INTERNATIONAL SECONDARY CERTIFICATE EXAMINATION
NOVEMBER 2023

FURTHER STUDIES MATHEMATICS (STANDARD): PAPER I

MARKING GUIDELINES

Time: 2 hours 200 marks

These marking guidelines are prepared for use by examiners and sub-examiners, all of whom are required to attend a standardisation meeting to ensure that the guidelines are consistently interpreted and applied in the marking of candidates' scripts.

The IEB will not enter into any discussions or correspondence about any marking guidelines. It is acknowledged that there may be different views about some matters of emphasis or detail in the guidelines. It is also recognised that, without the benefit of attendance at a standardisation meeting, there may be different interpretations of the application of the marking guidelines.

QUESTION 1

1.1 Solve:

(a) $\ln(2+e^{-x})=2$ Leave your answer in the form $x=\ln(....)$

$e^{2}=2+e^{-x}$
$e^{-x}=e^{2}-2$
$-x=\ln(e^{2}-2)$
$x=-\ln(e^{2}-2)$
$x=\ln\left(\dfrac{1}{e^{2}-2}\right)$

(b) $|2x+3|=3x+4$

$2x+3=3x+4$ or $2x+3=-3x-4$
$x=-1$ or $x=-\dfrac{7}{5}$
a check reveals $x=-1$ only

1.2 Give, in standard $ax^4+bx^3+cx^2+dx+e=0$ form, a quartic equation which has $x=2+\sqrt{3}$ and $2-i$ as roots. The values of *a, b, c, d* and *e* must be rational.

one quadratic has roots $2+\sqrt{3}$ and $2-\sqrt{3}$
the sum of these roots is 4 and the product is 1 so $x^2-4x+1=0$
the other has roots $2-i$ and $2+i$
the sum of these roots is 4 and the product is 5 so $x^2-4x+5=0$
$(x^2-4x+1)(x^2-4x+5)=0$
$x^4-8x^3+22x^2-24x+5=0$

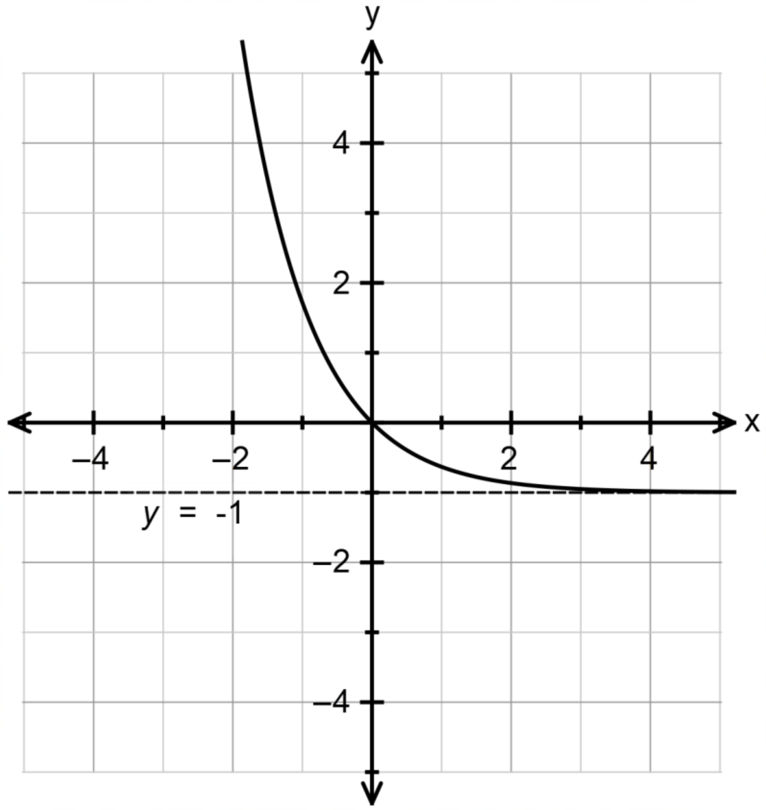
1.3 Determine positive real values of *a* and *b* if:

$(a+bi)(b+i)=(2b+a)i$
$\text{LHS}=ab-b+(b^2+a)i$

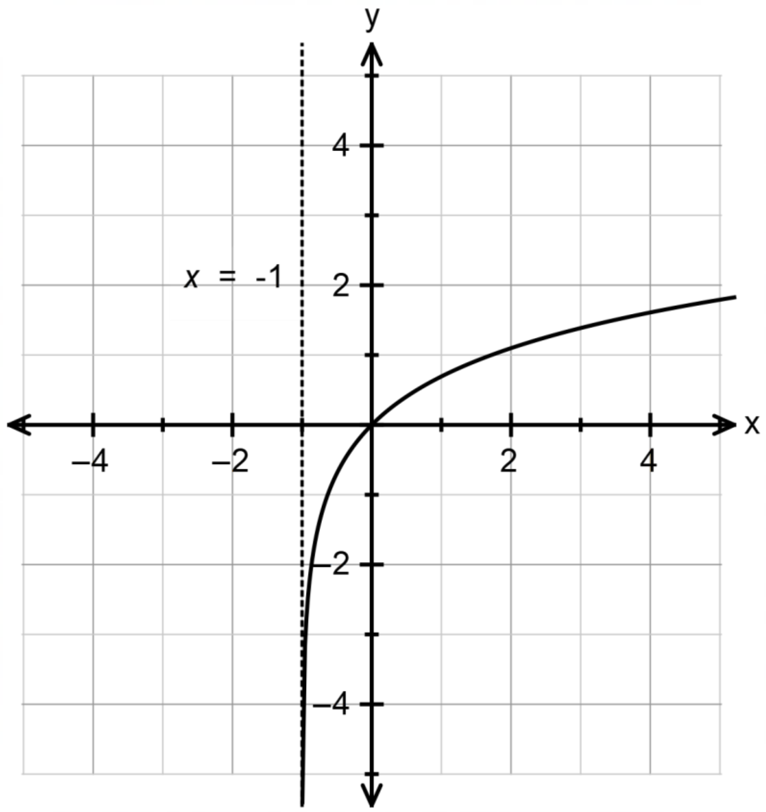
so, $ab-b=0$ and $b^2+a=2b+a$
so, $b(a-1)=0$
since $b\neq 0$, $a=1$
$\therefore b=2$

1.4. Sketch the following functions on the axes provided. You should draw and give the equations of any asymptotes as well as showing any intercepts with the axes.

(a) $y = e^{-x} - 1$



(b) $y = \ln(x+1)$



QUESTION 2

Use mathematical induction to prove that:

$$-1+4-9+16-25+.....(-1)^n n^2 = \frac{(-1)^n n(n+1)}{2} \text{ for } n \in \mathbb{N}$$

if $n=1$ LHS $=-1$ and $RHS=-1$ so it is true for $n=1$

assume true for $n=k$

$$-1+4-9+16-25+.....(-1)^k k^2 = \frac{(-1)^k k(k+1)}{2} (*)$$

now consider $n=k+1$

$$-1+4-9+16-25+.....(-1)^k k^2 + (-1)^{k+1}(k+1)^2 = \frac{(-1)^k k(k+1)}{2} + (-1)^{k+1}(k+1)^2$$

$$= (-1)^k \left[\left(\frac{k(k+1)}{2} - (k+1)^2\right]\right.$$

$$= (-1)^k \left[\left(\frac{k(k+1)}{2} - \frac{2(k+1)^2}{2}\right]\right.$$

$$= (-1)^k \left[\frac{(k+1)(k-2(k+1))}{2}\right]$$

$$= (-1)^k \left[\frac{(k+1)(-k-2)}{2}\right]$$

$$= (-1)^{k+1} \left[\frac{(k+1)(k+2)}{2}\right]$$

but this is just (*) with $n=k+1$

so, we have proved it true for $n=k+1$

$\therefore$ by *PMI* we have proved it for $n \in \mathbb{N}$

QUESTION 3

Determine $\frac{d}{dx}\sqrt{3x}$ by first principles

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{3(x+h)} - \sqrt{3x}}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{3(x+h)} - \sqrt{3x}}{h} \times \frac{\sqrt{3(x+h)} + \sqrt{3x}}{\sqrt{3(x+h)} + \sqrt{3x}}$$

$$f'(x) = \lim_{h \to 0} \frac{3(x+h) - 3x}{h\left(\sqrt{3(x+h)} + \sqrt{3x}\right)}$$

$$f'(x) = \lim_{h \to 0} \frac{3x + 3h - 3x}{h\left(\sqrt{3(x+h)} + \sqrt{3x}\right)}$$

$$f'(x) = \lim_{h \to 0} \frac{3h}{h\left(\sqrt{3(x+h)} + \sqrt{3x}\right)}$$

$$f'(x) = \lim_{h \to 0} \frac{3}{\left(\sqrt{3(x+h)} + \sqrt{3x}\right)}$$

$$f'(x) = \frac{3}{2\sqrt{3x}}$$

QUESTION 4

Consider the function $f(x)=\dfrac{x^2-5x+7}{x-2}$.

(a) Determine, with classification, the equations of any asymptotes.

vertical asymptote: $x=2$

$x^2-5x+7=(x-2)(x-3)+1$

so, $\dfrac{x^2-5x+7}{x-2}=x-3+\dfrac{1}{x-2}$

so, $y=x-3$ is an oblique asymptote

(b) Justify mathematically why the function does not have any x-intercepts.

$\dfrac{x^2-5x+7}{x-2}=0$

$\therefore x^2-5x+7=0$

but $\Delta=(-5)^2-4(1)(7)=-3$

$\therefore$ no real roots

(c) Determine the coordinates of any stationary points.

$f(x)=\dfrac{x^2-5x+7}{x-2}$

$f'(x)=\dfrac{(2x-5)(x-2)-1(x^2-5x+7)}{(x-2)^2}$

$f'(x)=\dfrac{x^2-4x+3}{(x-2)^2}=0$

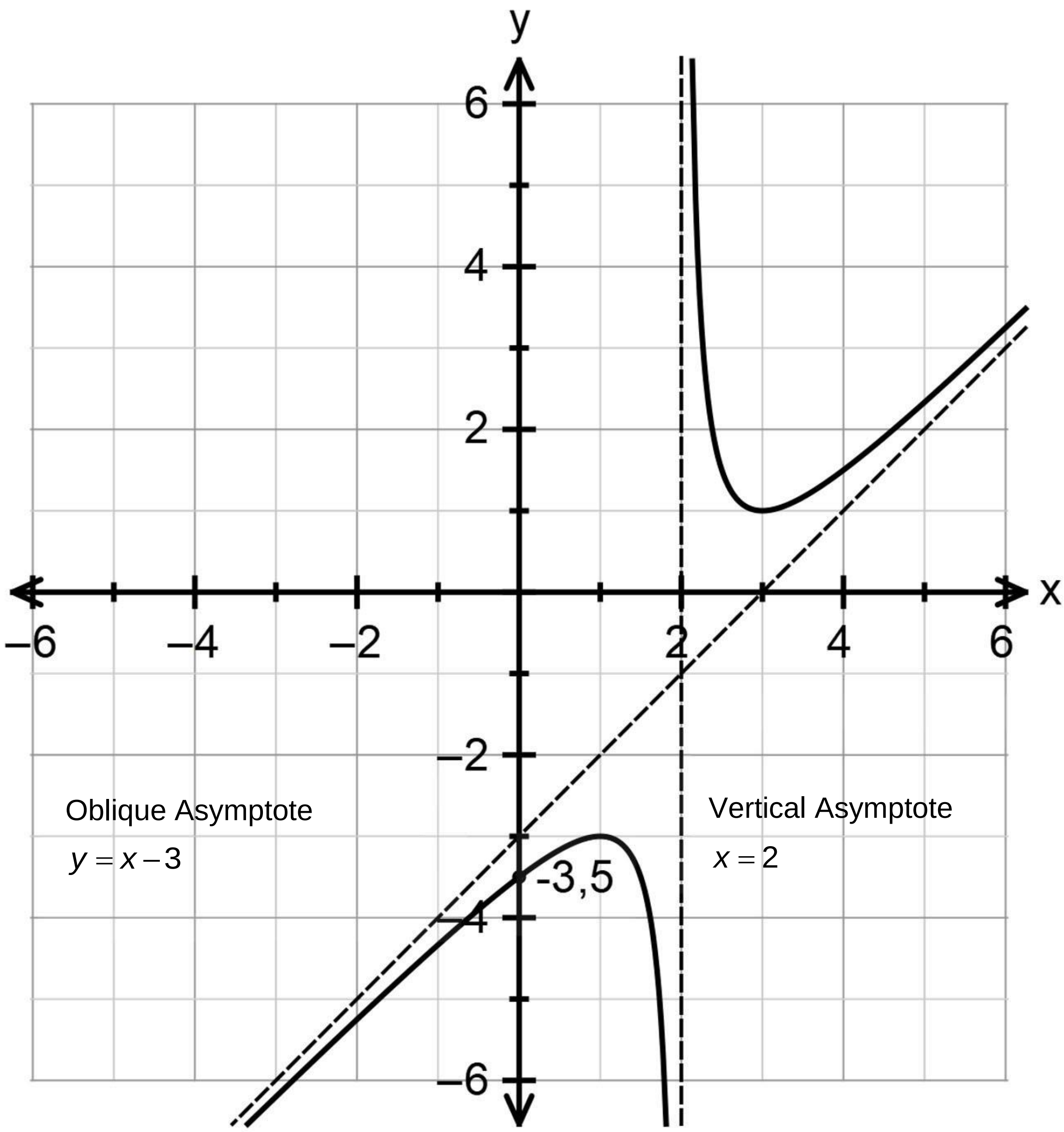
$\therefore x^2-4x+3=0$

$\therefore (x-1)(x-3)=0$

$\therefore x=1 \quad \text{or} \quad 3$

$\therefore (1\ ;-3) \quad \text{or} \quad (3\ ;1)$

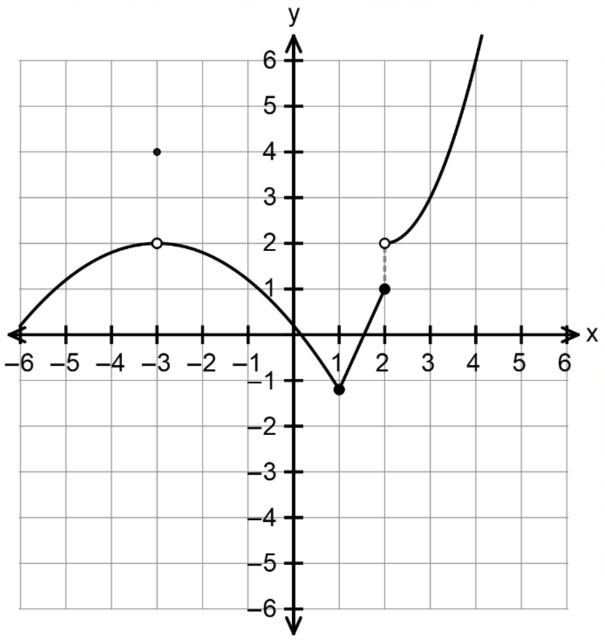
(d) Draw the graph of *f* on the axes provided showing all points of interest. You should draw and label any asymptotes.



QUESTION 5

5.1 On the axes provided draw a **function** *g* which satisfies the following:

- *g* is continuous for all values of *x* except at $x = -3$ and $x = 2$
- $g(-3) = 4$ and $\lim\limits_{x \to -3} g(x)$ exists
- $g(2) = 1$ and $\lim\limits_{x \to 2^-} g(x) = 1$ but there is a jump discontinuity at x = 2
- *g* is also not differentiable at $x = 1$



5.2 Express the following statements **using mathematical notation**:

(a) The left-hand and right-hand limits of *g* at *a* are unequal.

$$\lim_{x-a^-} g(x) \neq \lim_{x \to a^+} g(x)$$

(b) *h* is not differentiable at *p* despite being continuous at *p*.

$$\lim_{x \to p^-} h'(x) \neq \lim_{x \to p^+} h'(x)$$

5.3 Answer true or false to each of the following statements:

(a) If a function is differentiable at a point, then it is continuous at that point.

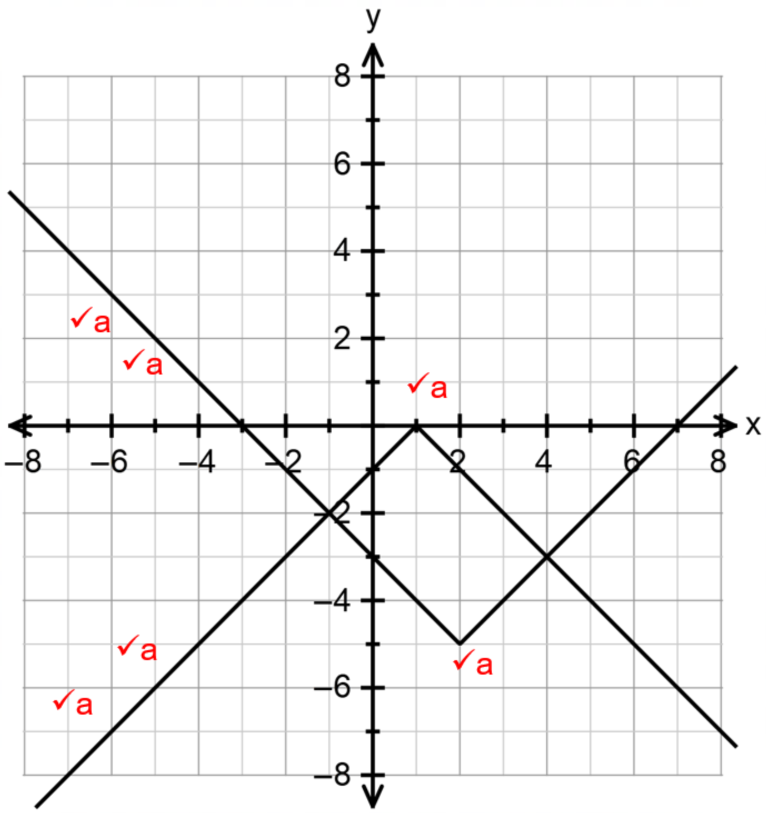
TRUE

(b) If a function is not differentiable at a point, then it is not continuous at that point.

FALSE

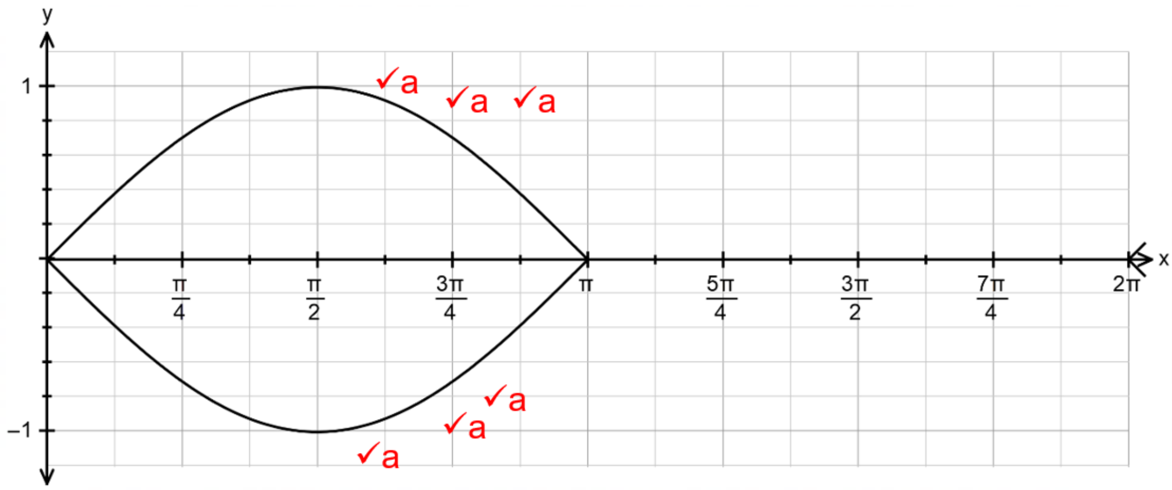
QUESTION 6

6.1 Use the axes below to solve $|x-2|-5 \geq -|x-1|$ sketching the **graphs of two functions**. You must label the graphs you have drawn with their equations.



$x \leq -1$ or $x \geq 4$

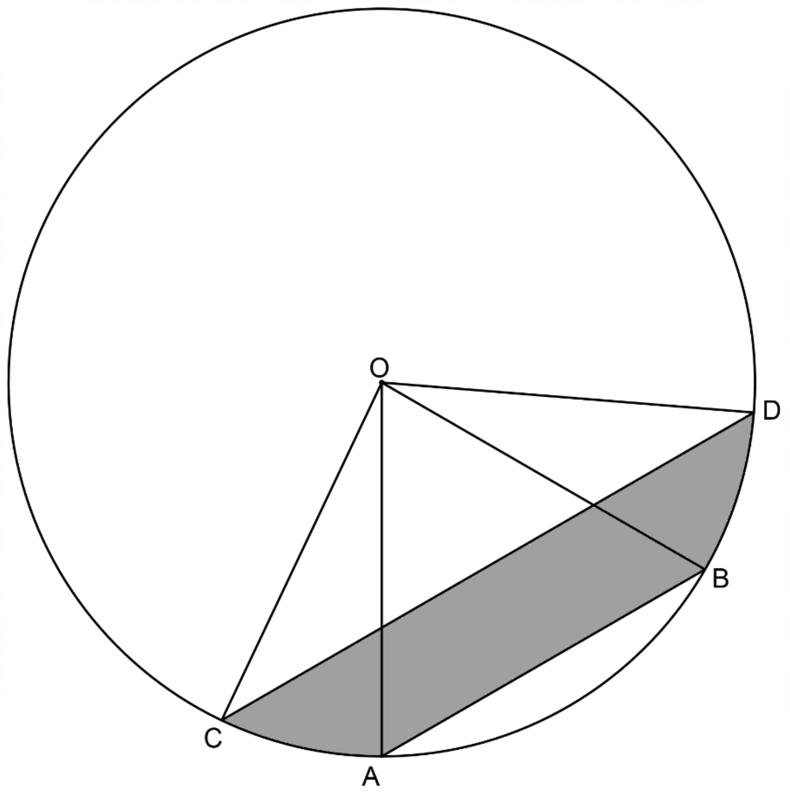
6.2 Draw the graph of $|y| = \sin x$ on the axes provided showing all points of interest.



QUESTION 7

In the diagram below triangle AOB is equilateral with sides of 1 unit.
O is the centre of the circle and $CD = \sqrt{3}$ units.

Determine the shaded area.



$$\cos\left(C\hat{O}D\right) = \left(\frac{1^2 + 1^2 - \left(\sqrt{3}\right)^2}{2(1)(1)}\right) = -\frac{1}{2}$$

$$\therefore C\hat{O}D = \pi - \cos^{-1}\left(\frac{1}{2}\right) = \frac{2\pi}{3}$$

Area sector $OCD = \frac{1}{2}(1^2)\left(\frac{2\pi}{3}\right) = \frac{2\pi}{6}$

from this we must subtract $\Delta OCD = \frac{1}{2}(1)(1)\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{4}$

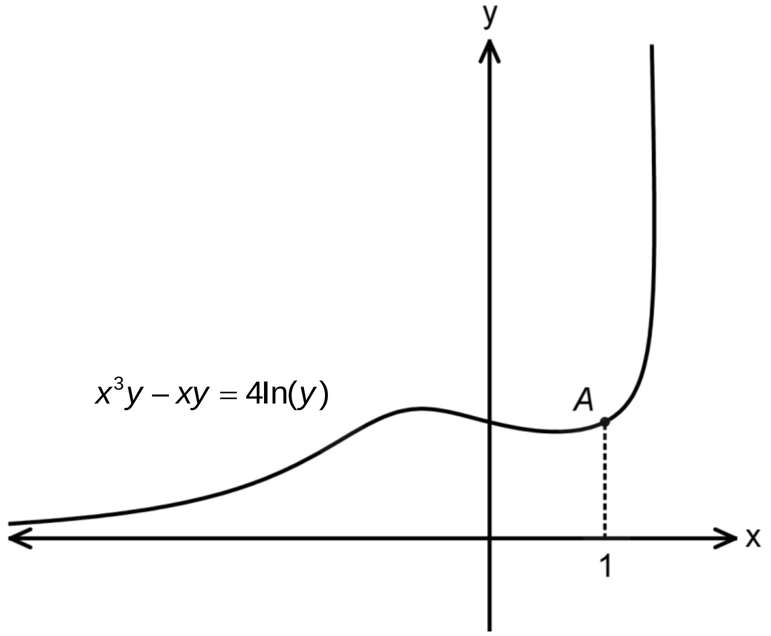
and the segment formed by $AB = \frac{1}{2}\left(1^2\,\frac{\pi}{3} - \sin\frac{\pi}{3}\right) = \frac{\pi}{6} - \frac{\sqrt{3}}{4}$

so, shaded area $= \left(\frac{2\pi}{6} - \frac{\sqrt{3}}{4}\right) - \left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right)$

so, shaded area = $\frac{\pi}{6} = 0{,}52$ units^2

QUESTION 8

A portion of the graph of the implicitly defined relationship $x^3y - xy = 4\ln(y)$ is shown below.



(a) Determine the *y*-coordinate of point A showing all working.

$$1^3y - 1y = 4\ln(y)$$
$$0 = 4\ln(y)$$
$$\ln(y) = 0$$
$$y = 1$$
$$\therefore A(1;1)$$

(b) Find the equation of the tangent to the curve at the point A.

$x^3y - xy = 4\ln(y)$

implicit differentiation yields:

$$3x^2y + x^3\frac{dy}{dx} - y - x\frac{dy}{dx} = \frac{4}{y}\cdot\frac{dy}{dx}$$

$$x^3\frac{dy}{dx} - x\frac{dy}{dx} - \frac{4}{y}\cdot\frac{dy}{dx} = y - 3x^2y$$

$$\frac{dy}{dx}\left(x^3 - x - \frac{4}{y}\right) = y - 3x^2y$$

$$\frac{dy}{dx} = \frac{y - 3x^2y}{x^3 - x - \frac{4}{y}}$$

at A, $\frac{dy}{dx} = \frac{1}{2}$

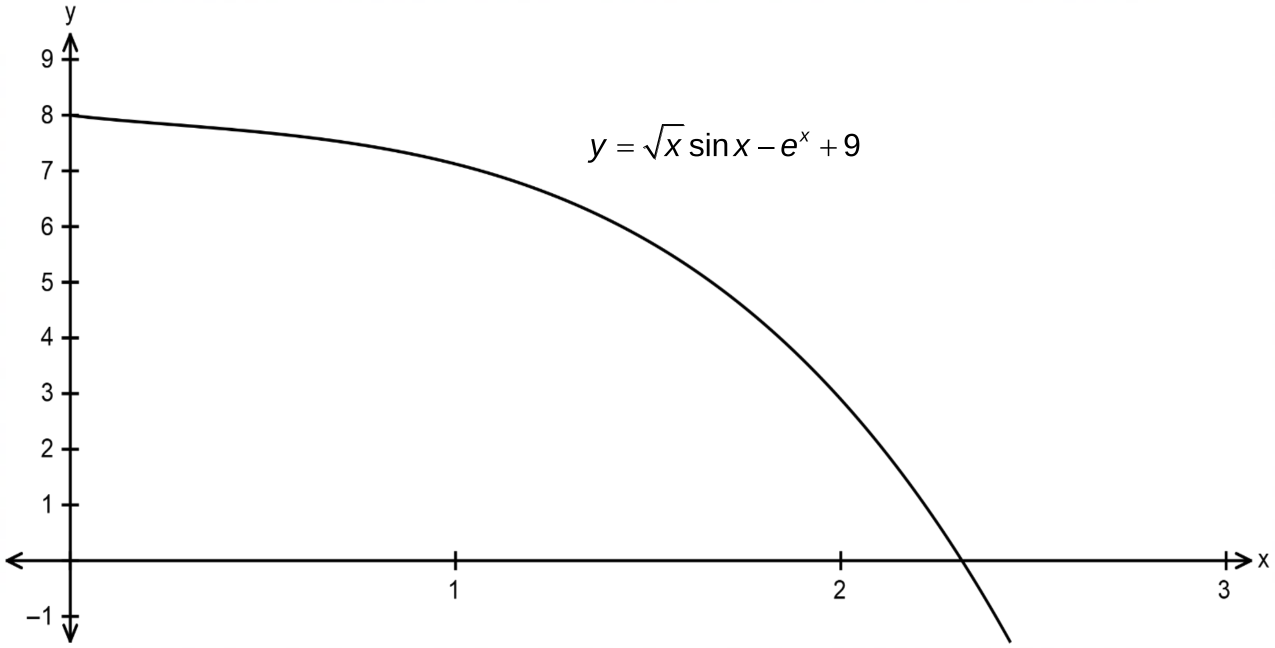
$$y - 1 = \frac{1}{2}(x - 1)$$

$$y = \frac{1}{2}x + \frac{1}{2}$$

QUESTION 9

The function $f(x) = \sqrt{x}\sin x - e^x + 9$ is shown below.

Use the Newton-Raphson method to find the *x*-intercept to 5 decimal places using $x_0 = 2$ as an initial guess.



You should show:

- the iterative formula you use.
- x_1 to 5 decimal places.

You do **not** need to write down all your approximations.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

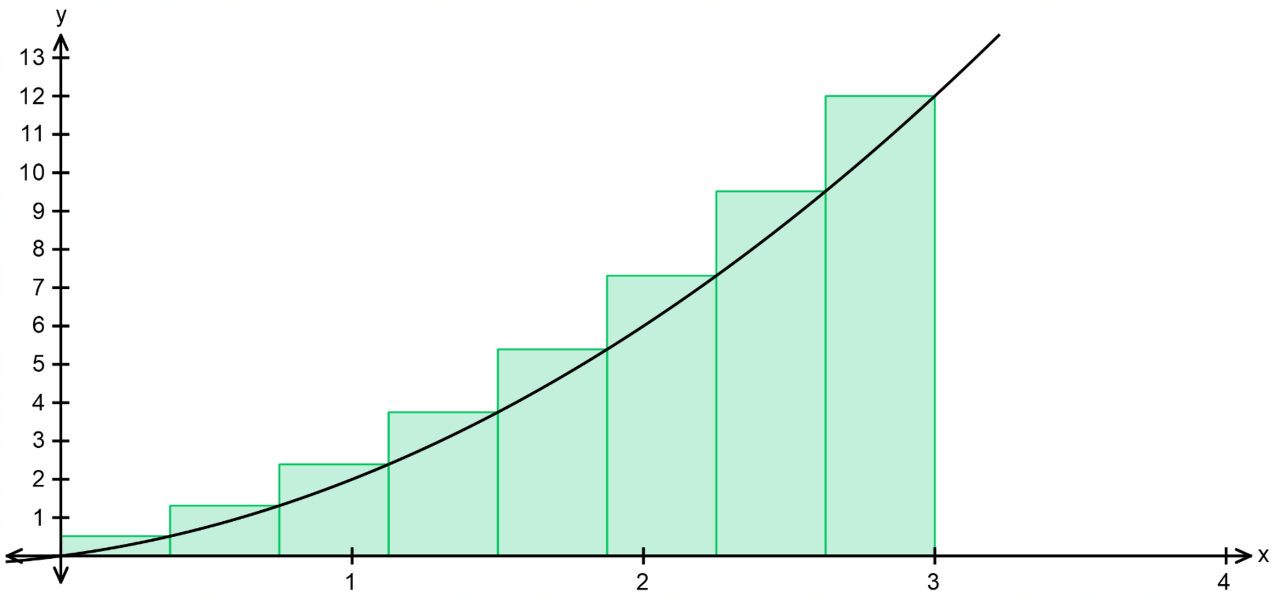
$$x_{n+1} = x_n - \frac{\sqrt{x}_n \sin x_n - e^{x_n} + 9}{\frac{1}{2}x^{-\frac{1}{2}}\sin x_n + \sqrt{x}_n \cos x_n - e^{x_n}}$$

$$x_1 = 2{,}37838$$

$$x = 2{,}31448$$

QUESTION 10

Kofi is attempting to work out the area under the curve $y = x^2 + x$ from $x = 0$ to $x = 3$ by partitioning it into rectangles as shown.



He has correctly worked out that when he uses n rectangles the area is given by:

$$A = 13,5 + \frac{18}{n} + \frac{27}{6n^2}.$$

He uses his formula and ends up with an error of $13\frac{2}{3}\%$. How many rectangles did he use?

exact answer $= \int_0^3 x^2 + x\, dx = 13,5$

$\therefore \frac{A - 13,5}{13,5} \times 100 = \frac{41}{3}$

$\therefore A = \frac{41}{300} \times 13,5 + 13,5$

$\therefore A = 15,345$

$\therefore 13,5 + \frac{18}{n} + \frac{27}{6n^2} = 15,345$

$\therefore 81n^2 + 108n + 27 = 92,07n^2$

$\therefore 11,07n^2 - 108n - 27 = 0$

$\therefore n = 10$

QUESTION 11

Determine the following integrals:

(a) $\int \sin^2 x\, dx$

$$\cos 2x = 1 - 2\sin^2 x$$
$$\therefore 2\sin^2 x = 1 - \cos 2x$$
$$\therefore \sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$$
$$= \int \frac{1}{2} - \frac{1}{2}\cos 2x\ dx$$
$$= \frac{x}{2} - \frac{\sin 2x}{4} + c$$

(b) $\int x\sqrt{x+1}\ dx$

let $u = x + 1$ then $x = u - 1$

and $du = dx$

$$= \int (u-1)u^{\frac{1}{2}}\ du$$
$$= \int u^{\frac{3}{2}} - u^{\frac{1}{2}}\ du$$
$$= \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + c$$
$$= \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + c$$

Alternatively:

$\int x\sqrt{x+1}\ dx$

by parts

let $f(x) = x$ then $f'(x) = 1$ and let $g'(x) = (x+1)^{\frac{1}{2}}$ then $g(x) = \frac{2}{3}(x+1)^{\frac{3}{2}}$

$$= \frac{2}{3}x(x+1)^{\frac{3}{2}} - \int \frac{2}{3}(x+1)^{\frac{3}{2}}\ dx$$
$$= \frac{2}{3}x(x+1)^{\frac{3}{2}} - \frac{2}{3}\int (x+1)^{\frac{3}{2}}\ dx$$
$$= \frac{2}{3}x(x+1)^{\frac{3}{2}} - \frac{2}{3}\times\frac{2}{5}(x+1)^{\frac{5}{2}}$$
$$= \frac{2}{3}x(x+1)^{\frac{3}{2}} - \frac{4}{15}(x+1)^{\frac{5}{2}} + c$$

(c) $\int \frac{2x+3}{x^2+6x+9}dx$

$$\frac{2x+3}{(x+3)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2}$$

$$\therefore 2x+3 = A(x+3)+B = Ax+(3A+B)$$

$$\therefore A = 2 \text{ and } 3(2)+B = 3 \text{ so } B = -3$$

$$= \int \frac{2}{x+3}dx - \int \frac{3}{(x+3)^2}dx$$

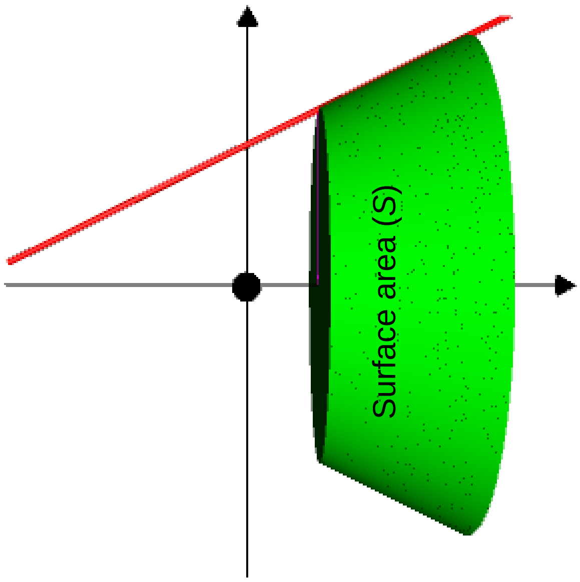
$$= 2\ln|x+3| - 3\int (x+3)^{-2}\,dx$$

$$= 2\ln|x+3| + 3(x+3)^{-1} + c$$

QUESTION 12

Consider the function $y = \frac{x}{2} + 4$.

(a) It is rotated about the x-axis from $x = 2$ to $x = b$ generating a volume of $\frac{436\pi}{3}$ units^3.



By setting up and evaluating an integral determine the value of b.

$$\pi\int_2^b \left(\frac{x}{2}+4\right)^2 dx = \frac{436\pi}{3}$$

$$\int_2^b \left(\frac{x}{2}+4\right)^2 dx = \frac{436}{3}$$

$$\int_2^b \frac{x^2}{4}+4x+16\,dx = \frac{436}{3}$$

$$\left[\frac{x^3}{12}+2x^2+16x\right]_2^b = \frac{436}{3}$$

$$\frac{b^3}{12}+2b^2+16b-\left(\frac{2^3}{12}+2(2^2)+16(2)\right) = \frac{436}{3}$$

$$\frac{b^3}{12}+2b^2+16b = \left(\frac{2^3}{12}+2(2^2)+16(2)\right)+\frac{436}{3}$$

$$\frac{b^3}{12}+2b^2+16b = 186$$

$$b^3+24b^2+192b-2\,232 = 0$$

$$b = 6$$

(b) The surface area (S) generated by rotating f about the x-axis from $x = a$ to $x = b$ is given by the formula:

$$S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Determine the surface area when the function is rotated about the x-axis from $x = 2$ to $x = 6$.

$$S = 2\pi \int_2^6 \left(\frac{x}{2} + 4\right) \sqrt{1 + (\tfrac{1}{2})^2}\, dx$$

$$S = \sqrt{5}\pi \int_2^6 \left(\frac{x}{2} + 4\right) dx$$

$$S = 24\sqrt{5}\pi \text{ units}^2$$

Total: 200 marks