,}77\left(1+\dfrac{0{,}092}{12}\right)^{12} + \dfrac{7\,000\left[\left(1+\dfrac{0{,}092}{12}\right)^{12}-1\right]}{\dfrac{0{,}092}{12}}$

$= 348\,879.36$

QUESTION 2

2.1 $0{,}115 = \left(1+\dfrac{r}{12}\right)^{12} - 1$

$r = 0{,}1093....$

2.2 $F = \dfrac{750\,000\left[(1+0{,}115)^{15}-1\right]}{0{,}115}$

$= R\,26\,858\,268{,}67$

2.3 $20\,000\,000(1+\dfrac{0{,}1093}{12})^{180} = \dfrac{x\left[\left(1+\dfrac{0{,}1093}{12}\right)^{173}-1\right]}{\dfrac{0{,}1093}{12}} + 26\,858\,268{,}67$

$x = R\,180\,906{,}94$

Interest = 173 × 180 906,94 + 26 858 268,67 – 20 000 000
= R38 155 169,57

QUESTION 3

$$20\,000\left(1+\frac{r}{4}\right)^{40}=\frac{50\left[\left(1+\frac{0,08}{365}\right)^{3651}-1\right]}{\frac{0,08}{365}}$$

$$r=0,27267....$$

$$r_{eff}=\left(1+\frac{0,27267}{4}\right)^{4}-1$$

$$=30,184\%$$

QUESTION 4

4.1

$$r=b-d$$

$$=\left[(3\times 5)\times 0,5\times 0,3\times 0,4\right]-\left[\tfrac{1}{3}\right]$$

$$=\frac{17}{30}$$

4.2 Malthusian growth (adjusted). No carrying capacity or natural predators.

4.3

$$P_{n+1}=P_n\left(1+\frac{17}{30}\right)-x$$

$$90=90\left(1+\frac{17}{30}\right)-x$$

$$x=51$$

QUESTION 5

5.1 (a) Pred = 5
Prey = 50

(b) $F_n = F_n + fbR_nF_n - cF_n$

$$R_E = \frac{c}{fb}$$

$$R_E = \frac{0,01}{0,02 \times 0,003}$$

$$= 167$$

$$R_n = R_n + aR_n\left(1 - \frac{R_n}{K}\right) - bR_nF_n$$

$$F_n = \frac{a}{b}\left(1 - \frac{c}{fbK}\right)$$

$$= \frac{0,04}{0,003}\left(1 - \frac{0,01}{0,02 \times 0,003 \times 1\ 200}\right)$$

$$= 11,48 = 11$$

(c) Prey = 75,
Pred = 13

5.2 (a) a has increased (a = 0,06)
No effect on R_e – More produced, more eaten.
F_e increases – more rapid production of prey can support a larger number of predator.
Reaches equilibrium more rapidly.

(b) K has decreased (K = 500)
R_e stays the same – not affected by carrying capacity.
F_e decreases.
More rapid approach to equilibrium – more confined space.

(c) f has increased (f = 0,04)
R_e decreases, F_e increases – predators attack more prey.
Slower to equilibrium.

(d) b has decreased (b = 0,001)
R_e increases, F_e decreases.
Quick approach to equilibrium.

QUESTION 6

6.1 $158 = 1{,}6T_0 - \frac{T_0^2}{5\,000}$

$T_0^2 - 8\,000T_0 + 790\,000 = 0$

$T_0 = 7\,900$ or $T_0 = 100$

6.2 $\frac{175}{2} = 50a - \frac{50^2}{b}$

$T_n = aT_n - \frac{T_n^2}{b}$

$1 = a - \frac{T_n}{b}$

$1 = a - \frac{800}{b}$

$\frac{175}{2} = 50\left(1 + \frac{800}{b}\right) - \frac{2\,500}{b}$

$\frac{175}{2} = 50 + \frac{40\,000}{b} - \frac{2\,500}{b}$

$75b = 75\,000$

$b = 1\,000$

$a = 1{,}8$ (9)

Total for Module 3: 100 marks

MODULE 4 MATRICES AND GRAPH THEORY

QUESTION 1

1.1 $\begin{pmatrix} 2 & 6 & 1 & : & 7 \\ 1 & 2 & -1 & : & -1 \\ 5 & 7 & -4 & : & 9 \end{pmatrix}$ and swop R_1 for R_2

$\begin{pmatrix} 1 & 2 & -1 & : & -1 \\ 2 & 6 & 1 & : & 7 \\ 5 & 7 & -4 & : & 9 \end{pmatrix} R_2 - 2R_1$

$\begin{pmatrix} 1 & 2 & -1 & : & -1 \\ 0 & 2 & 3 & : & 9 \\ 5 & 7 & -4 & : & 9 \end{pmatrix} R_3 - 5R_1$

$\begin{pmatrix} 1 & 2 & -1 & : & -1 \\ 0 & 2 & 3 & : & 9 \\ 0 & -3 & 1 & : & 14 \end{pmatrix} 2R_3 - 3R_2$

$\begin{pmatrix} 1 & 2 & -1 & : & -1 \\ 0 & 2 & 3 & : & 9 \\ 0 & 0 & 11 & : & 55 \end{pmatrix}$

$z = 5$

$y = -3$

$x + 2(-3) - 1(5) = -1$

$x = 10$

OR

$$\begin{pmatrix} x \\ y \\ Z \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 1 & -31 & 8 \\ 1 & 13 & -3 \\ 3 & -16 & 2 \end{pmatrix}\begin{pmatrix} 7 \\ -1 \\ 9 \end{pmatrix} = \begin{pmatrix} 10 \\ -3 \\ 5 \end{pmatrix}$$

1.2 $$\overset{A}{\begin{pmatrix} 2 & 1 \\ 6 & 3 \\ 2 & 5 \end{pmatrix}} \overset{A^T}{\begin{pmatrix} 2 & 6 & 2 \\ 1 & 3 & 5 \end{pmatrix}} = \begin{pmatrix} 5 & 15 & 9 \\ 15 & 45 & 27 \\ 9 & 27 & 29 \end{pmatrix}$$

1.3 $\det A = -1\begin{pmatrix}1 & 2 & 4\\ 0 & 2 & 3\\ 1 & 6 & 1\end{pmatrix} - 4\begin{pmatrix}5 & 1 & 2\\ -1 & 0 & 2\\ 1 & 1 & 6\end{pmatrix}$

$= -1\left(1\begin{pmatrix}2 & 3\\ 6 & 1\end{pmatrix} + 1\begin{pmatrix}2 & 4\\ 2 & 3\end{pmatrix}\right) - 4\left(-1\begin{pmatrix}-1 & 2\\ 1 & 6\end{pmatrix} - 1\begin{pmatrix}5 & 2\\ -1 & 2\end{pmatrix}\right)$

$= -1\big((2-18)+(6-8)\big) - 4\big(-(-6-2)-(10+2)\big)$

$= -1(-18) - 4(-4)$

$= 34$

QUESTION 2

2.1 (a) $\begin{pmatrix}\frac{3}{2} & 0\\ 0 & \frac{3}{2}\end{pmatrix}\begin{pmatrix}-1 & 3 & 0\\ 3 & 1 & 3\end{pmatrix} = \begin{pmatrix}-\frac{3}{2} & \frac{9}{2} & 0\\ -\frac{3}{2} & \frac{3}{2} & \frac{9}{2}\end{pmatrix}$

(b) $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}-1 & 3 & 0\\ -1 & 1 & 3\end{pmatrix} = \begin{pmatrix}-1 & 3 & 0\\ 1 & -1 & -3\end{pmatrix}$

(c) $\tan\sigma = 3 \qquad \sigma = 71{,}565$

$\begin{pmatrix}\cos 2(71{,}565) & \sin 2(71{,}565)\\ \sin 2(71{,}565) & -\cos 2(71{,}565)\end{pmatrix}\begin{pmatrix}-1 & 3 & 0\\ -1 & 1 & 3\end{pmatrix} = \begin{pmatrix}0{,}1999 & -1{,}7999 & 1{,}8\\ -1{,}4 & 2{,}6 & 2{,}399\end{pmatrix}$

$* \begin{bmatrix}0{,}2 & -1{,}8 & 1{,}8\\ -1{,}4 & 2{,}6 & 2{,}4\end{bmatrix}$

2.2 $\begin{pmatrix}1 & b\\ c & 1\end{pmatrix}\begin{pmatrix}-2 & 0 & -2 & -4\\ 1 & 1 & -1 & -1\end{pmatrix} = \begin{pmatrix}-2 & 0 & -2 & -4\\ -3 & 1 & -5 & -9\end{pmatrix}$

$-2 + b = -2 \quad \therefore b = 0$

$-2c + 1 = -3 \quad \therefore c = 2$

Shear with a factor of 2 and the *y*-axis the invariant line

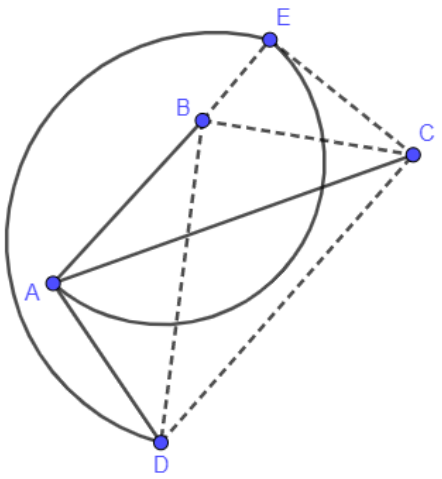
QUESTION 3

3.1 2
1 D_4 ; 5 D_2 ; 2 D_1

3.2 (a) 4

(b) 2

3.3 mark for each edge added. -1 for every error.



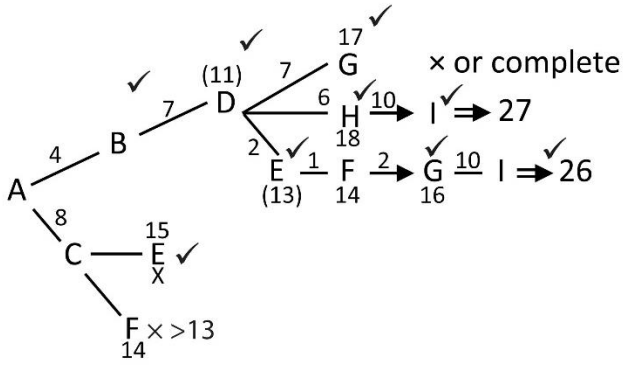
QUESTION 4

UT ; TR ; SR SP or RQ or RP ; PQ = 6

for seeing more than one case and not making a cycle

QUESTION 5

5.1 A ; B ; D Must be the start



A – B – D – E – F – G – I ⇒ 26

5.2 (a) J → M → **K** → L → N – J = 620

(b) M – K
N – L
K – L – 395
+ J – M
J – N – 225
Total: 620

(c) T = 620
∴ 10:00+10hr20min, 20:20 or (8:20 pm),

QUESTION 6

$3 \times 3 \; - \; 3(2 \times 2)$
$4 \times 4 \; - \; 4(3 \times 3) \; \rightarrow \; 4 \times 3\,(2 \times 2)$
$5 \times 5 \; - \; 5 \; (4 \times 4) \; \rightarrow \; 5 \times 4 \times 3\,(2 \times 2)$
$6 \times 6 \; - \; 6 \; (5 \times 5) \; \rightarrow \; 6 \times 5 \times 4 \times 3\,(2 \times 2)$
$7 \times 7 \; - \; 7(6 \times 6) \; - \; 7 \times 6 \times 5 \times 4 \times 3\,(2 \times 2)$

$\therefore$ 2 520 (2×2) determinants

Total for Module 4: 100 marks