



INTERNATIONAL SECONDARY CERTIFICATE EXAMINATION
NOVEMBER 2022

FURTHER STUDIES MATHEMATICS (EXTENDED): PAPER II

MARKING GUIDELINES

Time: 1 hour

100 marks

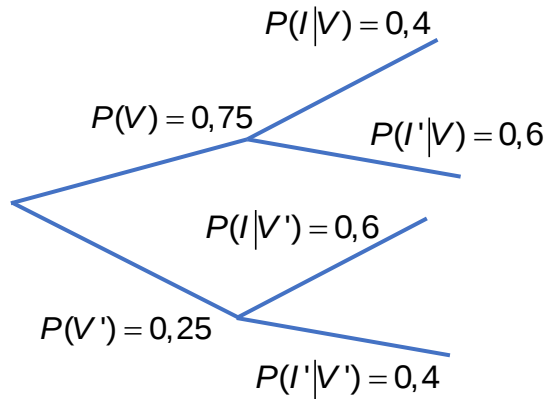
These marking guidelines are prepared for use by examiners and sub-examiners, all of whom are required to attend a standardisation meeting to ensure that the guidelines are consistently interpreted and applied in the marking of candidates' scripts.

The IEB will not enter into any discussions or correspondence about any marking guidelines. It is acknowledged that there may be different views about some matters of emphasis or detail in the guidelines. It is also recognised that, without the benefit of attendance at a standardisation meeting, there may be different interpretations of the application of the marking guidelines.

MODULE 2 STATISTICS

QUESTION 1

1.1 (a)



$$P(V|I) = \frac{(0,75)(0,4)}{(0,75)(0,4) + (0,25)(0,6)} = \frac{2}{3}$$

(b)

	Previous infection	No previous infection	Total
Vaccinated	30%	45%	75%
Not vaccinated	15%	10%	25%
Total	45%	55%	100%

$$P(V|I) = \frac{30\%}{45\%} = \frac{2}{3}$$

1.2 (a) $P(X \geq 2) = 0,49$

(b) $X \sim B(10; 0,49)$

$$\begin{aligned}
 P(X > 8) &= P(X = 9) + P(X = 10) \\
 &= \binom{10}{9} (0,49)^9 (0,51) + \binom{10}{10} (0,49)^{10} (0,51)^0 \\
 &= 0,0091
 \end{aligned}$$

(c) $Y \sim B(n; 0,13)$

$$P(Y \geq 1) > 0,7$$

$$1 - \binom{n}{0} (0,13)^0 (0,87)^n > 0,7$$

$$(0,87)^n < 0,3$$

$$n > \frac{\log 0,3}{\log 0,87}$$

$$n > 8,64$$

$$\therefore n = 9$$

$$1.3 \quad X \sim B\left(160; \frac{1}{8}\right) \rightarrow X \sim N\left(20; \sqrt{17,5}^2\right)$$

$$\begin{aligned} P(X < 25) &\rightarrow P(X < 24,5) \\ &= P\left(Z < \frac{24,5 - 20}{\sqrt{17,5}}\right) \\ &= P(Z < 1,08) \\ &= 0,5 + 0,3599 \\ &= 0,8599 \end{aligned}$$

QUESTION 2

$$2.1 \quad X \sim N(45; 2^2)$$

$$\begin{aligned} P(X < 41 \text{ or } X > 47) &= 1 - P(41 < X < 47) \\ &= 1 - P\left(\frac{41 - 45}{2} < Z < \frac{47 - 45}{2}\right) \\ &= 1 - P(-2 < Z < 1) \\ &= 1 - (0,4772 + 0,3413) \\ &= 0,1815 \end{aligned}$$

$$2.2 \quad P(X < a) = 0,9$$

$$\begin{aligned} 1,28 &= \frac{a - 45}{2} \\ a &= 47,56 \end{aligned}$$

QUESTION 3

3.1 $H_0 : \mu = 27$

$H_1 : \mu \neq 27$

Rejection region: reject H_0 if $|Z| > 2,58$

Test statistic: $Z = 2,4$

Conclusion: since $Z < 2,58$ fail to reject H_0 at the 1% level of significance and suggest insufficient evidence to support the claim.

3.2 (a) $\frac{k}{80} = \frac{0,468 + 0,682}{2}$

$$k = 80 \left(\frac{23}{40} \right)$$

$$k = 46$$

(b) $0,575 + Z \sqrt{\frac{(0,575)(0,425)}{80}} = 0,682$

$$Z = 1,94$$

$$\therefore P(-1,94 < Z < 1,94) = 2(0,4738)$$

$$= 0,9476$$

$$\therefore \alpha = 95\%$$

(c) 59,3% is in the confidence interval hence there is evidence to say that the proportion of people who voted has not changed.

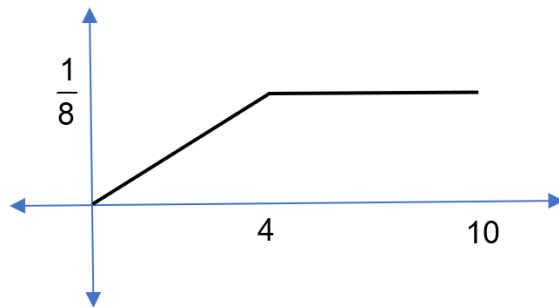
QUESTION 4

$$\begin{aligned} 4.1 \quad (a) \quad E[X] &= 1(0,3 - m) + 2(2m) + 3(0,7 - m) \\ &= 0,3 - m + 4m + 2,1 - 3m \\ &= 2,4 \end{aligned}$$

$$\begin{aligned} (b) \quad 0,3 - m &\geq 0 & 2m &\geq 0 & 0,7 - m &\geq 0 \\ m &\leq 0,3 & m &\geq 0 & m &\leq 0,7 \\ \therefore 0 &\leq m \leq 0,3 \end{aligned}$$

$$\begin{aligned} (c) \quad \text{var}(X) &= E[X^2] - (E[X])^2 \\ 0,72 &= E[X^2] - (2,4)^2 \\ E[X^2] &= 6,48 \\ 0,3 - m + 4(2m) + 9(0,7 - m) &= 6,48 \\ -2m &= -0,12 \\ m &= 0,06 \end{aligned}$$

4.2 (a)



$$(b) \quad P(X > 3) = 1 - \frac{1}{2}(3)\left(\frac{3}{32}\right) = \frac{55}{64}$$

$$(c) \quad 1 - \left(\frac{55}{64}\right)^2 = 0,2615$$

OR

$$\left(\frac{55}{64}\right)\left(\frac{9}{64}\right) \times 2 + \left(\frac{9}{64}\right)^2 = 0,2615$$

QUESTION 5

5.1 $(7-1)! \times 2 = 1440$

5.2 Let $P(X) = P(Y) = P(X' \cap Y') = x$

$$\begin{aligned}\therefore P(X \cup Y) &= P(X) + P(Y) - P(X \cap Y) \\ 1 - P(X' \cap Y') &= P(X) + P(Y) - P(X) \cdot P(Y) \\ 1 - x &= x + x - x^2 \\ x^2 - 3x + 1 &= 0 \\ x &= 0,382 \quad \text{or} \quad x = 2,618\end{aligned}$$

OR

Let $P(X|Y) = P(X) = P(Y) = P(X' \cap Y') = k$

$$P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$$

$$\therefore P(X \cup Y) = 2k - P(X \cap Y)$$

$$P(X' \cap Y') = 1 - P(X \cup Y)$$

$$\therefore k = 1 - 2k + P(X \cap Y)$$

$$P(X|Y) = \frac{P(X \cap Y)}{P(Y)}$$

$$\therefore P(X \cap Y) = k^2$$

$$\therefore k = 1 - 2k + k^2$$

$$k^2 - 3k + 1 = 0$$

$$k = 0,382$$

Total for Module 2: 100 marks

MODULE 3 FINANCE AND MODELLING

QUESTION 1

$$1.1 \quad 550\,000 \left(1 + \frac{0,08}{12}\right)^5 = \frac{6000 \left[1 - \left(1 + \frac{0,08}{12}\right)^{-n}\right]}{\frac{0,08}{12}}$$

$n = 150,34$ Accurate. But ca for $n = 6$ on LHS
 $\therefore$ 151 payments

1.2 O.B. after 150 payments =

$$OB(2\text{ years}) = 550\,000 \left(1 + \frac{0,08}{12}\right)^{155} - \frac{6000 \left[\left(1 + \frac{0,08}{12}\right)^{150} - 1\right]}{\frac{0,08}{12}}.$$

$$= R2090,18$$

$$\therefore \text{Final payment} = 2090,18 \left(1 + \frac{0,08}{12}\right)$$

$$= R2104,12$$

$$1.3 \quad OB(2\text{ years}) = 550\,000 \left(1 + \frac{0,08}{12}\right)^{24} - \frac{6000 \left[\left(1 + \frac{0,08}{12}\right)^{19} - 1\right]}{\frac{0,08}{12}}$$

$$= R523\,982,93$$

$$OB(3\text{ years}) = 550\,000 \left(1 + \frac{0,08}{12}\right)^{36} - \frac{6000 \left[\left(1 + \frac{0,08}{12}\right)^{31} - 1\right]}{\frac{0,08}{12}}$$

$$= R492\,773,70$$

1.4

$$12 \times 6000 - (523\,982,93 - 492\,773,70)$$

$$= R40\,790,77$$

QUESTION 2

2.1 Amount saved:

$$\frac{30\,000 \left[\left(1 + \frac{0,1}{12} \right)^{49} - 1 \right] \cdot \left(1 + \frac{0,1}{12} \right)^{36}}{\frac{0,1}{12}} = R2435295,52$$

2.2 $1\,750\,000(1+0,06)^7 - x(1-0,2)^7 = 2\,435\,295,52$ [spacing for millions]

2.3 (a) $\left(1 + \frac{r}{2} \right)^2 = \left(1 + \frac{0,1}{12} \right)^{12}$
 $r = 0,1021066 \dots$

(b) $2\,450\,000 = \frac{y \left[\left(1 + \frac{0,1}{12} \right)^{85} - 1 \right]}{\frac{0,1}{12}} + \frac{50\,000 \left[\left(1 + \frac{0,1021}{2} \right)^{14} - 1 \right]}{\frac{0,1021}{2}}$

$A = 2\,450\,000$

$q = 50\,000$

$m = 14$

$n = 85$

$i = 0,00833$

$j = 0,05105$

(c) $y = R11\,897,32$

QUESTION 3

3.1 6 litters per 12 months

$$\frac{12}{6} = 2 \text{ months}$$

3.2 $1/6$ or 16,67% (two decimals accepted)

3.3 $r = b - d$

$$= 0,55 \times 0,8 \times 4 - 1/6$$

$$= 1.5933$$

3.4 $P_{n+1} = 2,5933P_n$ $P_0 = 30$

$$P_0 = 2.5933P_0 - x$$

$$x = 1,5933 \times 30$$

$$= 47,799$$

$$\therefore 48 \text{ minimum}$$

QUESTION 4

4.1 $x = \frac{496,163 - 32,878}{2 \times 169,075}$

$$= 1,370$$

4.2 $\frac{729,835 - 632,054}{2y} = 0,072$

$$\therefore y = 679,035 \text{ (or answer can be given in 1000's)}$$

4.3 Show that $\frac{\Delta P}{P}$ plotted against P is a straight line with negative gradient
Credit also given for s-shaped curve mentioning inflection and asymptote

4.4 $O = -0,00339K + 2,335$

$$\therefore K = 689$$

$$r = 2,335$$

QUESTION 5

5.1 (A) OR (D)

$$5.2 \quad f.b.R_n.F_n = 10 \quad b = 0,04$$

$$0,0312 \times 0,04 \times R_0 \times 40 = 10$$

$$\therefore R_0 = 200$$

(C)

$$5.3 \quad F_{n+1} = F_n + fbR_nF_n - cF_n$$

$$F_{n+1} = F_n + 0,00296R_nF_n - 0,02F_n$$

$$Lifespan = \frac{1}{0,02}$$

$$= 50 \text{ months}$$

(E)

5.4 (D)

QUESTION 6

$$6.1 \quad P_{n+1} = P_n \left(1 + \frac{0,12}{12} \right) - m$$

$$P_{n+1} = 1,01P_n - m \quad P_0 = x$$

$$6.2 \quad P_1 = 1,01P_0 - m$$

$$1\,197\,000 = 1,01P_0 - m$$

$$P_3 = 1,01P_2 - m$$

$$P_3 = 1,01(1,01P_1 - m) - m$$

$$P_3 = 1,0201P_1 - 1,01m - m$$

$$1\,190\,909,70 = 1,01^2(1\,197\,000) - 2,01m$$

$$m = 15\,000$$

Total for Module 3: 100 marks

MODULE 4 MATRICES AND GRAPH THEORY**QUESTION 1**

$$1.1 \quad (a) \quad \left(\begin{array}{ccc|c} 2 & 3 & -5 & 7 \\ 0 & 3 & -2 & 10 \\ 0 & -2 & 3 & 20 \end{array} \right)$$

$$(b) \quad 3R_3 + 2R_2 \quad \left(\begin{array}{ccc|c} 2 & 3 & -5 & 7 \\ 0 & 3 & -2 & 10 \\ 0 & 0 & 5 & 80 \end{array} \right) \sim \left(\begin{array}{ccc|c} 2 & 3 & -5 & 7 \\ 0 & 3 & -2 & 10 \\ 0 & 0 & 1 & 16 \end{array} \right) \quad Z = 16$$

$$R_2 + 2R_3 \quad \left(\begin{array}{ccc|c} 2 & 3 & -5 & 7 \\ 0 & 3 & 0 & 42 \\ 0 & 0 & 1 & 16 \end{array} \right) \quad y = 14 \quad \quad \quad x = \frac{45}{2}$$

$$1.2 \quad (a) \quad \begin{pmatrix} 1 & 0 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x & -1 \\ -2 & 3 \end{pmatrix}$$

$$\begin{pmatrix} x+0 & -1+0 \\ 3x+4 & -3-6 \end{pmatrix} = \begin{pmatrix} x & -1 \\ 3x+4 & -9 \end{pmatrix}$$

$$(b) \quad \begin{pmatrix} 1 & -1 \\ 7 & -9 \end{pmatrix} \begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} -5 \\ -43 \end{pmatrix}$$

QUESTION 2

$$\begin{aligned} 2.1 \quad (a) \quad \det &= 3(x-6) - 8(x-5) + 2(6-5) \\ &= 3x - 18 - 8x + 40 + 2 \\ &= -5x + 24 \end{aligned}$$

$$(b) \quad \frac{1}{-5x+24} \begin{pmatrix} x-6 & -8x+12 & 6 \\ -x+5 & 3x-10 & -1 \\ 1 & 22 & -5 \end{pmatrix}$$

$$\frac{x-6}{-5x+24} = 1$$

$$\begin{aligned} x &= 5 & 22 &= 2y \\ & & y &= 11 \end{aligned}$$

$$\begin{aligned} 2.2 \quad 12 - 60i + 18i + 60i + 18i^2 + 12i \\ &= -6 + 30i \\ \therefore x &= -3 \quad y = 10 \end{aligned}$$

QUESTION 3

$$\begin{aligned} 3.1 \quad (a) \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} &= \begin{bmatrix} -1 & -3 & -1 \\ 1 & 1 & 2 \end{bmatrix} \\ a+b &= -1 \\ a+3b &= -3 \\ c+3d &= 1 \\ c+d &= 1 \\ a=0, b=-1; c=1 \text{ and } d=0 \\ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} &= \begin{bmatrix} -1 & -3 & -1 \\ 1 & 1 & 2 \end{bmatrix} \end{aligned}$$

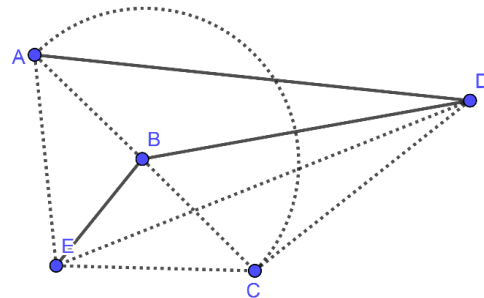
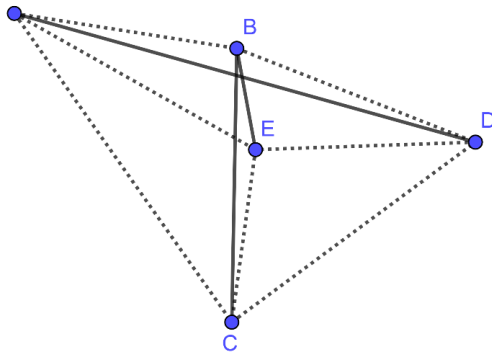
(b) Rotation of 90 anticlockwise, about the origin.

$$3.2 \quad \begin{pmatrix} \cos 30 & -\sin 30 \\ \sin 30 & \cos 30 \end{pmatrix} \begin{pmatrix} 4 & 12 & 0 \\ 8 & 16 & 8 \end{pmatrix} = \begin{pmatrix} -0.535 & 2.392 & -4 \\ 8.928 & 19.850 & 6.928 \end{pmatrix}$$

QUESTION 4

4.1 (a) Graph 2 and 3

(b)



(c) No, they aren't,

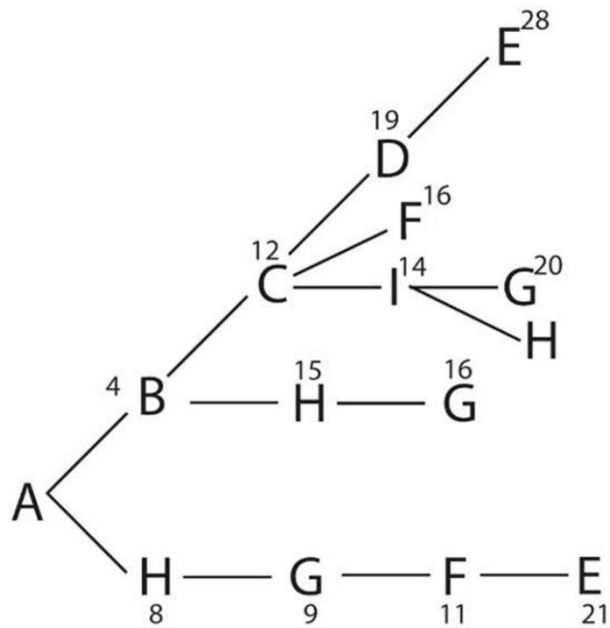
4.2 (a) A & F have odd degree edges

(b) $ABE = 26$, $ADE = 25.5$
 $25.5 < 20 + 2x < 26$
 $2.75 < x < 3$

(c) 11 edges starting at A and ending at F,
 then add returning to A
 Odd vertex A and F,
 Shortest path to repeat is FE, ED and DA
 Total weight $208 + 4x$,
 A-B-C-A-D-C-E-B-F-D -E-F -E-D-A

QUESTION 5

5.1



$$A H G F E = 21$$

5.2

- AB – 4
- AH – 8 / BC – 8
- HG – 1 / CI – 2
- GF – 2 / CF – 4
- FC – 4 / GF – 2
- CI – 2 / HG – 1
- CD – 7 / CD – 7
- DE – 9 / DE – 9

= 37

QUESTION 6

Let $n = k + 1$

$$\begin{aligned}\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}^{k+1} &= \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}^k \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1-2^k \\ 0 & 2^k \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -1+2(1-2^k) \\ 0 & 2^{k+1} \end{pmatrix} \\ &= \begin{pmatrix} +1 & -1+2-2^{k+1} \\ 0 & 2^{k+1} \end{pmatrix} \\ &= \begin{pmatrix} +1 & 1-2^{k+1} \\ 0 & 2^{k+1} \end{pmatrix}\end{aligned}$$

Therefore, true for $n = k + 1$

By the Principle of Mathematical Induction, true for all $n \in \mathbb{N}$.

Total for Module 4: 100 marks