INTERNATIONAL SECONDARY CERTIFICATE EXAMINATION
MAY 2024

FURTHER STUDIES MATHEMATICS (EXTENDED): PAPER II

MARKING GUIDELINES

Time: 1 hour 100 marks

These marking guidelines are prepared for use by examiners and sub-examiners, all of whom are required to attend a standardisation meeting to ensure that the guidelines are consistently interpreted and applied in the marking of candidates' scripts.

The IEB will not enter into any discussions or correspondence about any marking guidelines. It is acknowledged that there may be different views about some matters of emphasis or detail in the guidelines. It is also recognised that, without the benefit of attendance at a standardisation meeting, there may be different interpretations of the application of the marking guidelines.

MODULE 2 STATISTICS

QUESTION 1

1.1 (a) 0,5

(b) $X \sim N\left(12{,}5;\sigma^2\right)$

(1) $P\left(X > 14\right) = 0{,}06$

$1{,}55 = \dfrac{14 - 12{,}5}{\sigma}$ or $1{,}5548 = \dfrac{14 - 12{,}5}{\sigma}$

$\sigma = 0{,}97$ (or 0,96)

(2) $P\left(X > 12\right) = P\left(Z > \dfrac{12 - 12{,}5}{0{,}97}\right)$

$= P\left(Z > -0{,}52\right)$

$= 0{,}5 + 0{,}1985$

$= 0{,}6985$ or = 0,6969

(3) $P\left(X < 12{,}5 \middle| X > 12\right) = \dfrac{P\left(12 < X < 12{,}5\right)}{P\left(X > 12\right)}$

$= \dfrac{0{,}6985 - 0{,}5}{0{,}6985}$

$= 0{,}2842$

1.2 (a) A 94% CI for μ is

$331{,}25 \pm 1{,}88\left(\dfrac{3{,}3}{5}\right)$

$\left(330{,}01;\ 332{,}49\right)$

(b) 330 ml is not in the interval hence the brewer can conclude that the beer bottles are being overfilled. OR The interval is in excess of 330 ml, hence the brewer can be confident that he is complying with the law and ensuring the clients get at least 330 ml.

(c) The width of the confidence interval will decrease.

QUESTION 2

2.1 (a) $(0,5)(0,8) = 0,4$

(b) $X \sim B(5;0,4)$

$$P(X \leq 1) = \binom{5}{0}(0,4)^0(0,6)^5 + \binom{5}{1}(0,4)^1(0,6)^4$$
$$= 0,337$$

2.2 (a) $E[X] = 200(0,6) = 120$

$$\sigma_x = \sqrt{200(0,6)(0,4)} = 4\sqrt{3}$$

(b) $X \sim B(200;0,6)$ approximately $X \sim N\left(120;\sqrt{48}^2\right)$

$$P(111 < X < 130) \rightarrow P(111,5 < X < 129,5)$$
$$= P\left(\frac{111,5-120}{\sqrt{48}} < Z < \frac{129,5-120}{\sqrt{48}}\right)$$
$$= P(-1,23 < Z < 1,37)$$
$$= 0,3907 + 0,4147$$
$$= 0,8054$$

(c) It is a valid approximation as both $np = 200(0,6) > 5$ and $nq = 200(0,4) > 5$

QUESTION 3

3.1 (a) $3k+\frac{1}{3}=1$

$3k=\frac{2}{3}$

$k=\frac{2}{9}$

(b) $E[X]=1\left(\frac{1}{4}\right)+2\left(\frac{2}{9}\right)+3\left(\frac{4}{9}\right)=2,0278$

(c) $P(X=2\cap X=3)\times 2+P(X=3\cap X=3)$

$=\left(\frac{2}{9}\right)\left(\frac{4}{9}\right)\times 2+\left(\frac{4}{9}\right)^2$

$=\frac{32}{81}$

3.2 $(0,1;4)\ \ (0,5\ ;0)$

$m=\frac{4-0}{0,1-0,5}=-10$

$y=-10(x-0,5)$

$=-10x+5$

$\therefore \int_m^{0,5} -10x+5\ dx=0,5$

$-5x^2+5x\Big|_m^{0,5}=0,5$

$\frac{5}{4}-\left(-5m^2+5m\right)=\frac{1}{2}$

$5m^2-5m+0,75=0$

$m\neq 0,82 \quad or \quad m=0,1838$

QUESTION 4

4.1 $$-2,05 < \frac{q-50}{\frac{10}{6}} < 2,05$$

$$-3,42 < q - 50 < 3,42$$

$$46,6 < q < 53,4$$

4.2 $$-2,05 < \frac{48-m}{\frac{10}{6}} < 2,05$$ or $$-3,42 < 48 - m < 3,42$$

$$44,6 < m < 51,4$$

QUESTION 5

5.1 $$\frac{\binom{3}{2}\binom{4}{1}+\binom{3}{3}\binom{4}{0}+\binom{2}{2}\binom{5}{1}\times 2}{\binom{7}{3}} = \frac{23}{35}$$

5.2 (a) $$\frac{5\times 2\times 8!}{10!} = \frac{1}{9}$$

(b) (12) (23) (34) (45)
(67) (78) (89) (910)

Together is $4\times 2\times 2\times 8!$

Hence apart is $10! - 16(8!) = 2\,983\,680$

Total for Module 2: 100 marks

MODULE 3 FINANCE AND MODELLING

QUESTION 1

(a) $$6\,500\,000\left(1+\frac{0{,}125}{12}\right)^{8}=\frac{x\left[1-\left(1+\frac{0{,}125}{12}\right)^{-292}\right]}{\frac{0{,}125}{12}}$$

$$x=\text{R}77\,311{,}31$$

(b) $$OB=\frac{77\,311{,}31\left[1-\left(1+\frac{0{,}125}{12}\right)^{-252}\right]}{\frac{0{,}125}{12}}$$

$$=\text{R}6\,876\,896{,}48$$

(c) $$3\,876\,896{,}49=\frac{x\left[1-\left(1+\frac{0{,}125}{12}\right)^{-192}\right]}{\frac{0{,}125}{12}}$$

$$=\text{R}46\,781{,}34$$

(d) $$\Delta OB=7\,061\,831{,}755-6\,876\,896{,}48$$
$$=184\,935{,}28$$

$$Interest=40\times 77\,311{,}31\ -184\,935{,}28$$
$$=2\,907\,517{,}12$$

QUESTION 2

(a) $0{,}15 = \left(1+\frac{r}{12}\right)^{12} - 1$

$r = 0{,}1406$

$$F = \frac{15000\left[\left(1+\frac{0{,}1406}{12}\right)^{240}-1\right]}{\frac{0{,}1406}{12}} \times \left(1+\frac{0{,}1406}{12}\right) + \frac{15000\left[(1+0{,}15)^{20}-1\right]}{0{,}15} \times (1+0{,}15)$$

$= \text{R}21\ 673\ 253{,}41$

(b) (1) $$21673253 = \frac{700000\left[1-\left(1+\frac{0{,}12}{4}\right)^{-n}\right]}{\frac{0{,}12}{4}}$$

$n = 89{,}41....$

i.e. 90 *payments*

(b) (2) *OB after* 89 :

$$= 21673253\left(1+\frac{0{,}12}{4}\right)^{89} - \frac{700000\left[\left(1+\frac{0{,}12}{4}\right)^{89}-1\right]}{\frac{0{,}12}{4}}$$

$= \text{R}284\ 808{,}92$

$FP = 284808{,}92\left(1+\frac{0{,}12}{4}\right)$

$= \text{R}293\ 414{,}99$

QUESTION 3

(a) Future Annuity

(b) $40\,200{,}00 = a(20000) + b$
$60602{,}00 = a(40200) + b$

$\therefore 20200a = 20402$
$\therefore a = 1{,}01 \text{ and } b = 20\,000$

$F_4 = 81208{,}01 \times 1{,}01 + 20\,000$
$= \text{R}102\ 020{,}09$

(c) $0{,}01 \times 12 = 0{,}12$

$$\therefore r_{eff} = \left(1 + \frac{0{,}12}{12}\right)^{12} - 1$$
$$= 12{,}68\%$$

QUESTION 4

(a) $r = b - d$
$8{,}5 = 0{,}4 \times x \times 50 - 1$
$x = 0{,}475$

(b) (1) $P_{n+1} = P_n$
$\therefore P_4 = P_4(9{,}5) - x$
$\therefore x = 8{,}5 \times 3258025$
$\therefore x = 27693213$

(2) $$\frac{3258025}{9{,}5^4} = 400$$

QUESTION 5

(a) Pred = 20; Prey = 100

(b) 22 – 13 = 9

(c) $\frac{1}{0,05} = 20 \text{ cycles}$

(d) $R_E = \frac{c}{fb}$

$= \frac{0,05}{0,0008}$

≈ 63

(e) $F_E = \frac{a}{b}\left(1 - \frac{c}{fbK}\right)$

$\therefore 12 = \frac{0,15}{b}\left(1 - \frac{0,05}{0,0008 \times 30}\right)$

$\therefore b = 0,009895$

$\therefore bF_0R_0 \approx 20$

OR

$F_E = \frac{a}{b}\left(1 - \frac{R_E}{K}\right)$

$\therefore 12 = \frac{0,15}{b}\left(1 - \frac{63}{300}\right)$

$\therefore b = 0,009875$

$\therefore bF_0R_0 \approx 20$

QUESTION 6

(a) K = 180

$$50 = \frac{180}{1+P}$$

$$\therefore 1+P = \frac{180}{50}$$

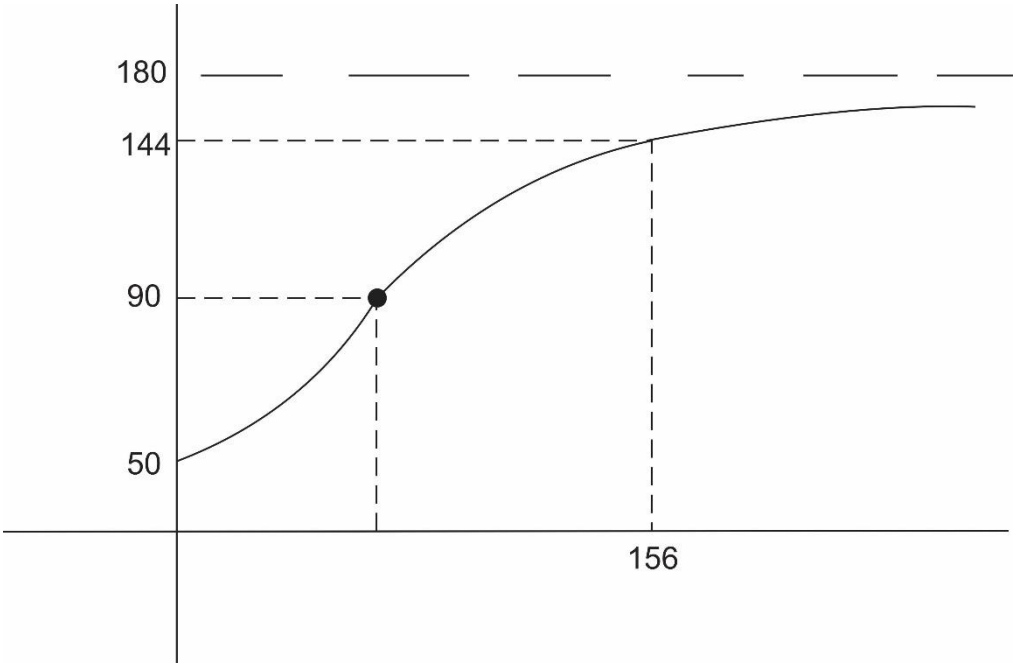
$$\therefore P = 2,6$$

$$\therefore 144 = \frac{180}{1+2,6e^{-156Q}}$$

$$\therefore e^{-156Q} = 0,09615...$$

$$Q = 0,015$$

(b)



(c) $$90 = \frac{180}{1+2,6e^{-0,015t}}$$

$$\therefore t \approx 64 \text{ months}$$

Total for Module 3: 100 marks

MODULE 4 GRAPH THEORY AND MATRICES

QUESTION 1

1.1 (a) C a square matrix

(b) B $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

(c) C Adjacency Matrix

1.2 (a) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 2 & 6 & 4 \\ 2 & 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 2 & 6 & 4 \\ -2 & -1 & -2 & -3 \end{pmatrix}$

matrix
order

(b) signs
matrix
order

$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 2 & 6 & 4 \\ 2 & 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} -2 & -1 & -2 & -3 \\ 1 & -2 & -6 & -4 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 2 & 6 & 4 \\ 2 & 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} -3 & 6 & 18 & 12 \\ 2 & 1 & 2 & 3 \end{pmatrix}$

order
matrix

1.3 order matrix

$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 0 & 3 & 7 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0{,}574 & 2{,}457 & 6{,}308 \\ 0{,}819 & -1{,}721 & -3{,}196 \end{pmatrix}$

1. $-\sin\theta = 0{,}574$ any 2 equations
2. $3\cos\theta = 2{,}457$
3. $7\cos\theta - \sin\theta = 6{,}308$
4. $\cos\theta = 0{,}819$
5. $3\sin\theta = -1{,}721$
6. $7\sin\theta + \cos\theta = -3{,}196$

Solve for θ
Identify $\theta = 35; -35$
Choosing $\theta = -35$ or $\theta = 325$

1.4 $A = \begin{pmatrix} 3 & k & 8 \\ -2 & 2 & 8 \\ 1 & 1 & 1 \end{pmatrix}$

sign

$\det A = 1(8k - 16) - 1(24 + 16) + 1(6 + 2k) = 0$

$0 = -50 + 10k \quad \therefore k = 5$

QUESTION 2

2.1 $\text{adj A} = \begin{bmatrix} -1 & -2 & -1 \\ -38 & -4 & 22 \\ -31 & -2 & 17 \end{bmatrix}$

2.2 $\det A = \begin{vmatrix} 2 & -3 & 4 \\ 3 & 4 & -5 \\ 4 & -5 & 6 \end{vmatrix}$

$\det A = 2(24 - 25) + 3(18 + 20) + 4(-15 - 16)$
$= 2(-1) + 3(38) + 4(-31)$
$= -12$

2.3 $A^{-1} = -\frac{1}{12}\begin{pmatrix} -1 & -2 & -1 \\ -38 & -4 & 22 \\ -31 & -2 & 17 \end{pmatrix}$

2.4 $A^{-1}A = -\frac{1}{12}\begin{pmatrix} -1 & -2 & -1 \\ -38 & -4 & 22 \\ -31 & -2 & 17 \end{pmatrix}\begin{pmatrix} 2 & -3 & 4 \\ 3 & 4 & -5 \\ 4 & -5 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{12}\begin{pmatrix} -1 & -2 & -1 \\ -38 & -4 & 22 \\ -31 & -2 & 17 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\therefore x = 1$
$y = 0$
$z = 0$

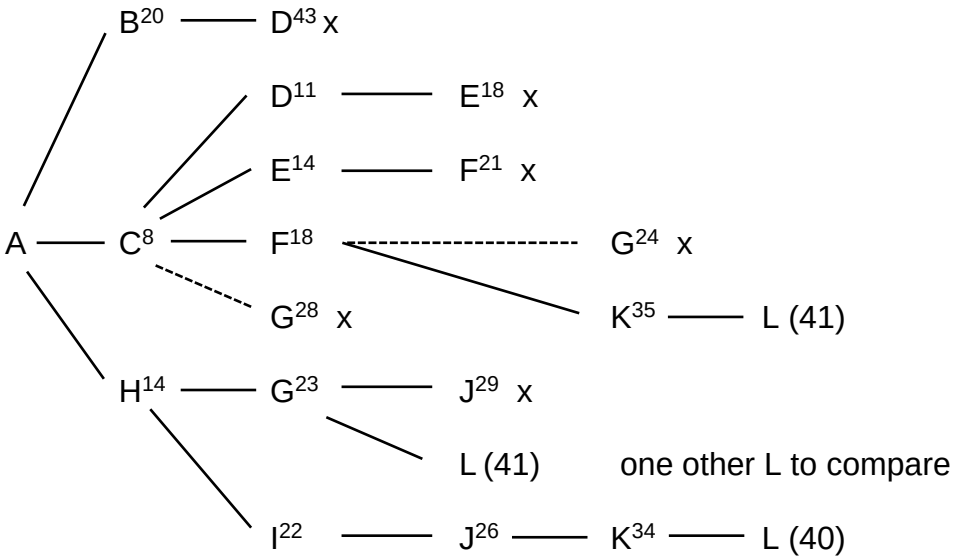
QUESTION 3

3.1 B All graphs must have nodes

3.2 A Their compliments are isomorphic

3.3 C has the same vertices as P, and only has all the edges not contained in P

QUESTION 4



A – H – I – J – K – L (40)

QUESTION 5

12 30 40 53 72 85 99 120 137 154

5.1 A – E – I – H – G – C – D – B – A – F – A

5.2 [This should be a list of edges not vertices. AH, HI, etc.]

HI	10		
AH	12		
AE	12		
GH	13		
CD	13		
BD	14		
IF	17	(or AF)	only one of IF and AF
CG	19		

Weight: 110

5.3 Repeat odd degree vertices: I; E; H; G; C; D

CD	13
EI	18
GH	13

Weight of all edges: 215

Add back, total: 259

QUESTION 6

6.1 $B^TB = I$

$(B^TB)B^{-1} = I(B^{-1})$ for B^{-1} for post multiply

$B^T(BB^{-1}) = B^{-1}$

$B^TI = B^{-1}$

$\therefore B^T = B^{-1}$

6.2 $A^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$A^TA = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$A^TA = \begin{pmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \cos^2\theta + \sin^2\theta \end{pmatrix}$

$A^TA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

Total for Module 4: 100 marks