JULY EXAM – AP. MEMO

2015:

SECTION A

QUESTION 1

$2! + (5)2! + (10)3! + \dots + (n^2+1)n! = n(n+1)!$

Prove true for $n=1$ ✓

LHS $= 2!$ RHS $= 1(2)!$ ✓

$\therefore$ LHS $=$ RHS ✓

Assume true for $n=k$ ✓

$2! + 5(2)! + \dots + (k^2+1)k! = k(k+1)!$ ✓

Prove true for $n=k+1$ ✓

$2! + 5(2)! + \dots + (k^2+1)k! + ((k+1)^2+1)(k+1)!$ ✓

$= k(k+1)! + (k^2+2k+2)(k+1)!$ ✓

$= (k+1)!(k + k^2+2k+2)$ ✓

$= (k+1)!(k+1)(k+2)$ ✓

$(k+2)! = (k+2).(k+1)!$ ✓

$(k+2)!(k+1)$ 13

By PMI it is true for $n \in \mathbb{N}$ ✓

QUESTION 2

$f(x) = x^4 - 6x^2 - 12x - 8$

$x = 1 - \sqrt{5}$

$x = 1 + \sqrt{5}$ ✓✓

$x^2 - 2x - 4 = 0$

$(x^2-2x-4)(x^2+ax+2) = f(x)$

$ax^3 - 2x^3 = 0$

$a = 2$ ✓ ✓✓ 7

$\therefore x^2+2x+2=0$

$x = -1 \pm i$ ✓✓

b) $f(x) = 1 - x^2$ $g(x) = \sin 4x$

$f(g(x)) = 1 - \sin^2 4x$ ✓✓ 4

$= \cos^2 4x$ ✓✓

c) $\left|\dfrac{2x+3}{x-1}\right| < 1$ 5

$x \neq 1$

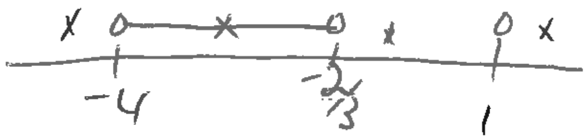
$2x+3 = x-1$

$x = -4$ ✓

$2x+3 = -x+1$

$3x = -2$

$x = -2/3$ ✓✓



$-4 < x < -2/3$ ✓✓

2) $\log_3(2-3x) = \log_9(6x^2-19x+2)$

$\log_3(2-3x) = \dfrac{\log(6x^2-19x+2)}{2\log 3}$

$\log_3(2-3x) = \log_3(6x^2-19x+2)^{1/2}$

$(2-3x) = (6x^2-19x+2)^{1/2}$ (7)

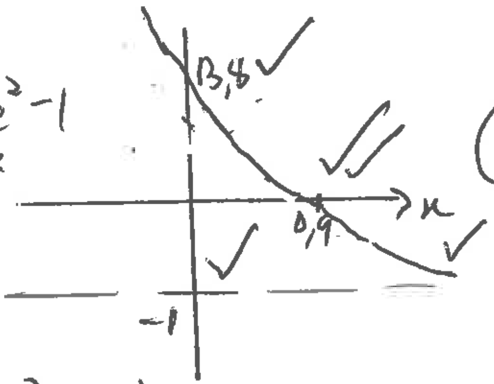
$4-12x+9x^2 = 6x^2-19x+2$

$3x^2+7x+2=0$

$x = -\frac{1}{3}$ $x=-2$

d. $f(x) = 2.e^{2-3x}-1$

$f(0) = 2.e^2-1$
$= 13,8$



(5)

$\ln(\frac{1}{2})-2 = -3x$

$x = 0,4$

2) $y = 2e^{2-3x}-1$

$x+1 = 2e^{2-3y}$

$\frac{x+1}{2} = e^{2-3y}$ (4)

$\ln\left|\frac{x+1}{2}\right| = 2-3y$

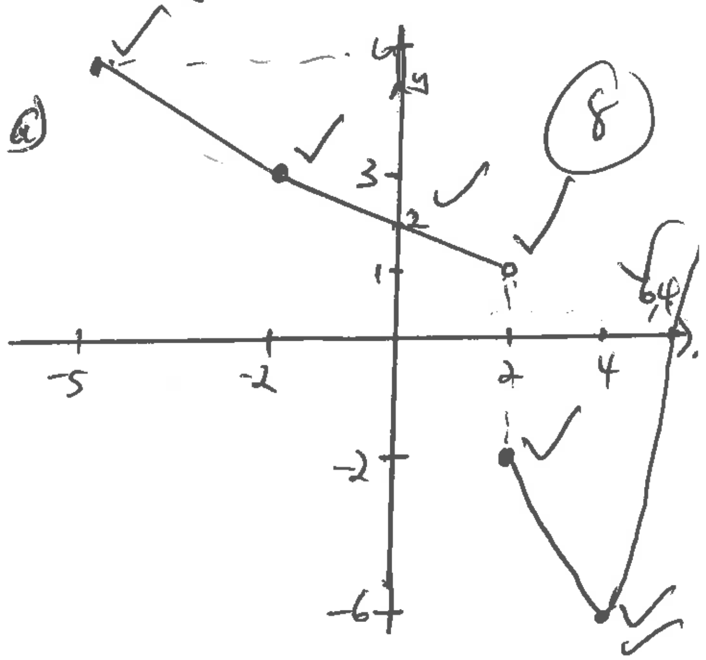
$3y = 2 - \ln\left|\frac{x+1}{2}\right|$

$f^{-1}(x) = y = \frac{2}{3} - \frac{1}{3}\ln\left|\frac{x+1}{2}\right|$

QUESTION 3

$g(x) = \begin{cases} |x+1|+2 & -5 \le x < -2 \\ 2-\frac{x}{2} & \text{for } -2 \le x < 2 \\ x^2-8x+10 & \text{for } x \ge 2 \end{cases}$

a)



(8)

b) Continuous and differentiable

$x > 2.$

$-5 < x < 2$ (4)

c) Discontinuous at $x = 2$. (jump discontinuity)

$\lim_{x\to 2^-}(2-x/2) = 1$ (4)

$\lim_{x\to 2^+}(x^2-8x+10) = -2$

d). $g'(x) \le 0$

$-5 \le x \le 4$ $x \ne 2$ (4)

QUESTION 4.

a) 1. $\lim_{x\to 1}\left(\frac{1}{x-1}-\frac{2}{x^2-1}\right)$

$=\lim_{x\to 1}\left(\frac{x+1-2}{x^2-1}\right)$

$=\lim_{x\to 1}\frac{\cancel{x-1}}{(x+1)\cancel{(x-1)}}$

$=\frac{1}{2}$

(6)

2. $\lim_{\theta\to 0}\frac{\sin(\theta/2)}{2\theta}$

$=\lim_{\theta\to 0}\frac{\frac{1}{2}\cos(\theta/2)}{2.}$

$=\frac{1}{4}$

(6)

3. $\lim_{n\to\infty}\frac{3^n+2^n}{3^n-2^n}$

$\lim_{n\to\infty}\frac{1+(2/3)^n}{1-(2/3)^n}$

$=1$

(6)

b) $\frac{8x^2+4x+1}{(x^2+1)(2x-1)}=\frac{Ax+B}{x^2+1}+\frac{C}{2x-1}$

$8x^2+4x+1=(Ax+B)(2x-1)+C(x^2+1)$

Sub $x=1/2$ $\quad 5=5/4\,C$

$4=C$

Sub $x=0$: $\quad 1=-B+C$

$B=3$

Sub $x=1$. $\quad 13=A+B+2C$

$13-3-8=A$

$2=A$

(8)

$=\frac{2x+3}{x^2+1}+\frac{4}{2x-1}$

c) $f(x)=x^k \quad k>n$

$f'(x)=k.\,x^{k-1}$

$f''(x)=k.(k-1)x^{k-2}$

$f'''(x)=k.(k-1)(k-2)x^{k-3}$

$f^n(x)=\frac{k!}{(k-n)!}x^{k-n}$

(10)

d) $x=\frac{1+t}{1-2t}$; $y=\frac{1+2t}{1-t}$

$\frac{dx}{dt}=\frac{(1-2t)1-(1+t)(-2)}{(1-2t)^2}$

$=\frac{1-2t+2+2t}{(1-2t)^2}$

$=\frac{3}{(1-2t)^2}$

$$\frac{dy}{dt} = \frac{(1-t)2 - (1+2t)(-1)}{(1-t)^2}$$

$$= \frac{2-2t+1+2t}{(1-t)^2}$$

$$= \frac{3}{(1-t)^2}$$

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$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= \frac{3}{(1-t)^2} \cdot \frac{(1-2t)^2}{3}$$

$$= \frac{(1-2t)^2}{(1-t)^2} \quad (t=0)$$

$$= \underline{\underline{1}}$$

e) $2x - y = 0$

$y = 2x$ Grad $= 2$.

$$4x^2 - 4xy + y^2 - 4x - 8y + 10 = 0$$

$$8x - 4\left(x\frac{dy}{dx} + y\right) + 2y\frac{dy}{dx} - 4 - 8\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} - 4x\frac{dy}{dx} - 8\frac{dy}{dx} = 4y - 8x + 4$$

$$\frac{dy}{dx} = \frac{4y - 8x + 4}{2y - 4x - 8}$$

$$= \frac{4(2x) - 8x + 4}{2(2x) - 4x - 8}$$

12

$$= \underline{\underline{-\tfrac{1}{2}}}$$

$2x - \frac{1}{2} = -1$

QUESTION 5:

$$f(x) = \frac{x^3 - 3x^2}{x^2 - 1}$$

a) x interept

$\underline{x = 0}$ $\underline{x = 3}$

$\underline{y = 0}$.

$$f'(x) = \frac{(x^2-1)(3x^2-6x) - (x^3-3x^2)2x}{x^2-1}$$

$$(x^2-1)(3x^2-6x) - 2x(x^3-3x^2] = 0$$

$$3x^4 - 6x^3 - 3x^2 + 6x - 2x^4 + 6x^3 = 0$$

$$x^4 - 3x^2 + 6x = 0$$

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$\underline{x = 0}$ $x^3 - 3x + 6 = 0$

$\underline{y = 0}$

$\underline{x = -2,4}$

$\underline{\underline{y = -6,5}}$

b) VA: $\underline{x = 1}$ $\underline{x = -1}$

OA: $\underline{y = x - 3}$

6

c) $$\frac{x^3 - 3x^2}{x^2 - 1} = x - 3$$

$$x^3 - 3x^2 = (x-3)(x^2-1)$$

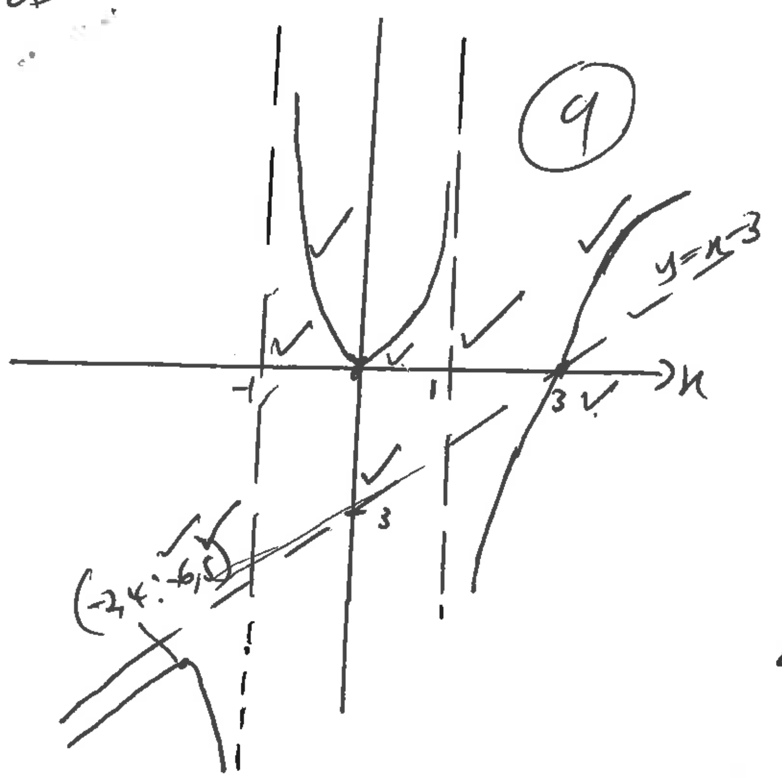
$$x^2(x-3) = (x-3)(x^2-1)$$

4

$x - 3 = 0$

$\underline{\underline{x = 3}}$ The graph intereects..

(ii)



QUESTION 6

a 1. $\int \sin 2x \cos 4x \, dx$ (6)

$= \frac{1}{2}\int(\sin 6x - \sin 2x)\,dx$

$= \frac{1}{2}\left[-\frac{\cos 6x}{6} + \frac{\cos 2x}{2}\right] + c$

2. $\int \frac{2x}{(x+1)^3}\,dx$

$u = x+1$

$du = dx$ (8)

$\int 2(u-1)\cdot u^{-3}\,du$

$= \int(2u^{-2} - 2u^{-3})\,du$

$= -2u^{-1} + \frac{2u^{-2}}{2} + c$

$= -2(x+1)^{-1} + (x+1)^{-2} + c$

3. $\int x \sin(x/2)\,dx$ (10)

$f(x) = x$ $\quad g'(x) = \sin(x/2)$

$f'(x) = 1$ $\quad g(x) = -2\cos(x/2)$

$= -2x\cos(x/2) + 2\int \cos(x/2)\,dx$

$= -2x\cos(x/2) + 2\cdot 2\sin(x/2)$

4. $\int \frac{dx}{\sin^2 x \cdot \cos^2 x}\,dx$

$= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x}\,dx$

$= \int\left(\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}\right)dx$ (9)

$= \int(\sec^2 x + \operatorname{cosec}^2 x)\,dx$

$= \tan x - \cot x + c$

b) $g(\theta) = 1 + 2\cos\theta$

Area (A1) $= \int_0^{2\pi/3}(1 + 2\cos\theta)\,d\theta$

$= \theta + 2\sin\theta \Big|_0^{2\pi/3}$ (8)

$= 2\pi/3 + 2\sin 2\pi/3$

Area (A2) $= \int_0^{2\pi/3}(-1{,}43\theta + 3)\,d\theta$

$\therefore$ Area $= (A_1) - A_2$

$= \underline{0{,}68}$

d)

$$V = \pi\left[-\int_0^1 (2x^2)^2 dx + \int_0^1 (x^4+1)^2 dx\right]$$

$$= \pi\left[\int_0^1 (x^8+2x^4+1) dx - \int 4x^4 dx\right]$$

$$= \pi\left[\frac{x^9}{9} + \frac{2x^5}{5} + x \Big|_0^1 - \frac{4x^5}{5}\Big|_0^1\right]$$

$$= \pi\left(32/45\right)$$

8

$$= 32/45\,\pi \quad (2.234)$$

SECTION B

QUESTION 1

a) $\frac{{}^5C_3}{{}^7C_3} = \frac{2}{7}$ (5)

b) $\frac{{}^5C_1}{{}^7C_3} = \frac{1}{7}$ (5)

QUESTION 2

a) $1 = 3k + 2k + k + 9k + 8k + 7k + 6k$

$1 = 36k$ (10)

$k = \frac{1}{36}$

b) mode: $x = 1$ (6)

$P(x=1) = 9 \cdot \frac{1}{36} = \frac{1}{4}$

QUESTION 3

1.1) $P(X > 1) = 1 - P(X \leq 1)$

$= 1 - \left(\frac{{}^6C_0 \, {}^{14}C_5}{{}^{20}C_5} + \frac{{}^6C_1 \, {}^{14}C_4}{{}^{20}C_5} \right)$

$= \frac{937}{1938} = 0{,}4834$ (8)

2) $P(X > 7) = P(X=9) + P(X=8)$

$= \frac{{}^9C_9 \, {}^{21}C_1}{{}^{30}C_{10}} + \frac{{}^9C_8 \, {}^{21}C_2}{{}^{30}C_{10}}$

$= \frac{7}{110055}$ $(0{,}000064)$ (6)

b i) $P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$

$= {}^{12}C_0 (0{,}1)^0 (0{,}9)^{12} + {}^{12}C_1 (0{,}1)^1 (0{,}9)^{11} + {}^{12}C_2 (0{,}1)^2 (0{,}9)^{10}$

$= 0{,}8891$ (8)

2. $P(X \geq 3)$

$1 - 0{,}889 = 0{,}111$

$X \sim Bin(6 ; 0{,}111)$

$P(X \geq 1) = 1 - P(X=0)$

$= 1 - {}^6C_0 (0{,}111)^0 (0{,}889)^6$

$= 0{,}494$ (10)

QUESTION 4

a) $\mu = 14 \pm 1{,}96 \frac{2}{\sqrt{50}}$

$= [13{,}45 ; 14{,}55]$

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Conclusion: the population mean depth of lake is within the given interval

b) $\hat{p} = \frac{380}{400}$ $1-\hat{p} = \frac{20}{400}$

9

$p = \frac{380}{400} \pm 2{,}17 \sqrt{\frac{\frac{380}{400} \cdot \frac{20}{400}}{400}}$

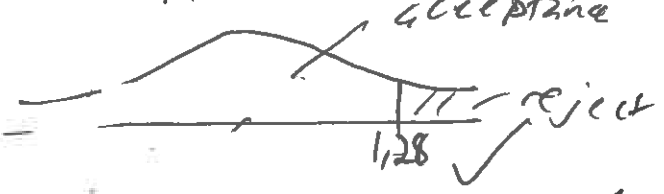
$[0{,}9264 ; 0{,}9736]$

QUESTION 5:

a) $H_0: \mu = 7$

$H_1: \mu \leq 7$

$\alpha = 0.10$



$z = \frac{8{,}7 - 7}{\frac{2{,}7}{\sqrt{11}}} = 2{,}088$

There is not enough evidence to reject H_0, at $\alpha = 10\%$ and conclude that the mean wait time for callers is at most 7 minutes.

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b) $H_0: \mu_1 = \mu_2$

$H_1: \mu_1 > \mu_2$

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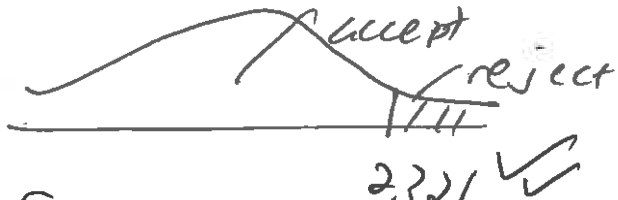
$\bar{x}_1 = 99$ $\bar{x}_2 = 80{,}5$



$\delta_{x_1} = 25$ $\delta_{x_2} = 10.$

$z = \frac{99 - 80{,}5}{\sqrt{\frac{25^2}{15} + \frac{10^2}{10}}} = 2{,}57$

There is enough evidence to reject H_0 at 1% significance level, to conclude that the true mean alkilinity of water in the lower reaches of the river is greater than that in the upper reaches.