SEPTEMBER 2015 ROEDEAN

AP MATHS MATRIC

CALCULUS AND ALGEBRA

QUESTION 1

$$\sum_{i=1}^{n} i^2$$

RTP: $(1)^2+(2)^2+(3)^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$ ✓

For $n=1$:

LHS $=(1)^2=1$; RHS $= \frac{(1)(1+1)(2(1)+1)}{6}$ ✓

$\therefore$ RHS $= \frac{6}{6} = 1$ ✓

$\therefore$ True for $n=1$.

Assume true for $n=k$; $(k \in \mathbb{N})$

i.e. Assume:

$1^2+2^2+3^2+\dots+(k-1)^2+k^2 = \frac{k(k+1)(2k+1)}{6}$ ✓ ... ①

RTP: True for $n=k+1$

i.e. RTP: $1^2+2^2+3^2+\dots k^2+(k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$ ✓ ... ②

Sub ① into LHS of ②

LHS $= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$ ✓

$= \frac{k(k+1)(2k+1)+6(k+1)^2}{6}$ ✓

$= \frac{(k+1)(k(2k+1)+6(k+1))}{6}$ ✓

$= \frac{(k+1)(2k^2+7k+6)}{6}$ ✓

$= \frac{(k+1)(k+2)(2k+3)}{6}$ ✓

$=$ RHS

$\therefore$ True for $n=k+1$ if true for $n=k$ and true for $n=1$ ✓

[12]

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(2)

QUESTION 2

2.1 $\ln a = 3$

$\ln x^3 + 6\log_a x$

$= \ln x^3 + \log_a x^6$ ✓

$= \ln x^3 + \dfrac{\ln x^6}{\ln a}$ ✓

$= \ln x^3 + \dfrac{\ln x^6}{3}$ ✓

$= \ln x^3 + \frac{1}{3}\ln x^6$

$= \ln x^3 + \ln x^2$ ✓

$= \ln x^5$ ✓ (5)

$= \underline{5\ln x}$

2.2 $2e^x - 1 = e^{-x}$

$\therefore 2e^x - 1 = \dfrac{1}{e^x}$ ✓

Let $e^x = k$

$\therefore 2k - 1 = \dfrac{1}{k}$

$\therefore 2k^2 - k - 1 = 0$ ✓

$\therefore (2k+1)(k-1) = 0$

$\therefore k = -\frac{1}{2}$ or $k = 1$

$\therefore e^x = -\frac{1}{2}$ or $e^x = 1$ ✓

no soln ✓ $\therefore x = 0$ ✓ (5)

OR

$2e^{2x} - e^x - 1 = 0$ ✓

$\therefore (2e^x+1)(e^x-1) = 0$

$\therefore e^x = -\frac{1}{2}$ or $e^x = 1$ ✓

no solution ✓ $\therefore$ ~~lnd~~ or $x = 0$ ✓

2.3 $M = 25e^{-0{,}0012t}$

(a) $M = 25e^0$

$\therefore \underline{M = 25 \text{ grams}}$ (1)

(b) 1 hour $= 1 \times 60 \times 60 = 3600$ seconds ✓

$M = 25e^{-0{,}0012(3600)}$ ✓

$\therefore \underline{M = 0{,}33 \text{ grams}}$ ✓ (3)

(c) Initial mass $= 25$ g

Half-life mass $= 12{,}5$ g

$M = 25e^{-0{,}0012t}$

$\therefore 12{,}5 = 25e^{-0{,}0012t}$ ✓

$\therefore \dfrac{1}{2} = e^{-0{,}0012t}$

$\therefore \ln(\tfrac{1}{2}) = -0{,}0012t$ ✓

$\therefore t = \dfrac{\ln(\frac{1}{2})}{-0{,}0012}$ ✓

$\therefore \underline{t = 577{,}62 \text{ seconds}}$ ✓ (4)

(d) ~~$M = 25e^{-0{,}0012t}$ (t=0) ... calc: $M = 24{,}282$ (any $\neq 24{,}282$)~~

✓ $M = 0$ ✓ (2)

2.4 See Answer Sheet. (5)

[25]

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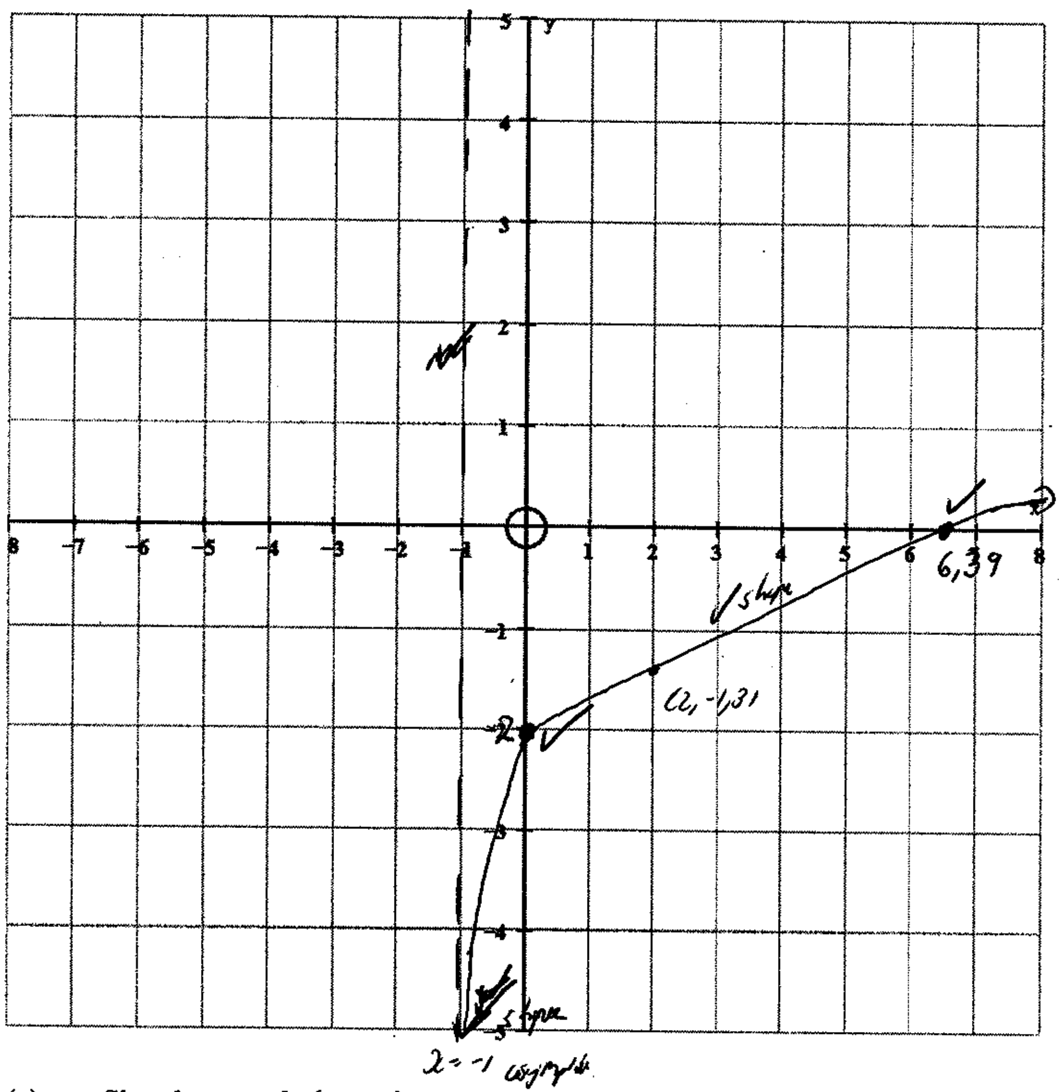
ANSWER SHEET

NAME: MEMO

pg 3

QUESTION 2.4



(a) Sketch $y = \ln(x+1) - 2$ on the axes above.
Clearly label all intercepts with the axes as well as all asymptote(s). (5)

$x=0$
$\Rightarrow y = \ln(1) - 2$
$\therefore y = -2$

$y=0$
$\therefore 0 = \ln(x+1) - 2$
$\therefore 2 = \ln(x+1)$
$\therefore e^2 = x+1$
$\therefore x = e^2 - 1$
$\therefore x = 6,39$

$x=1$
$y = \ln(2) - 2$
$\approx -1,31$

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QUESTION 3

3.1 $3|x| + 2x^2 - 2 = 0$

If $x \geq 0$ then $3x + 2x^2 - 2 = 0$
$\therefore 2x^2 + 3x - 2 = 0$
$\therefore (2x - 1)(x + 2) = 0$
$\therefore x = \frac{1}{2}$ or $x \neq -2$ N/A

If $x < 0$: $3(-x) + 2x^2 - 2 = 0$
$\therefore 2x^2 - 3x - 2 = 0$
$\therefore (2x + 1)(x - 2) = 0$
$x = -\frac{1}{2}$ or $x \neq 2$

(7)

3.2. $2 - \frac{|x+1|}{2} \geq 0$

$\therefore \frac{-|x+1|}{2} \geq -2$

$\therefore \frac{|x+1|}{2} \leq 2$

$\therefore |x+1| \leq 4$

$\therefore -4 \leq x+1 \leq 4$

$\therefore -5 \leq x \leq 3$ ... ①

(3)

and

$\frac{2x+1}{x-1} \geq 1$

$\therefore \frac{2x+1}{x-1} - 1 \geq 0$

$\therefore \frac{2x+1 - 1(x-1)}{x-1} \geq 0$

$\therefore \frac{x+2}{x-1} \geq 0$

+ 0 − u +
−2 1

$\therefore x \leq -2$ or $x > 1$ ②

$[x \in (-\infty; -2] \cup (1; \infty)$ (6)

Combining ① and ②:

−5 −2 1 3

$\therefore x \in [-5; -2] \cup (1; 3]$ (4)

(13)

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(5)

QUESTION 4

4.1. $x = 5 - 2i$

$\therefore x = 5 + 2i$ is a solution ✓

$\therefore x - 5 + 2i$ is a factor

$\therefore x - 5 - 2i$ is a factor

$\therefore (x - 5 + 2i)(x - 5 - 2i)$ is a factor ✓

$\therefore (x-5)^2 - 4i^2$ is a factor

$\therefore x^2 - 10x + 25 + 4$ is a factor

$\therefore \underline{x^2 - 10x + 29}$ ✓ is a factor.

$x = 2 + \sqrt{3}$

$\therefore x = 2 - \sqrt{3}$ is a solution ✓

$\therefore x - 2 - \sqrt{3}$ is a factor

and $x - 2 + \sqrt{3}$ is a factor

$\therefore (x - 2 - \sqrt{3})(x - 2 + \sqrt{3})$ ✓ is a factor

$\therefore (x-2)^2 - 3$ is a factor

$\therefore x^2 - 4x + 4 - 3$ " "

$\therefore x^2 - 4x + 1$ ✓ is a factor.

$\therefore (x^2 - 4x + 1)(x^2 - 10x + 29)$ ✓ is factor

but constant = 58

$\therefore \frac{58}{29} = 2 \Rightarrow$ co-efficient

$\therefore 2(x^2 - 4x + 1)(x^2 - 10x + 29) = 0$ ✓

$\therefore 2(x^4 - 10x^3 + 29x^2 - 4x^3 + 40x^2$

$- 116x + x^2 - 10x + 29) = 0$

$\therefore 2(x^4 - 14x^3 + 70x^2 - 126x + 29) = 0$

$\therefore 2x^4 - 28x^3 + 140x^2 - 256x + 58 = 0$ ✓

(10)

4.2. $i^2 = -1$

$\frac{3+2i}{5-i} + i^{13}$

$\left.\begin{array}{l} i = i \\ i^2 = -1 \\ i^3 = -i \\ i^4 = 1 \end{array}\right\}$ ✓m

$i^{(13)} = (i^4)^3 \times i$

$= \underline{i}$; ✓a

$\therefore \frac{3+2i}{5-i} + i$

$= \frac{3 + 2i + 5i - i^2}{5 - i}$ ✓a ✓m

$= \frac{(4 + 7i)}{(5 - i)} \frac{(5 + i)}{(5 + i)}$ ✓a ✓m

$= \frac{20 + 39i - 7}{26}$ ✓ ✓

$= \frac{13}{26} + \frac{39i}{26}$

$= \underline{\frac{1}{2} + \frac{3}{2}i}$ ✓ ✓

(9)

[19]

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6

QUESTION 5

$$f(x) = \begin{cases} 3\cos x & \text{for } x \le 0 \\ 3 - 4x + x^2 & \text{for } x > 0 \end{cases}$$

5.1. $\lim_{x \to 0^-} f(x)$

$= \lim_{x \to 0} 3\cos x$ ✓

$= 3\cos 0$

$= 3$ ✓

$\lim_{x \to 0^+} f(x)$

$= \lim_{x \to 0} (3 - 4x + x^2)$ ✓

$= 3 - 4(0) + (0)^2$

$= 3$ ✓

$\therefore \lim_{x \to 0^-} f(x) = 3 = \lim_{x \to 0^+} f(x)$

$\therefore \lim_{x \to 0} f(x) = 3$ ✓

and $f(0)$

$= 3\cos 0$

$= 3$ ✓

$\therefore \lim_{x \to 0} f(x) = f(0)$ ✓

$\therefore f$ is continuous at $x = 0$ ✓ (8)

5.2. $\lim_{x \to 0^-} f'(x)$

$= \lim_{x \to 0} [D_x (3\cos x)]$ ✓

$= \lim_{x \to 0} -3\sin x$ ✓

$= -3\sin 0$

$= 0$ ✓

$\lim_{x \to 0^+} f'(x)$

$= \lim_{x \to 0} [D_x (3 - 4x + x^2)]$ ✓

$= \lim_{x \to 0} (-4 + 2x)$ ✓

$= 2(0) - 4$

$= 0 - 4$ ✓

$\therefore \lim_{x \to 0^-} f'(x) \neq \lim_{x \to 0^+} f'(x)$ ✓

$\therefore$ f is not differentiable at $x = 0$ ✓ (8)

[16]

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QUESTION 6

$$f(x) = \frac{x^2-1}{x-3}$$

6.1. Deg num - deg denom $= 2-1 = 1$ (linear)
$\therefore$ oblique asymptote

Eqn oblique $f(x) = \frac{x^2-1}{x-3}$ ✓

$$= \frac{x(x-3)+3x-1}{x-3}$$

$$= \frac{x(x-3)+3(x-3)+8}{x-3}$$

$$f(x) = x+3 + \frac{8}{x-3}$$ ✓ ✓

$\therefore$ oblique asymptote: $y = x+3$ ✓ ✓

Vertical Asymptote: $x = 3$ ✓ ✓ (6)

6.3 x-int: $0 = \frac{x^2-1}{x-3}$ ✓

$\therefore (x-1)(x+1) = 0$ ✓

$\therefore x = 1$ or $x = -1$ ✓
$(1;0)$ or $(-1;0)$

y-int: $\frac{0-1}{0-3} = \frac{1}{3}$ ✓ $(0;\frac{1}{3})$ (4)

6.2. $f(x) = \frac{x^2-1}{x-3}$

Turning points when gradient $= 0$ ✓

$\therefore f'(x) = 0$

$\therefore \frac{(2x)(x-3) - (1)(x^2-1)}{(x-3)^2} = 0$ ✓ ✓

$\therefore 2x^2 - 6x - x^2 + 1 = 0$

$\therefore x^2 - 6x + 1 = 0$ ✓

$\therefore x = 5,83$ ✓ or $x = 0,17$ ✓ (6)

$y = \frac{(5,83)^2-1}{5,83-3} = 6,83$ | $y = \frac{(0,17)^2-1}{0,17-3} = 0,34$

$(5,83; 6,83)$ ✓ $(0,17; 0,34)$ ✓ (2)

(x^2-6x+1) $f'(x)$	$0,17$	(x^2-6x+1) $f'(x)$	3	$5,83$	$f'(x)$
$+$		$-$			$+$

✓m local max ✓m local min

(Sign of $f'(x)$ = ✓)

$\therefore (0,17; 0,34) \rightarrow$ local max ✓ $(5,83; 6,83) \rightarrow$ local min ✓ (4)

OR

$f(x) = x+3+\frac{8}{x-3}$

$\therefore f(x) = x+3+8(x-3)^{-1}$

$\therefore f'(x) = 1 - 8(x-3)^{-2}$ ✓

$\therefore f'(x) = 1 - \frac{8}{(x-3)^2}$ ✓ (3)

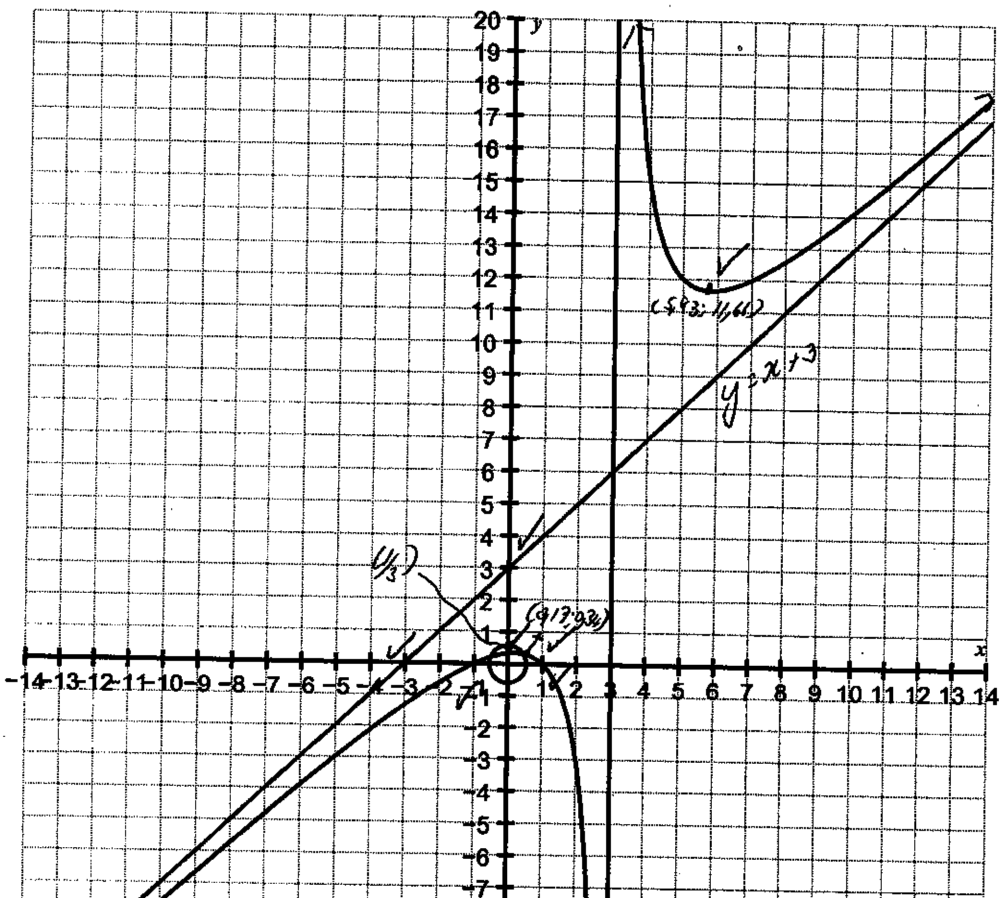
$\therefore f'(x) = (x-3)^2 - 8$

$\therefore f'(x) = \frac{x^2-6x+1}{(x-3)^2}$ (12)

$\therefore 0 = x^2 - 6x + 1$ ✓ $\therefore x = 5,83$ ✓ or $x = 0,17$ ✓

⑧

QUESTION 6.4



Equation 1: $y=(x^2-1)/(x-3)$
Equation 2: $x=3$
Equation 3: $y=x+3$

$x = 3$ ✓

← (8)

QUESTION 7

7.1 $\lim_{x\to 0} \dfrac{\sin 6x}{\sin 2x}$

$= \lim_{x\to 0} \left(\dfrac{6x}{6x}\right)\left(\dfrac{\sin 6x}{\sin 2x}\right)$ ✓m

$= \lim_{x\to 0} \left(\dfrac{\sin 6x}{6x}\right)\left(\dfrac{6x}{\sin 2x}\right)$ ✓m

$= 1 \lim_{x\to 0} \left(3\right)\left(\dfrac{2x}{\sin 2x}\right)$ ✓

$= 3 \lim_{x\to 0} \left(\dfrac{2x}{\sin 2x}\right)$ ✓

$= 3(1) = 3$ ✓ (5)

7.2 $\lim_{x\to\infty} \left(\dfrac{1}{x} - \dfrac{1}{x+3}\right)$

$= \lim_{x\to\infty} \left(\dfrac{x+3-x}{x(x+3)}\right)$ ✓

$= \lim_{x\to\infty} \left(\dfrac{3}{x^2+3x}\right)$ ✓

$= \lim_{x\to\infty} \dfrac{x^2(3/x^2)}{x^2(1+\frac{3}{x})}$ ✓ ✓

$= \dfrac{0}{1} = 0$ ✓ (5)

[10]

QUESTION 8

8.1 $f(x) = 1 - x + \cos^2 x \;;\; x \in [0;4]$

$f(1) = 1 - 1 + \cos^2 1$

$= 0{,}29$ ✓a

> 0

$f(2) = 1 - 2 + (\cos 2)^2$

$= -0{,}83$ ✓a

< 0

$f(1) > 0$ and $f(2) < 0$ ✓a

and f is continuous ∴ x-int between 0 and 2

(4)

8.2 $x_{r+1} = x_r - \dfrac{f(x_r)}{f'(x_r)}$

Let $x_1 = 1{,}2$ (between 1 and 1,5 from graph) ✓

$f(x) = 1 - x + \cos^2 x$

$\therefore f'(x) = -1 + 2\cos x(-\sin x)$

$\therefore f'(x) = -1 - 2\cos x \sin x$ ✓

$\therefore x_{r+1} = x_r - \left[\dfrac{(1 - x + \cos^2 x)}{(-1 - 2\cos x \sin x)}\right]$ ✓✓

$\therefore x_2 = (1{,}2) - \dfrac{(1 - (1{,}2) + (\cos(1{,}2))^2)}{(-1 - 2\cos(1{,}2)\sin(1{,}2))}$ ✓

$\therefore x_2 = 1{,}158998285$ ✓ $x_4 = 1{,}15969$

$x_3 = 1{,}15969\ldots$

$x = 1{,}1597$ ✓ ✓ (8)

[12]

QUESTION 9

9.1 $x^3+y^3=9xy$

$D_x(x^3+y^3)=D_x(9xy)$ ✓ ✓

$\therefore 3x^2+3y^2\left(\frac{dy}{dx}\right)=(9)(y)+(9x)\left(\frac{dy}{dx}\right)$

$\therefore 3y^2\left(\frac{dy}{dx}\right)-9x\left(\frac{dy}{dx}\right)=9y-3x^2$

$\therefore \frac{dy}{dx}\left(3y^2-9x\right)=9y-3x^2$ ✓

$\therefore \frac{dy}{dx}=\frac{9y-3x^2}{3y^2-9x}$ ✓

$\therefore \frac{dy}{dx}=\frac{3y-x^2}{y^2-3x}$ ✓ (10)

9.2 $\frac{dy}{dx}=\frac{3y-x^2}{y^2-3x}$

At $(2;4)$

$m=\frac{dy}{dx}=\frac{3(4)-(2)^2}{(4)^2-3(2)}$ ✓

$\therefore m=\frac{-8}{10}$

$\therefore m=\frac{4}{5}$ ✓

$\therefore y=\frac{4}{5}x+c$

Sub in $(2;4)$

$\therefore 4=\frac{4}{5}(2)+c$ ✓

$\therefore 4-\frac{8}{5}=c$

$\therefore \frac{12}{5}=c$ ✓

$\therefore y=\frac{4x}{5}+\frac{12}{5}$ ✓

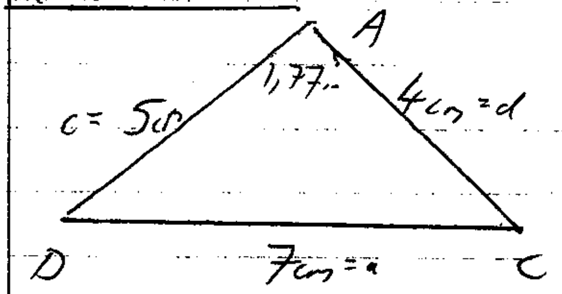
$\therefore 5y=4x+12$

$\therefore -4x+5y=-12$ ✓ (6)

[16] (or $4x-5y=12$)

QUESTION 10

(1)



In $\triangle ACD$

$a^2 = c^2 + d^2 - 2cd\cos A$

$\therefore \cos A = \dfrac{c^2 + d^2 - a^2}{2cd}$

$\therefore \cos A = \dfrac{5^2 + 4^2 - 7^2}{2(5)(4)} = -\dfrac{1}{5}$ ✓m

$\therefore \hat{A} = 1,77215$ ✓r

$\therefore \dfrac{\sin C}{c} = \dfrac{\sin A}{a}$ ✓m

$\therefore \sin C = \dfrac{c\sin A}{a}$

$\therefore \sin C = \dfrac{5 \times \sin 1,77\ldots}{7}$

$\therefore A\hat{C}D = 0,775\ldots$ ✓a

but $\triangle ACD \equiv \triangle BCD$ (SSS) ✓m

$\therefore A\hat{C}D = D\hat{C}B = 0,775\ldots$ ($\triangle ACD \equiv \triangle BCD$)

$\therefore A\hat{C}B = 2(A\hat{C}D) = 2(0,775\ldots)$

$\therefore A\hat{C}B = 1,55$ ✓a

$\dfrac{\sin D}{d} = \dfrac{\sin A}{a}$

$\therefore \dfrac{\sin D}{1} = \dfrac{d\sin A}{a}$

$\therefore \sin D = \dfrac{4\sin 1,77\ldots}{7}$ ✓m

$\therefore A\hat{D}C = 0,594\ldots$ ✓a

but $A\hat{D}C = B\hat{D}C$ ($\triangle ACD \equiv \triangle BCD$)

$\therefore ADB = 2 \times 0,59$ ✓m

$\therefore ADB = 1,19$ ✓a (10)

(2) Shaded Area

= Area sector ABD + Area Sector ABC ✓ ✓

$-$ (2 × Area $\triangle ACD$) ✓

$= \left(\frac{1}{2}\right)(4)^2(1,55)$ ✓

$+ \left(\frac{1}{2}\right)(5)^2(1,19)$ ✓

$-2\left(\frac{1}{2}\right)(4)(7)\sin 0,775$ ✓

$= \underline{7,68\text{ cm}^2}$ ✓✓ (8)

[18]

(11)

11.1 Points of intersection:

$f(x) = g(x)$ ✓m

$\therefore 2x^2+4x-7 = -x^2-2x+2$ ✓a

$\therefore 3x^2+6x-9=0$

$\therefore x^2+2x-3=0$ ✓a

$\therefore (x+3)(x-1)=0$

$\therefore x=-3$ or $x=1$ ✓a

$\therefore \text{Area} = \int_{-3}^{1} \left((-x^2-2x+2) - (2x^2+4x-7)\right)dx$ ✓a

$= \int_{-3}^{1} (-3x^2-6x+9)\,dx$ ✓a

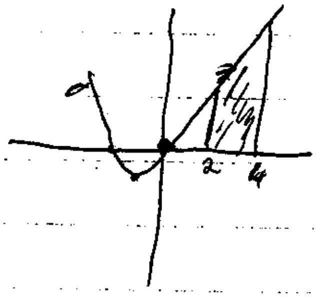
$= 32$ ✓a (8)

11.2 $y = x^2+x$

$y = x(x+1)$

x-ints $(0;0)$ and $(-1;0)$

TP: $x_{TP} = -\frac{1}{2}$ $y_{TP} = -\frac{1}{4}$



$V = \int_a^b y^2\,dx$ ✓✓m

$\therefore V = \int_2^4 (x^2+x)^2\,dx$ ✓a

$\therefore V = 337{,}07$ units^2 (5)

[13]

(-1 no constants (c) for all Q12)

12.1 $\int \frac{1}{(3x-2)^2}\,dx$

$= \int (3x-2)^{-2}\,dx$ ✓

$= \frac{(3x-2)^{-1}}{(-1)(3)} + c$ ✓ ✓ ✓

$= \frac{-1}{3(3x-2)} + c$ ✓ (5)

$= \frac{-1}{9x-6} + c$

12.2 $\int \sqrt{x}\sqrt{1+x^{3/2}}\,dx \ldots$ (*)

let $1+x^{3/2} = u$ ✓m

$\therefore \frac{du}{dx} = \frac{3}{2}x^{1/2}$ ✓

$\therefore \frac{2\,du}{3} = x^{1/2}dx = \sqrt{x}\,dx$ ✓

$\therefore$ (*) $= \int \left(\frac{2}{8}\,du\right)\sqrt{u}$ ✓

$= \frac{2}{3}\int u^{1/2}\,du$

$= \left(\frac{2}{3}\right)\frac{u^{3/2}}{(3/2)}$ ✓ $+ c = \frac{4}{9}(1+x^{3/2})^{3/2} + c$ ✓ (6)

12.3 $\int (2x+1)\sin 3x\,dx$

let $f(x) = 2x+1$ | let $g'(x) = \sin 3x$

$\therefore f'(x) = 2$ ✓ | $\therefore g(x) = \frac{-\cos 3x}{3}$ ✓

✓✓m

$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$

$= (2x+1)\left(\frac{-\cos 3x}{3}\right) - \int\left(\frac{-\cos 3x}{3}\right)2\,dx$ ✓ ✓

$= \frac{-2x\cos 3x}{3} - \frac{\cos 3x}{3} + \frac{2}{3}\left(\frac{\sin 3x}{3}\right) + c$ ✓✓

$= \frac{-2x\cos 3x}{3} - \frac{\cos 3x}{3} + \frac{2\sin 3x}{9} + c$ ✓ (8)

12.4 $\int (\tan x + \cot x)^2\,dx$

$= \int (\tan^2 x + 2\tan x\cot x + \cot^2 x)\,dx$ ✓

$= \int (\tan^2 x + 2 + \cot^2 x)\,dx$

$= \int [(\tan^2 x + 1) + (1 + \cot^2 x)]\,dx$ ✓

$= \int (\sec^2 x + \operatorname{cosec}^2 x)\,dx$ ✓ ✓

$= \tan x - \cot x + c$ ✓ ✓ (7)

[26]

QUESTION 13

13.1 $PA = r\theta$ ✓m

$\therefore PA = 2\theta$ ✓a (2)

13.2 (a) $V(\theta) = \dfrac{\theta}{\theta^2+1}$

Max when $V'(\theta) = 0$

$\therefore \dfrac{(1)(\theta^2+1) - (2\theta)(\theta)}{(\theta^2+1)^2} = 0$

$\therefore \theta^2+1-2\theta^2 = 0$ ✓

$\therefore 1 = \theta^2$

$\therefore \theta = \pm 1$ ✓

but from diagram $\theta = 1$ ✓ (9)

(b) $S = PA = 2\theta$ ✓

$\therefore PA = 2(1)$

$\therefore PA = 2$ ✓ (2)

[13]