QUESTION 1

$7^{2n-1}+1$ divisible by 8 for all $n \in N$

① Prove true for $n=1$

$7^{2(1)-1}+1$ ③

$= 7^1+1$

$= 8$ $\therefore$ true for $n=1$

② Assume true for $n=k$

$\therefore 7^{2k-1}+1 = 8r$ $\therefore 7^{2k-1} = 8r-1$

$7^{2k}.7^{-1} = 8r-1$ ③

$7^{2k} = 7(8r-1)$

③ Prove true for $n=k+1$

ie Prove $7^{2(k+1)-1}+1$ divisible by 8

$= 7^{2k+2-1}+1$

$= 7^{2k+1}+1$

$= 7^{2k}.7+1$

$= 7(8r-1).7+1$

$= 49(8r-1)+1$ ⑤

$= 49.8r-49+1$

$= 49.8r-48$

$= 8(49r-6)$

which is divisible by 8.

We have proved by mathematical induction that $7^{2n-1}+1$ is divisible by 8 for all $n \in N$. ②

Question 2

2.1. $2\log_3 x + \log_x 27 = \log_3 243$

$$\frac{2\log x}{\log 3} + \frac{3\log 3}{\log x} = 5\log_3 3$$

$$2(\log x)^2 + 3(\log 3)^2 = 5\log x\log 3$$

$$2(\log x)^2 - 5\log x\log 3 + 3(\log 3)^2 = 0$$

$$(2\log x - 3\log 3)(\log x - \log 3) = 0$$

$2\log x = 3\log 3$

$\log x^2 = \log 3^3$

$x^2 = 3^3$

$x^2 = 27$

$x = \pm\sqrt{27}$

$\log x = \log 3$

$\log x = \log 3$

$\therefore x = 3$

7

2.2. $f: y = 2\log_3(x-1)$

a) $f^{-1}: \quad x = 2\log_3(y-1)$

$3^x = (y-1)^2$

$\pm\sqrt{3^x} = y-1$

$y = 1 \pm \sqrt{3^x}$

but $y > 1$

can only be +ve

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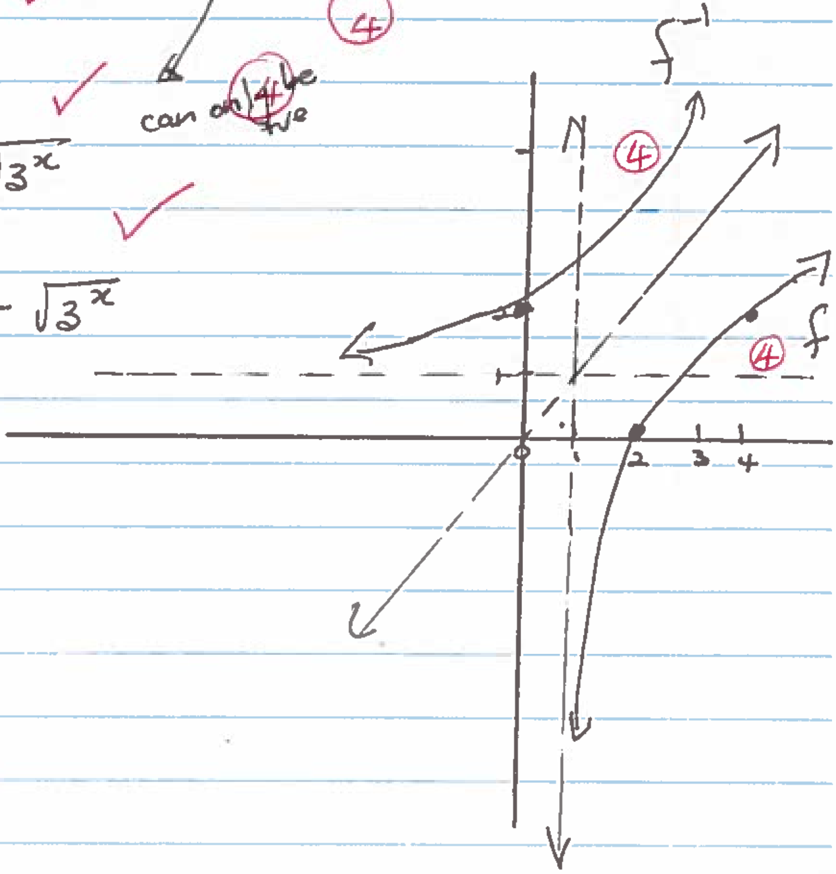
$\therefore f^{-1}(x) = 1 + \sqrt{3^x}$

b)



23 $g(x) = \frac{2x}{(x-3)^2}$

$$\frac{2x}{(x-3)^2} = \frac{A}{(x-3)} + \frac{B}{(x-3)^2}$$

$2x = A(x-3) + B$

let $x=3$

$\therefore 6 = B$

let $x=4$

$8 = A(1) + 6$

$\therefore 2 = A$

$$\therefore \frac{2x}{(x-3)^2} = \frac{2}{(x-3)} + \frac{6}{(x-3)^2}$$

6

Roots

25. $x = 2-i$ Factor $x-2+i$

$x = 2+i$ Factor $x-2-i$

$x = 5$ Factor $x-5$

$\therefore (x-2+i)(x-2-i)(x-5) = 0$

$(x^2 - 2x - \cancel{ix} - \cancel{2x} + 4 + \cancel{2i} + \cancel{ix} - \cancel{2i} - i^2)(x-5) = 0$

$(x^2 - 4x + 5)(x-5) = 0$

$x^3 - 5x^2 - 4x^2 + 20x + 5x - 25 = 0$

$x^3 - 9x^2 + 25x - 25 = 0$

7

Question 3

3.1 $y = x$ intersecting $y = -x^2 - 2x + 2$

$x = -x^2 - 2x + 2$

$x^2 + 3x - 2 = 0$

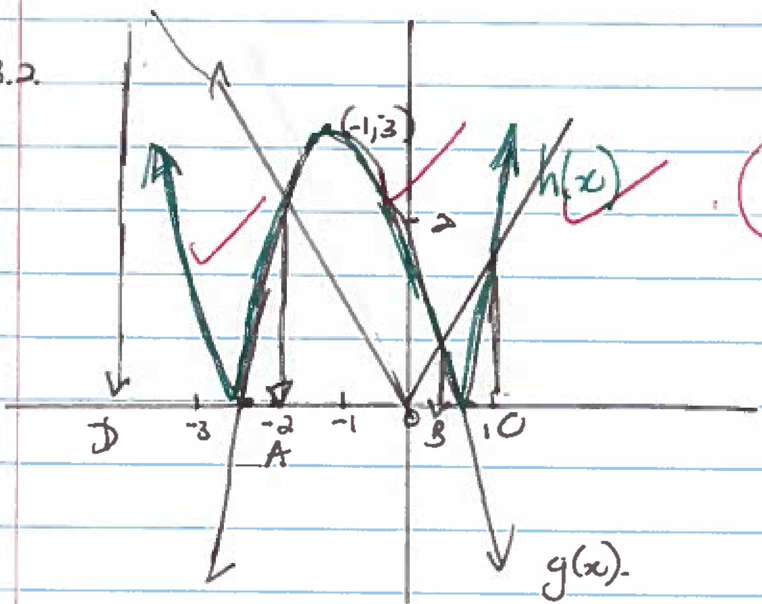
$x = \frac{-3 \pm \sqrt{9 - 4(1)(-2)}}{2(1)}$

$= \frac{-3 \pm \sqrt{17}}{2}$

and B $x = \frac{-3 + \sqrt{17}}{2}$

4

3.2.



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3.3. A: $y = -x$ $y = -x^2 - 2x + 2$

$-x = -x^2 - 2x + 2$

$x^2 + x - 2 = 0$

$(x + 1)(x + 2) = 0$

$x = 1$ or $x = -2$

B. $y = x$ $y = -x^2 - 2x + 2$

$-x^2 - 2x + 2 = x$

$-x^2 - 3x + 2 = 0$

$x^2 + 3x - 2 = 0$

$x = 0{,}56$

C: $y = x$ $y = x^2 + 2x - 2$. (Reflect g about x-axis)

$x^2 + 2x - 2 = x$

$x^2 + x - 2 = 0$

$(x - 1)(x + 2) = 0$

$x = 1$ or $x = -2$

D: $y = -x \qquad y = x^2 + 2x - 2$

$$-x = x^2 + 2x - 2$$

$$0 = x^2 + 3x - 2$$

$$\underline{x = -3{,}56}$$

8

Question 4

4.1. Jump discontinuity

2

4.2 1) Not differentiable at $x = -3$ (because not continuous)

2

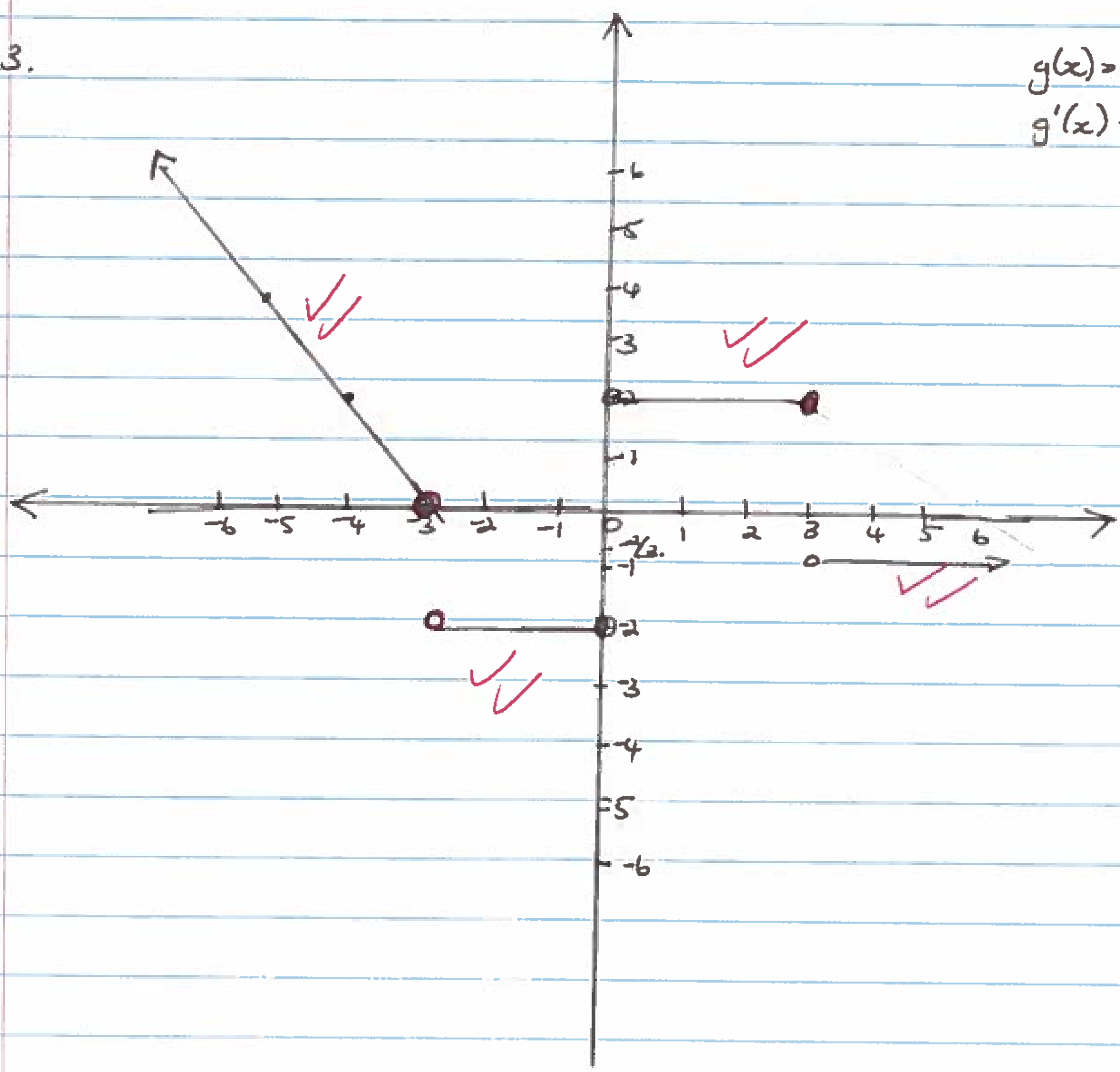
2) Not diff at $x = 3$ → is continuous but $\lim_{x \to 3^+} g'(x) \neq \lim_{x \to 3^-} g'(x)$

2

4.3.

$g(x) = -x^2 - 6x - 5$

$g'(x) = -2x - 6 . (x \leq -$



10

Question 5

5.1. $$\lim_{x\to 1} \frac{(\sqrt{x^2+8}-3)}{(x^2-1)} \times \frac{(\sqrt{x^2+8}+3)}{(\sqrt{x^2+8}+3)}$$

$$= \lim_{x\to 1} \frac{x^2+8-9}{(x^2-1)(\sqrt{x^2+8}+3)}$$

6

$$= \lim_{x\to 1} \frac{\cancel{x^2-1}}{(\cancel{x^2-1})(\sqrt{x^2+8}+3)}$$

$$= \frac{1}{6}$$

5.2. a) $f(x) = \sin^2(3x)$

$f'(x) = 2\sin(3x) \cdot \cos 3x \cdot 3$

$= 6\sin 3x \cos 3x$

4

b) $g(x) = \sqrt{(x^2+2)^2+4x^2}$

$= \left[(x^2+2)^2+4x^2\right]^{\frac{1}{2}}$

$$= \frac{1}{2}\left[(x^2+2)^2+4x^2\right]^{-\frac{1}{2}} \cdot \left[2(x^2+2) \cdot 2x + 8x\right]$$

$$= \frac{1}{2\sqrt{(x^2+2)^2+4x^2}} \cdot 4x^3+8x+8x$$

8

$$= \frac{4x^3+16x}{2\sqrt{(x^2+2)^2+4x^2}}$$

c) $$h(x) = \frac{\sqrt[3]{1-2x}}{x^3} = \frac{(1-2x)^{1/3}}{x^3}$$

$$= \frac{x^3 \cdot \frac{1}{3}(1-2x)^{-2/3} \cdot -2 - (1-2x)^{1/3} \cdot 3x^2}{x^6}$$

6

$$= \frac{-\frac{2}{3}x^3(1-2x)^{-2/3} - 3x^2(1-2x)^{1/3}}{x^6}$$

OR can use product rule

5.4 $2(x^2+y^2)^2 = 25(x^2-y^2)$

$$2(x^2+y^2)^2 - 25x^2 + 25y^2 = 0$$

$$4(x^2+y^2)' . \left[2x + 2y\frac{dy}{dx}\right] - 25 . 2x + 50y\frac{dy}{dx} = 0$$

$$8x(x^2+y^2) + 8y(x^2+y^2)\frac{dy}{dx} - 50x + 50y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\left(8y(x^2+y^2) + 50y\right) = 50x - 8x(x^2+y^2)$$

$$\frac{dy}{dx} = \frac{50x - 8x(x^2+y^2)}{8y(x^2+y^2)+50y}$$

At $x=3$; $y=1$

$$= \frac{50(3) - 8(3)(3^2+1^2)}{8(1)(3^2+1^2)+50(1)}$$

$$= \frac{150 - 24(10)}{8(10)+50}$$

$$= \frac{-90}{130}$$

$$= -\frac{9}{13}$$

10

5.3. $f(x) = (1+7x)^{-1}$

$f'(x) = -1(1+7x)^{-2} . 7$

$f''(x) = -1.(-2)(1+7x)^{-3} . 7.7$

$f'''(x) = (-1)(-2)(-3)(1+7x)^{-4} . 7.7.7$

$$f^n(x) = (-1)^n n! (1+7x)^{-(n+1)} . 7^n$$

5

Question 6

6.1. a) let $x = 0$

$$y\text{ int} = \frac{-3}{(-1)^2} = \frac{-3}{1} = -3$$

1

b) $$g(2) = \frac{8-4-2-3}{(2-1)^2} = \frac{-1}{1} = -1$$

$$g(3) = \frac{27-9-3-3}{(3-1)^2} = \frac{12}{4} = {}^{+}3$$

2

because $g(2)$ is $-$ve and $g(3)$ +ve there must be an x intercept between 2 and 3.

c) $$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}$$

$$= x_n - \frac{\dfrac{x^3 - x^2 - x - 3}{(x-1)^2}}{\dfrac{(x-1)^2(3x^2-2x-1) - (x^3-x^2-x-3).\,3(x-1)}{(x-1)^4}}$$

$$= x_n - \frac{x^3 - x^2 - x - 3}{(x-1)^2} \times \frac{(x-1)^4}{(x-1)\left[3x^3 - 2x^2 - x - 3x^2 + 2x + 1\right]}$$

$-(3x^3 - 3x^2 \ldots$

$$= x_n - \frac{x_n^3 - x_n^2 - x_n - 3}{(x_n - 1)\left[-2x_n^2 + 4x_n + 10\right]}$$

8

$$\approx \underline{2{,}1303}$$

6.2.
$$\begin{array}{r l}
 & x + 1 \\
x^2 - 2x + 1 \;\big) & x^3 - x^2 - x - 3 \\
 & x^3 - 2x^2 + x \\
\hline
 & x^2 - 2x - 3 \\
 & x^2 - 2x + 1 \\
\hline
 & -4
\end{array}$$

$$\therefore g(x) = x + 1 - \frac{4}{(x-1)^2}$$

4

6.3 Vertical asymptote : $x = 1$ ✓ ①

oblique asymptote : $y = x + 1$. ✓ ①

6.4 a) $g(x) = x + 1 - 4(x-1)^{-2}$

$g'(x) = 1 + 8(x-1)^{-3}$ ✓

$= 1 + \frac{8}{(x-1)^3}$ ✓ ③

$g''(x) = -24(x-1)^{-4}$ ✓ ②

b) T.P $g'(x) = 0$

$1 + 8(x-1)^{-3} = 0$

$\frac{8}{(x-1)^3} = -1$ ✓

$8 = -(x-1)^3$ ✓

$-8 = (x-1)^3$

$x - 1 = -2$

$x = -1$ ✓

$y = -1 + 1 - 4(-1-1)^{-2}$

$= \frac{-4}{(-2)^2}$

$= \frac{-4}{4} = -1$

$\therefore$ TP $(-1; -1)$ ✓

$g''(-1) = -24(-1-1)^{-4}$ ✓

$= \frac{-24}{(-2)^4}$

$= \frac{-24}{16}$

$= -ve$ ✓ $\therefore$ local ✓ max

⑦ Bonus 1

c) $g'(x) > 0$ (increasing)

$1 + 8(x-1)^{-3} > 0$

$1 + \frac{8}{(x-1)^3} > 0$ ✓ ①

$\frac{(x-1)^3 + 8}{(x-1)^3} > 0$

$$\frac{[(x-1)+2]\left[(x-1)^2-2(x-1)+4\right]}{(x-1)^3} > 0$$

$$\frac{(x+1)(x^2-2x+1-2x+2+4)}{(x-1)^3} > 0$$

$$\frac{(x+1)(x^2-4x+7)}{(x-1)^3} > 0 \quad ✓ \; 3$$

$$\frac{(x+1)((x-2)^2-4+7)}{(x-1)^3} > 0$$

$$\frac{(x+1)((x-2)^2+3)}{(x-1)^3} > 0$$

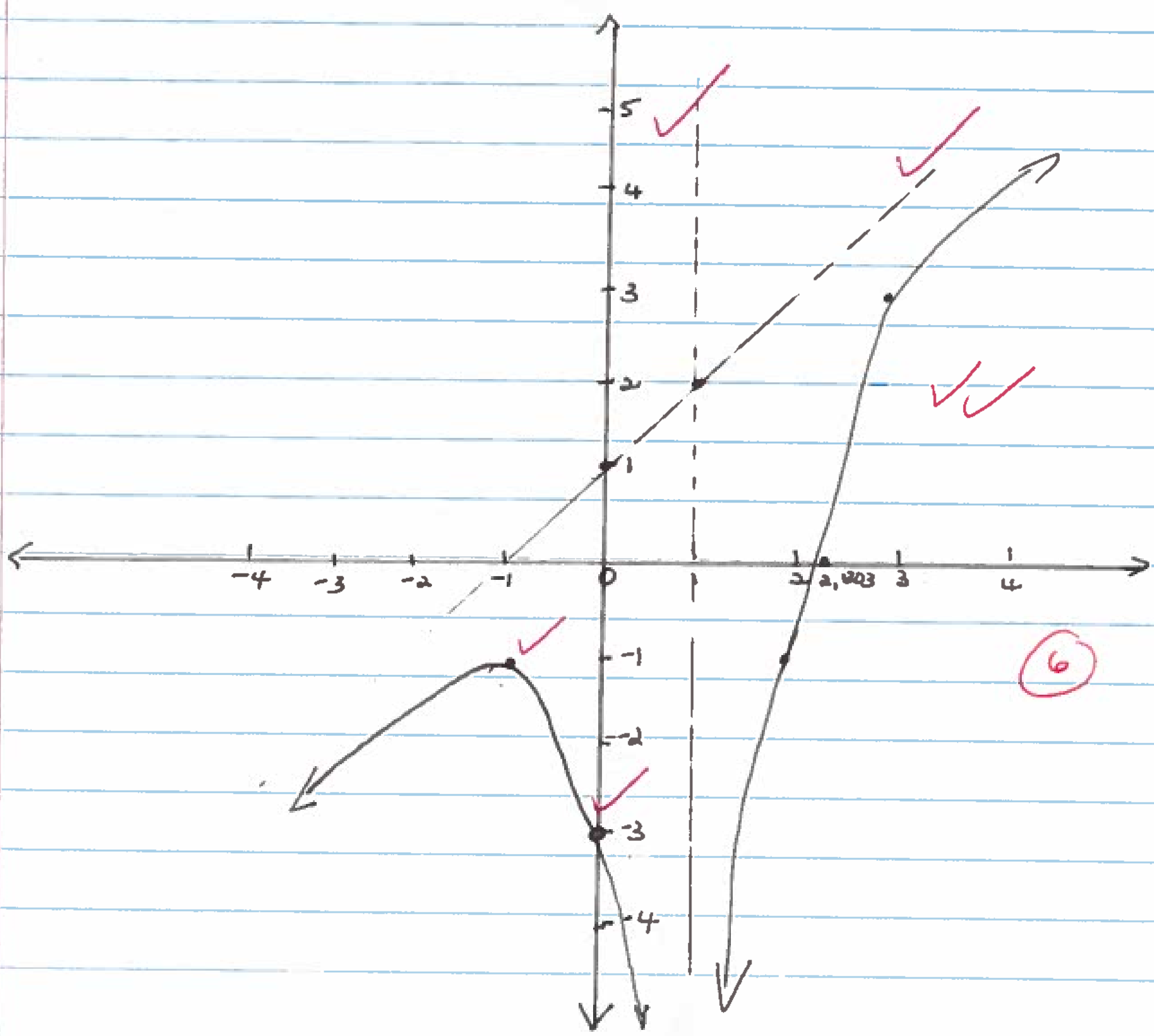
→ Always +ve

+	+	0	−	−	−	u/D	+	+	+			2
		−1				1						

✓

$\therefore \quad x < -1$ ✓ or $x > 1$

fn is increasing



Question 7

7.1 a) $\int \sin 2x \cdot \sin 5x \, dx$

$= \frac{1}{2}\int \left[\cos(2x-5x) - \cos(2x+5x)\right] dx$

$= \frac{1}{2}\int (\cos(-3x)) - \cos(7x)) \, dx$

$= \frac{1}{2}\int(\cos 3x - \cos 7x) \, dx$

$= \frac{1}{2}\left[\sin 3x \cdot \frac{1}{3} - \sin 7x \cdot \frac{1}{7}\right] + c$

$= \frac{1}{6}\sin 3x - \frac{1}{14}\sin 7x + c$

6

b) $\int \frac{x}{\sqrt{4-x^2}} \, dx \longrightarrow = \int \frac{-\frac{1}{2}du}{\sqrt{u}}$

Let $u = 4 - x^2$

$\frac{du}{dx} = -2x$

$du = -2x \, dx$

$-\frac{1}{2}du = x \, dx$

$= -\frac{1}{2}\int u^{-1/2} \, du$

$= -\frac{1}{2}\left[2u^{1/2}\right] + c$

$= -u^{1/2} + c$

$= -\sqrt{4-x^2} + c$

7

7.2 $\int x\cos 2x \, dx$

$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx + c$

$f(x) = x$

$g(x) = \frac{1}{2}\sin 2x$

$f'(x) = 1$

$g'(x) = \cos 2x$

$\int x\cos 2x \, dx = \frac{1}{2}x\sin 2x - \int \frac{1}{2}\sin 2x \cdot 1 \, dx$

$= \frac{1}{2}x\sin 2x - \frac{1}{2}\cdot(-\cos 2x)\cdot\frac{1}{2} + c$

$= \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + c$

7

7.3 a) $\partial_x \left[\cos^4 x - \sin^4 x\right]$

$= 4\cos^3 x \cdot (-\sin x) - 4\sin^3 x \cdot \cos x$

$= -4\left[\cos^3 x \cdot \sin x + \sin^3 x \cdot \cos x\right]$

$= -4 \cos x \sin x \left(\cos^2 x + \sin^2 x\right)$

$= -4\cos x \sin x$

$= -2\left(2\sin x \cos x\right)$

$= -2\sin 2x$

6

b) $\int(\cos^4 x - \sin^4 x)\,dx$

$= \int -2\sin 2x \, dx$

$= -2\int \sin 2x \, dx$

$= -2(-\cos 2x)\cdot\frac{1}{2} + C$

$= \cos 2x + C$

4

c) LHS $\cos^4 x - \sin^4 x$

$= (\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x)$

$= \cos^2 x - \sin^2 x$

$= \cos 2x$

$=$ RHS

3

Question 8

8.1 $= \frac{40}{50}$

$= \frac{4}{5}$

$\frac{1}{2}\theta = 0{,}6435\ldots$

$\theta = 1{,}287$ radians

4

Area circle Area sector Area Δ

8.2 Area below water $= \pi r^2 - \frac{1}{2} r^2 \theta + \frac{1}{2} \times (60)\, 40$

$= \pi (50)^2 - \frac{1}{2} (50)^2 . 1{,}287 + \frac{1}{2} \times 60 \times 40$

$= 7445.231 \text{ cm}^2.$

8