INTERNATIONAL SECONDARY CERTIFICATE EXAMINATION
NOVEMBER 2022

FURTHER STUDIES MATHEMATICS (STANDARD): PAPER I

MARKING GUIDELINES

Time: 2 hours | 200 marks

These marking guidelines are prepared for use by examiners and sub-examiners, all of whom are required to attend a standardisation meeting to ensure that the guidelines are consistently interpreted and applied in the marking of candidates' scripts.

The IEB will not enter into any discussions or correspondence about any marking guidelines. It is acknowledged that there may be different views about some matters of emphasis or detail in the guidelines. It is also recognised that, without the benefit of attendance at a standardisation meeting, there may be different interpretations of the application of the marking guidelines.

QUESTION 1

1.1 (a) $\ln(x-2)=1$ or $\ln(x-2)=-1$ ✓ m – 2 cases

✓a $\therefore x-2=e$ or $x-2=e^{-1}$ ✓a

$\therefore x=e+2$ or $x=e^{-1}$ ✓ca $+2$ ✓ca

$\therefore x=4{,}72$ ✓ca or $x=2{,}37$ ✓ca (6)

(b) $\therefore e^{x}\left(e-1\right)$ ✓a $=12$ ✓m – factoring

$\therefore e^{x}=\dfrac{12}{e-1}$ ✓a

$\therefore x=\ln\dfrac{12}{e-1}=1.94$ ✓m – use of ln ✓ca (5)

(c) $\therefore -2\left|2x+4\right|\geq -16$ ✓a

$\therefore \left|2x+4\right|\leq 8$ ✓✓a

$\therefore -8\leq 2x+4\leq 8$ ✓m ✓a

$\therefore -12\leq 2x\leq 4$ ✓ca

$\therefore -6\leq x\leq 2$ ✓ca (7)

1.2 $\dfrac{15-5ai}{a+2i}\times\dfrac{a-2i}{a-2i}$ ✓m $=\dfrac{15a-30i-5a^{2}i+10ai^{2}\ \text{✓a}}{a^{2}-4i^{2}\ \text{✓a}}=-1-7i$

$\therefore \dfrac{5a}{a^{2}+4}+\dfrac{-30-5a^{2}}{a^{2}+4\ \text{✓a}}i=-1-7i$ ✓m

$\therefore \dfrac{5a}{a^{2}+4}=-1$

$\therefore 5a=-a^{2}-4$ ✓a

$\therefore a^{2}+5a+4=0$ ✓m

$\therefore \left(a+1\right)\left(a+4\right)=0$

$\therefore a=-1$ or $a=-4$ ✓a

but, a check reveals that $a=-1$ is the only option which generates the correct imaginary part

$\therefore a=-1$ ✓a

ALTERNATE SOLUTION 1

$\therefore 15-5ai=(-1-7i)(a+2i)$ ✓m

$\therefore 15-5ai=-a-7ai-2i-14i^{2}$ ✓a

$\therefore 15-5ai=13-(7a+2)i$ ✓m ✓a

$\therefore 5a=7a+2$ ✓m – equating coefficients

$\therefore -2a=2$ ✓m – solving for a

$\therefore a=-1$ ✓a ✓a

ALTERNATE SOLUTION 2

$\therefore 15 - 5ai = (-1 - 7i)(a + 2i)$ ✓m ✓a
$15 = -a - 14i^2$ ✓a ✓a ✓m – real
$\therefore a = -1$ ✓a ✓a ✓a (8)

1.3 (a) if $2 + i$ is a root then so is $2 - i$
sum of roots is 4 and product of roots is 5 ✓m – ✓m – sum and product
$\therefore f(x) = (x^2 - 4x + 5)$ is a factor ✓a
$\therefore f(x) = (x^2 - 4x + 5)(x^2 + x - 20)$ ✓m – other factor ✓a
$\therefore f(x) = (x^2 - 4x + 5)(x + 5)(x - 4)$ ✓ca (7)

(b) $(x^2 - 4x + 5)(x + 5)(x - 4) = 0$
$\therefore x = 2 + i$ ✓a or $2 - i$ ✓ca or -5 or 4 ✓ca (3)

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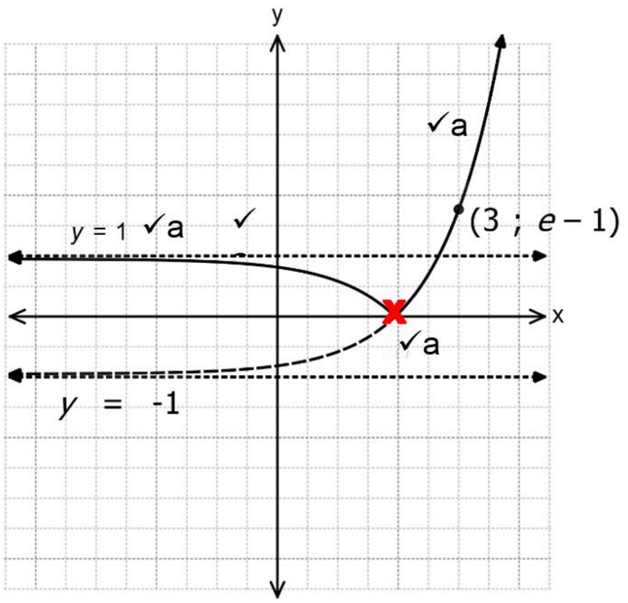
QUESTION 2

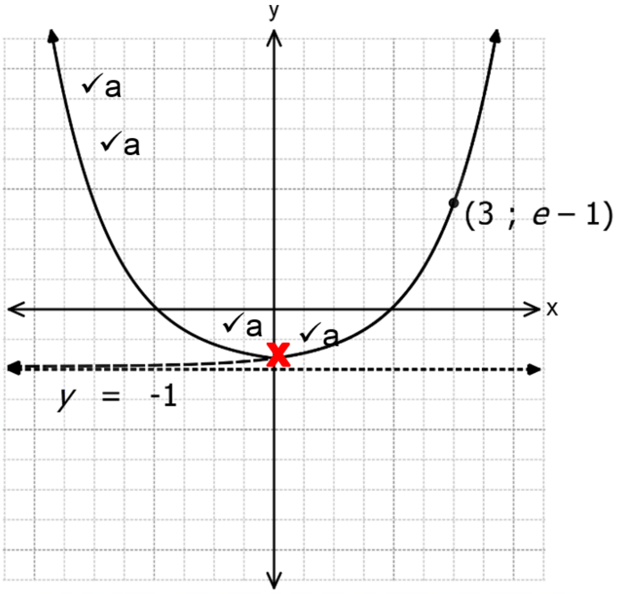
2.1 (a) $c = -1$ ✓a ✓a
$e - 1 = e^{3+p} - 1$ ✓m – substitution ✓a
$\therefore e = e^{3+p}$ ✓ca
$\therefore p = -2$ ✓ca (6)

(b) (i)



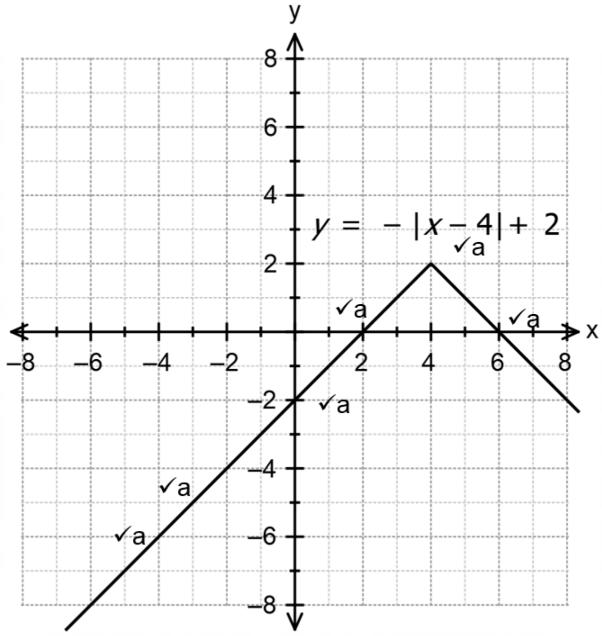
(4)

(ii)



(4)

2.2



(6)

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QUESTION 3

3.1 $ax - 8 = 0$ when $x = 2$ ✓m – denominator

$\therefore a = 4$ ✓a

now $2x^2 + bx - 10 = (4x - 8)\left(\frac{1}{2}x \text{ ✓m} + 4 \text{ ✓a}\right) + R$

$\therefore 2x^2 + bx - 10 = 2x^2 + 12x - 32 + R$ ✓a

$\therefore b = 12$ ✓ca

(6)

3.2
- A *y*-intercept of 3
- *x*-intercepts of –2 and 3
- Two vertical asymptotes, one of which must be $x = 1$
- A horizontal asymptote of $y = -2$ ✓✓a – y-intercept of 3

✓✓aa – x-intercepts

$$y = \frac{-2(x+2)(x-3)}{(x-1)(x+4)}$$

✓✓aa – vertical asymptotes

✓a – numerator and denominator of = degree
✓a – horizontal asymptote of –2 (8)

3.3 $g'(x) = \frac{(2x+3) \checkmark a (3x-13) - 3(x^2+3x-10) \checkmark a}{(3x-13)^2 \checkmark a} = 0$ ✓m – quotient rule

$\therefore 3x^2 - 26x - 9 = 0$ ✓m – equating to zero

$\therefore (3x+1)(x-9) = 0$

$\therefore x = -\frac{1}{3}$ or 9 ✓ca

$\therefore \left(-\frac{1}{3} ; 0{,}78\right)$ ✓ca and $(9 ; 7)$ ✓ca (8)

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QUESTION 4

4.1 (a) $p = 2$ ✓a ✓a (2)

(b) we need f to be continuous at $x = -2$ ✓m – continuity

now $\lim_{x \to -2^+} f(x) = 1$ ✓a

$\therefore \lim_{x \to -2^-} f(x) = 1$ ✓m – equating left and right-hand limits

$\therefore -2m + c = 1$ ✓a

also, $\lim_{x \to -2^+} f'(x) = \lim_{x \to -2^+} (-2x) = 4$ ✓m – equating left and right-hand derivatives

$\therefore \lim_{x \to -2^-} f'(x) = 4$ ✓a

$\therefore m = 4$ ✓ca

$\therefore c = 9$ ✓ca (8)

4.2 (a) $x^2 + 10y^2 + 14x + 16y = 2$ ✓m – implicit differentiation

$\therefore 2x + 20y \checkmark a \frac{dy}{dx} + 14 + 16\frac{dy}{dx} \checkmark a = 0$

$\therefore \frac{dy}{dx}(20y + 16) = -2x - 14$ ✓m – factoring ✓a

$\therefore \frac{dy}{dx} = \frac{-2x - 14}{20y + 16}$ ✓ca (6)

(b) $\frac{dy}{dx} = \frac{-2x-14}{20y+16}$

$\left.\frac{dy}{dx}\right|_{(-2;1)} = \frac{-10}{36} = -\frac{5}{18}$ ✓m – gradient

$\therefore$ gradient of normal $= \frac{18}{5}$ ✓ca

$\therefore y - 1 = \frac{18}{5}(x+2)$ ✓m – equation of line ✓ca

$\therefore y = \frac{18}{5}x + \frac{41}{5}$ (this line need not be shown) (4)

[20]

QUESTION 5

5.1 (a) $f(x) = x^3 - 2x + 2 = 0$ ✓m – equating to zero

$\therefore f'(x) = 3x^2 - 2$ ✓m – ✓a

$\therefore x_{n+1} = x_n$ ✓a $- \frac{x_n^{\ 3} - 2x_n + 2}{3x_n^{\ 2} - 2 \text{ ✓a}}$ ✓m – formula (6)

(b) The tangent at $x-1$intersects the x-axis at 0 and the tangent at $x = 0$ intersects the x-axis at 1 so the answers cycle rather than converging. ✓a ✓a

Note that we will accept any comment suggesting 'cycling'/looping. (4)

(c) $x = -1{,}76929$ ✓a ✓a (2)

5.2 (a) $\cos\theta = \frac{3^2+4^2-6^2}{2(3)(4)} = -\frac{11}{24}$ ✓a ✓a ✓m – cos rule

$\therefore \theta = 2{,}047$ ✓ca (4)

(b) shaded area = area of sector – area of triangle ✓m – subtraction

$= \frac{5^2(2{,}047) \text{ ✓a}}{2} - \frac{1}{2}(3)(4)$ ✓a $\sin 2{,}047$ ✓m – sector ✓m – triangle

$= 20{,}25 \text{ m}^2$ ✓ca

20,26 if you use the rounded 2,047 (6)

[22]

QUESTION 6

$\sum_{i=1}^{n}(3i-1)(3i+2)=3n^3+6n^2+n$ ✓m – proving for $n-1$ ✓a ✓m – assumption

if $n=1$ then $LHS=10$ and $RHS=10$ ✓a

so it is true for $n=1$ ✓m – considering $n=k+1$ ✓a

Assume true for $n=k$ viz. ✓a

$(2)(5)+(5)(8)+(8)(11)+...+(3k-1)(3k+2)=k^3+6k^2+k\ (*)$ ✓a

now if $n=k+1$ then:

$(2)(5)+(5)(8)+(8)(11)+...+(3k-1)(3k+2)+(3k+2)(3k+5)=3k^3+6k^2+k+(3k+2)(3k+5)$

$=3k^3+6k^2+k+9k^2+21k+10$ ✓m – attempting to write in correct form

✓a $=3k^3+9k^2+9k+3+6k+12k+6+k+1$ ✓a ✓a

but this is just $(*)$ with $k+1$ for k

so, it is true for $n=k+1$

$\therefore$ by the principle of mathematical induction it is true for $n\in\mathbb{N}$

[12]

QUESTION 7

7.1 ✓a $10200 = 10000e^k$

$\therefore e^k = \frac{10200}{10000}$ ✓a

$\therefore k = \ln\frac{102}{100} = 0.0198$ ✓m – using ln ✓ca (4)

7.2 $y = y_0 e^{kt}$

$\therefore \frac{y}{y_0} = e^{kt}$ ✓m – division

$\therefore kt = \ln\frac{y}{y_0}$ ✓m – ln

$\therefore t = \frac{\ln\frac{y}{y_0}}{k}$ ✓m – division

$t = \frac{\ln 10}{0{,}0198} = 116{,}29$ ✓a

$\therefore t = 117$ months ✓a (6)

[10]

QUESTION 8

shaded area $= \int_1^4 f(x)dx - \int_1^4 g(x)dx$ ✓m – subtraction

$\therefore \frac{21}{2} = \frac{14}{3} - \int_1^4 f(x) - kx - k\ dx$ ✓a

$\therefore \frac{21}{2} = \frac{14}{3} - \left(\int_1^4 f(x)dx - \int_1^4 kx + k\ dx\right)$ ✓a

$\therefore \frac{21}{2} = \frac{14}{3} - \frac{14}{3} + \left[\frac{kx^2}{2} + kx\right]_1^4$ ✓a ✓m – integration

$\therefore \frac{21}{2} = 12k - \left(\frac{k}{2} + 1\right)$ ✓m – evaluation

$\therefore 21 = 24k - k - 2$ ✓a

$23 = 23k$

$\therefore k = 1$ ✓ca

ALTERNATE METHOD

$$\checkmark\text{a}\ \frac{21}{2}=\int_1^4 f(x)-g(x)\ dx\ \checkmark\text{m}-\text{subtraction}$$

$$\frac{21}{2}=\int_1^4 f(x)-f(x)+kx+k\ \ dx\ \checkmark\text{m}-\text{substitution}$$

$$\frac{21}{2}=\int_1^4 kx+k\ \ dx\ \checkmark\text{a}$$

$$\frac{21}{2}=\left[\frac{kx^2}{2}+kx\right]_1^4\ \checkmark\text{m}-\text{integration}\ \checkmark\text{a}$$

$$\frac{21}{2}=8k+4k-\frac{k}{2}-k\ \checkmark\text{a}$$

$$21=16k+8k-k-2k$$

$$k=1\checkmark\text{a}$$

[8]

QUESTION 9

9.1 $$\int -\text{cosec}^2\theta\cot\theta d\theta\ \checkmark\text{m}-\text{identifying derivative}$$

$$=\int \text{cosec}\theta\left(-\text{cosec}\theta\cot\theta\right)d\theta\ \checkmark\text{a}$$

$$=\frac{\text{cosec}^2\theta}{2}+c\ \checkmark\text{a}\checkmark\text{a}\checkmark\text{a}$$

(5)

9.2 $$\int\frac{3x}{\sqrt{2x+5\ \checkmark\text{a}}}dx\ \checkmark\text{m}-\text{taking 3 out}$$

$$=3\int x\left(2x^2+5\right)^{-\frac{1}{2}}dx\ \checkmark\text{a}$$

$$=\frac{3\left(2x^2+5\right)^{\frac{1}{2}}}{\frac{1}{2}\times 4}+c\ \ \checkmark\text{a}$$

$$=\frac{3\left(2x^2+5\right)^{\frac{1}{2}}}{2}+c\ \checkmark\text{a}\checkmark\text{a}$$

ALTERNATE

$= 3\int x\left(2x^2+5\right)^{-\frac{1}{2}} dx$ ✓m – substitution

let $u = 2x^2+5$ ✓a then $\dfrac{du}{dx}$ ✓m $= 4x$ *so* $dx = \dfrac{du}{4xdx}$ ✓a

$= \dfrac{3}{4}\int u^{-\frac{1}{2}} du$ ✓m – integrating a power

$= \dfrac{3}{4}\left(\dfrac{u^{\frac{1}{2}}}{\frac{1}{2}}\right)+c$ ✓a

$= \dfrac{3u^{\frac{1}{2}}}{2}+c$ ✓ca

$= \dfrac{3\left(2x^2+5\right)^{\frac{1}{2}}}{2}+c$ (7)

9.3 $\dfrac{2x^2+3x+8}{x^2-x-6}$ ✓m – division by inspection ✓a

$= \dfrac{2\left(x^2-x-6\right)+5x+20}{x^2-x-6}$ ✓a

$= \dfrac{2\left(x^2-x-6\right)}{x^2-x-6}+\dfrac{5x+20}{x^2-a-6}$ ✓a

$= 2+\dfrac{5x+20}{(x-3)(x+2)}$ ✓m – partial fractions ✓a

now $\dfrac{5x+20}{(x-3)(x+2)} = \dfrac{A}{x-3}+\dfrac{B}{x+2}$ ✓a ✓a

$A = 7$ and $B = -2$ both by cover-up method

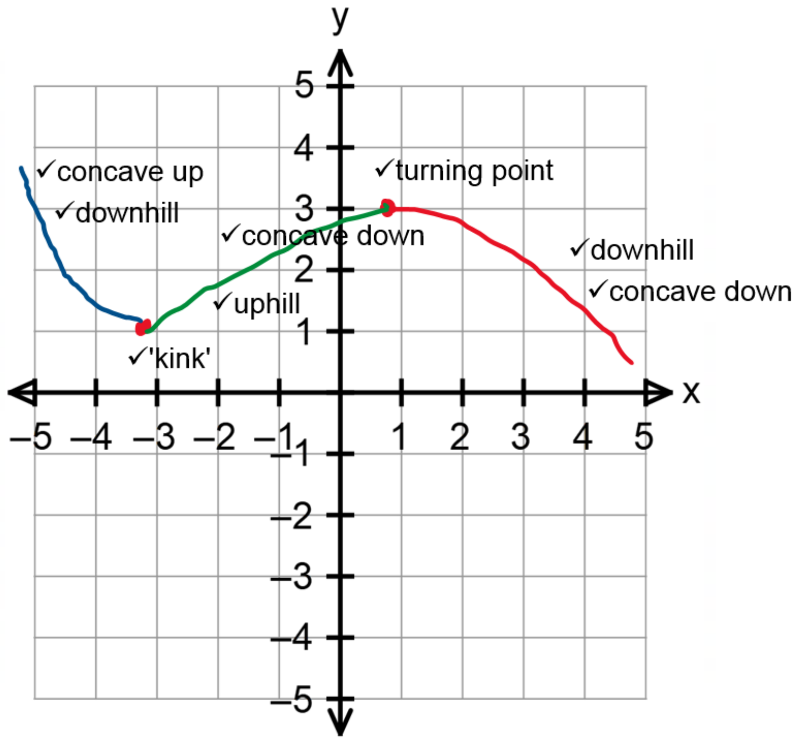
$\therefore \int \dfrac{2x^2+3x+8}{x^2-x-6}dx$ ✓ca $= \int 2+\dfrac{7}{x-3}-\dfrac{2}{x-2}\,dx$ ✓ca

$= 2x+7\ln|x+2|-2\ln|x+2|+c$ (10)

[22]

QUESTION 10

10.1



(8)

10.2 $g'(x) = \cos x \tan x + \sin x \sec^2 x = 0$ ✓m – derivative ✓m – derivative = 0

✓a $\therefore \sin x + \sin x \sec^2 x = 0$ ✓a

$\therefore \sin x(1 + \sec^2 x) = 0$ ✓m – factoring

$\therefore \sin x = 0$ ✓a

$\therefore x = \pi$

$\therefore (\pi; 0)$ ✓a

(8)

[16]

QUESTION 11

11.1 $P\hat{Q}R = 90°$ ($\angle$ ub semi-circle) ✓m – trig ratio

$\therefore \cos\theta = \frac{PQ}{4}$ ✓a

$\therefore PQ = 4\cos\theta$

time $= \frac{\text{distance}}{\text{speed}}$ ✓a time to row $= \frac{4}{3}\cos\theta$

Note: PQ can also be established using $\perp$ ✓a bisector of chord

$Q\hat{O}R = 2\theta$ ($\angle$ at centre)

$\therefore QR = r\theta = 4\theta$

so, time to walk $= \frac{4\theta}{5}$ ✓a ✓m – arclength formula ✓a

total time $t = \frac{4}{3}\cos\theta + \frac{4\theta}{5}$ ✓a

(8)

11.2 $t=\frac{4}{3}\cos\theta+\frac{4\theta}{5}$ ✓m derivative ✓m $=0$

$\frac{dt}{d\theta}=-\frac{4}{3}\sin\theta+\frac{4}{5}=0$

$\therefore -\frac{4}{3}\sin\theta=-\frac{4}{5}$ ✓ca

$\therefore \sin\theta=\frac{3}{5}$ ✓ca

$\therefore \theta=\arcsin\left(\frac{3}{5}\right)=0{,}644$ (4)

[12]

Total: 200 marks