HERZLIA 2016 PRELIM (CALCULUS & ALGEBRA)

1.1. $|2x^2 - 3x| = 1$

$\therefore 2x^2 - 3x = -1$ or $2x^2 - 3x = +1$

$\therefore 2x^2 - 3x + 1 = 0$ $\qquad$ $\therefore 2x^2 - 3x - 1 = 0$

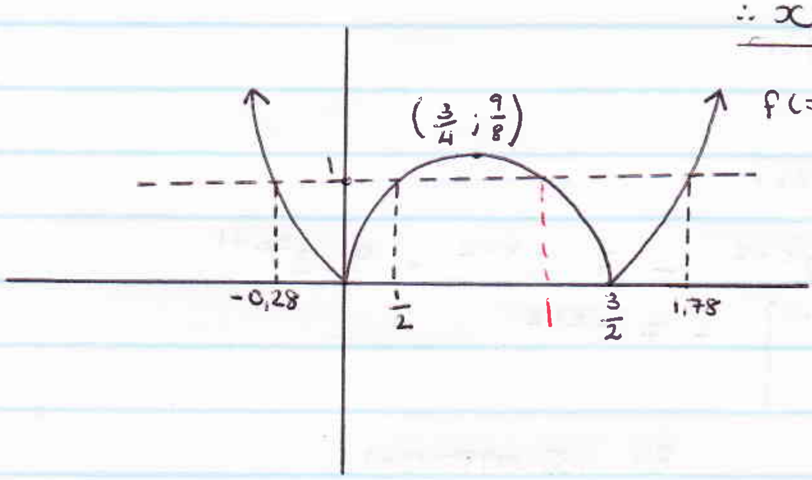
~~$\therefore 2x^2 - 3x$~~

$\therefore x = \dfrac{3 \pm \sqrt{9+8}}{4}$

$\therefore (2x-1)(x-1) = 0$

$\therefore x = \frac{1}{2}$ or $x = 1$ $\qquad$ $= \dfrac{3 \pm \sqrt{17}}{4}$

$\therefore x = -0{,}28$ or $x = 1{,}78$



$f(x) = |2x^2 - 3x|$

1.2. $\ln(e^x - 1) = \ln 6 - x$

~~$\ln(e^x - 1) = e$~~

~~$\therefore e$~~

$\therefore e^{\ln(e^x - 1)} = e^{(\ln 6 - x)} = e^{\ln 6} \cdot e^{-x}$

$\therefore e^x - 1 = \dfrac{6}{e^x}$

$\therefore e^{2x} - e^x - 6 = 0$

$\therefore (e^x - 3)(e^x + 2) = 0$

$x = \ln 3$ or $x \neq \ln(-2)$

2. $2^{n+2} + 3^{2n+1}$ div by 7

$n=1$: $2^3 + 3^3 = 8 + 27$

$= 35 = 5.7$

$\therefore$ true for $n=1$

Assume true for some $k \in \mathbb{N}$; $k \geq 1$

Assume: ~~i.e.~~ $2^{k+2} + 3^{k+1} = 7p \quad p \in \mathbb{N}$

$2^{(k+1)+2} + 3^{2(k+1)+1}$

$= 2^{k+3} + 3^{2k+3}$

$= 2.2^{k+2} + 3^2.3^{2k+1}$

$= 2.2^{k+2} + 7.2^{k+2} - 7.2^{k+2} + 9.3^{2k+1}$

$= 9\left\{2^{k+2} + 3^{2k+1}\right\} - 7.2^{k+2}$

$= 9(7p) - 7.2^{k+2}$ BY ASSUMPTION

$= 7(9p - 2^{k+2})$ which is clearly divisible by 7

$\therefore$ If true for some $k \in \mathbb{N}$; $k \geq 1$ then it is true for $k+1$.

It is true for $n=1$ $\therefore$ true for $\forall n \in \mathbb{N}$ by the principal of mathematical induction.

3. $D_x\left[(x^2+y^2)^2\right] = D_x(4xy)$

$\therefore\ 2(x^2+y^2)(2x+2y.y') = 4y+4xy'$

$\therefore\ 4y(x^2+y^2)y' - 4xy' = 4y - 4x(x^2+y^2)$

$\therefore\ y' = \dfrac{y - x(x^2+y^2)}{y(x^2+y^2)-x}$

$\therefore$ at $(1;1)$ $\quad y' = \dfrac{1-2}{2-1} = -1$

$\therefore$ tangent: $\quad y = -x + c$

sub in $(1;1)$: $\quad 1 = -1 + c$

$\therefore c = 2$

$\therefore\ \underline{y = -x + 2}$

(as indicated on given graph)

④ $f(x) = \dfrac{x^2 - 4x + 3}{x + 2}$

$= \dfrac{(x-3)(x-1)}{(x+2)}$

4.1.

$$\begin{array}{r|l} & x - 6 \\ \hline x + 2 & x^2 - 4x + 3 \\ & x^2 + 2x \\ \hline & \quad -6x + 3 \\ & \quad -6x - 12 \\ \hline & \qquad 15 \end{array}$$

4.2. Verticle : $x = -2$

Oblique : $y = x - 6$

4.3. $f'(x) = 1 - \dfrac{15}{(x+2)^2}$

$\therefore$ Stationary points at $1 = \dfrac{15}{(x+2)^2}$

$\therefore (x+2)^2 = 15$

$\therefore x = \pm\sqrt{15} - 2$

$x = -5{,}87$ or $x = 1{,}87$

$f''(x) = \dfrac{30}{(x+2)^2}$

NOTE: $f''(x) < 0$ when $x < -2$ $\therefore$ local max at $x = -5{,}87$

$f''(x) > 0$ when $x > -2$ $\therefore$ local min at $x = 1{,}87$

$f(-5{,}87) = -15{,}75$ $\qquad f(1{,}87) = -0{,}25$

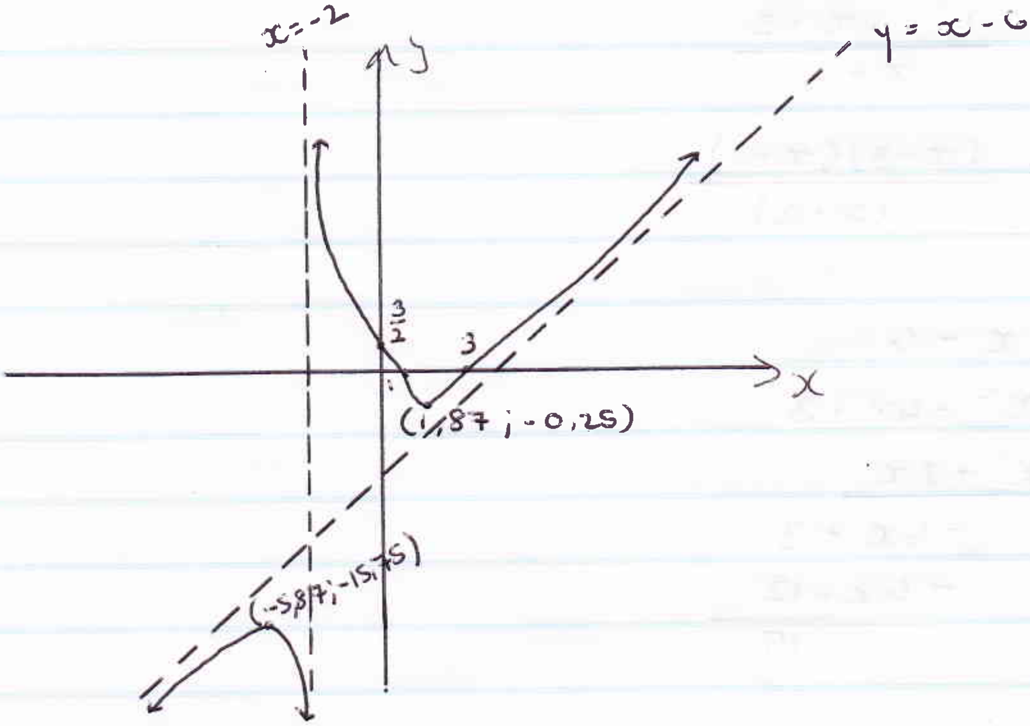
4.4) x-ints : $x = 3$ and $x = 1$

y-int : $y = \frac{3}{2}$

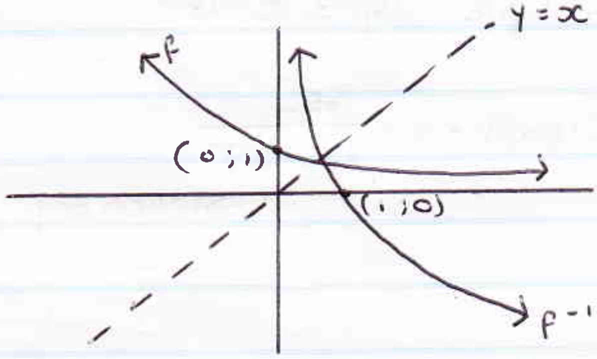
$\lim\limits_{x \to -2^-} f(x) = \dfrac{(-2^- - 3)(-2^- - 1)}{-2^- + 2} = \dfrac{+}{0^-} = -\infty$

$\lim\limits_{x \to -2^+} f(x) = \dfrac{+}{0^+} = +\infty$

4.4.



⑤ $f(x) = 3^{-x}$. $f^{-1}(x) = -\log_3 x$



Point of int: solve $3^{-x} = x$

$$3^{-x} - x = 0$$

Let $g(x) = 3^{-x} - x$

$$\therefore g'(x) = (\ln 3)3^{-x}(-1) - 1$$

$$= -(\ln 3)3^{-x} - 1$$

$$\therefore x_{n+1} = x_n - \frac{3^{-x_n} - x_n}{-(\ln 3)3^{-x_n} - 1} \quad \text{with } x_0 = \frac{1}{2}$$

$x_1 = 0{,}54733$

$x_2 = 0{,}54781$

$x_3 = 0{,}54781$

$\therefore x = 0{,}5478$

6. $y = x^{\frac{1}{x}}$

$\therefore \ln y = \ln x^{\frac{1}{x}}$

$\therefore \ln y = \frac{1}{x} \ln x$ ("plog" rule)

$\frac{d}{dx}$ both sides: $\therefore \left(\frac{1}{y}\right) y' = \left(-\frac{1}{x^2}\right) \ln x + \left(\frac{1}{x}\right)\left(\frac{1}{x}\right)$

→ Product Rule

$\therefore y' = y \left[\frac{-\ln x + 1}{x^2}\right]$

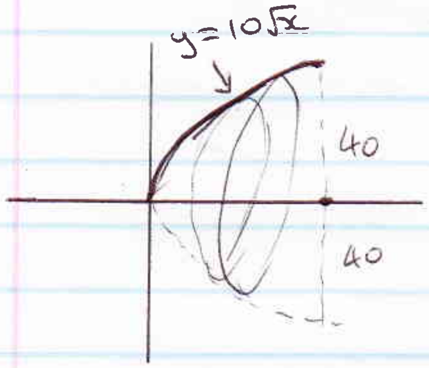
$= x^{\frac{1}{x}} \left[\frac{1 - \ln x}{x^2}\right]$

$= 0$ when $1 - \ln x = 0$

$\therefore \ln x = 1$

$\therefore x = e$

7.



$40 = 10\sqrt{x}$

$\therefore x = 16$

$V = \pi \int_0^{16} (10\sqrt{x})^2 \, dx$

$= \pi \int_0^{16} 100x \, dx$

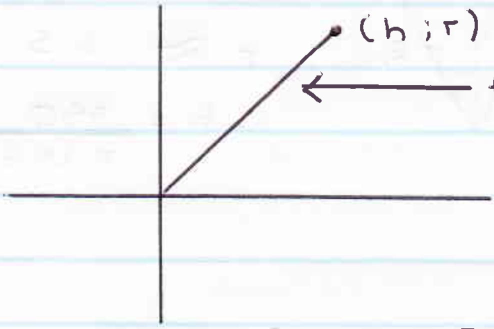
$= 50\pi \left[x^2\right]_0^{16}$

$= 12\,800\pi$ ml

≈ 40 l

8

8.1. $SA = \int 2\pi\, f(x) \sqrt{1 + (f'(x))^2}\, .\, dx$



$f(x) = \frac{r}{h} x$

$\therefore f'(x) = \frac{r}{h}$

$\therefore SA = \int_0^h 2\pi \left(\frac{r}{h} x\right) \sqrt{1 + \left(\frac{r}{h}\right)^2}\, .\, dx$

$= 2\pi \left(\frac{r}{h}\right) \sqrt{1 + \frac{r^2}{h^2}} \int_0^h x \,.\, dx$

$= 2\pi \left(\frac{r}{h}\right) \sqrt{\frac{r^2 + h^2}{h^2}} \left[\frac{1}{2} x^2\right]_0^h$

$= 2\pi \left(\frac{r}{h}\right) \frac{1}{h} \sqrt{r^2 + h^2} \left(\frac{1}{2} h^2\right)$

$= \pi r \sqrt{r^2 + h^2}$

8.2. $V = \frac{\pi}{3} r^2 h$ $\qquad V = 250\ ml\ (cm^3)$

$\therefore h = \frac{3V}{\pi r^2} = \frac{750}{\pi r^2}$

$\therefore SA(r) = \pi r \sqrt{r^2 + \left(\frac{750}{\pi r^2}\right)^2}$

for min / max DER = 0

$SA(r) = \pi r \left(\frac{1}{\pi r^2}\right) \sqrt{\pi^2 r^6 + 750^2}$

$= (r^{-1})(\pi^2 r^6 + 750^2)^{1/2}$

$\therefore SA'(r) = -\frac{1}{r^2} \sqrt{\pi^2 r^6 + 750^2} + \frac{1}{r} \cdot \frac{1}{2} (\pi^2 r^6 + 750^2)^{-\frac{1}{2}} \cdot (6\pi^2 r^5)$

$\therefore SA'(r) = \frac{-\sqrt{\pi^2 r^6 + 750^2}}{r^2} + \frac{3\pi^2 r^4}{\sqrt{\pi^2 r^6 + 750^2}}$

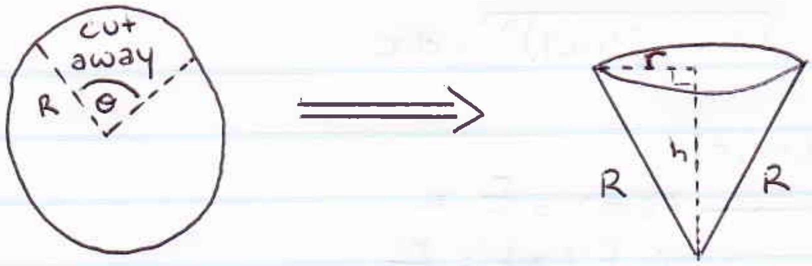
$= \frac{-(\pi^2 r^6 + 750^2) + 3\pi^2 r^6}{r^2 \sqrt{\pi^2 r^6 + 750^2}}$

$\therefore SA'(r) = 0$ when $3\pi^2 r^6 - \pi^2 r^6 - 750^2 = 0$

$\therefore 2\pi^2 r^6 = 750^2$

$\therefore r = \sqrt[6]{\frac{750^2}{2\pi^2}} = 5{,}53$ cm

$\therefore h = \frac{750}{\pi (5{,}53)^2} = 7{,}82$ cm

8.3.1. 

$r \approx 5,5$

$\therefore h = \dfrac{750}{\pi (5,5)^2}$

$$R = \sqrt{r^2 + h^2}$$
$$= \sqrt{(5,5)^2 + \left[\frac{750}{\pi (5,5)^2}\right]^2}$$
$$= 9,62$$

8.3.2. Major arc length of disc = circumf. of base of cone.

$\therefore \cancel{R\theta = 2\pi r}$ $\quad R(2\pi - \theta) = 2\pi r$

$$\therefore 2\pi - \theta = \frac{2\pi(5,5)}{9,62}$$

$$\therefore \theta = 2,69 \text{ rad.}$$

8.3.3. $\frac{1}{2} R^2 (2\pi - \theta) = \pi r \sqrt{r^2 + h^2}$ $\longrightarrow$ RTP (required to prove)

Area of cut disc $= \frac{1}{2}(9,62)^2(2\pi - 2,69)$

$= 166,26$ (with rounding)

Cone SA $= \pi(5,5)(9,62)$

$= 166,22$

(9)

9.1. <u>GOAL:</u> $\sqrt{r^2-(x-\mu)^2} = D_x\left[\dfrac{(x-\mu)\sqrt{r^2-(x-\mu)^2} - r^2\arccos\left(\dfrac{x-\mu}{r}\right)}{2}\right]$

$$D_x\left[\ldots \text{RHS} \ldots\right] = \frac{1}{2}\left\{(1)\sqrt{(r^2-(x-\mu)^2)} - \frac{(x-\mu)^2}{\sqrt{r^2-(x-\mu)^2}} + \frac{\cancel{r^2}}{\sqrt{1-\left(\frac{x-\mu}{r}\right)^2}} \times \frac{1}{\cancel{r}}\right\}$$

$$* \quad \sqrt{1-\frac{(x-\mu)^2}{r^2}} = \sqrt{\frac{r^2-(x-\mu)^2}{r^2}}$$

$$= \frac{1}{r}\sqrt{r^2-(x-\mu)^2}$$

$$= \frac{1}{2}\left\{\sqrt{r^2-(x-\mu)^2} - \frac{(x-\mu)^2}{\sqrt{r^2-(x-\mu)^2}} + \frac{r^2}{\sqrt{r^2-(x-\mu)^2}}\right\}$$

$$= \frac{1}{2}\left\{\frac{r^2-(x-\mu)^2-(x-\mu)^2+r^2}{\sqrt{r^2-(x-\mu)^2}}\right\}$$

$$= \frac{1}{2}\left\{\frac{2\left[r^2-(x-\mu)^2\right]}{\sqrt{r^2-(x-\mu)^2}}\right\}$$

$$= \left[r^2-(x-\mu)^2\right]\cdot\left[r^2-(x-\mu)\right]^{-\frac{1}{2}}$$

$$= \left[r^2-(x-\mu)^2\right]^{\frac{1}{2}}$$

$$= \sqrt{r^2-(x-\mu)^2}$$

9.2. $\int r^2 - (x-\mu)^2 \, dx = \dfrac{(x-\mu)\sqrt{r^2-(x-\mu)^2} - r^2 \arccos\frac{(x-\mu)}{r}}{2} + C$

Let $x - \mu = r\cos\theta$ (polar co-ordinate substitution)

$\therefore dx = -r\sin\theta \, d\theta$

$$\therefore \int \sqrt{r^2 - (x-\mu)^2} \, dx = \int \sqrt{r^2 - r^2\cos^2\theta} \, (-r\sin\theta \, d\theta)$$

$$= \int (-r^2) \sqrt{1-\cos^2\theta} \, \sin\theta \, d\theta$$

$$= -r^2 \int \sin^2\theta \, d\theta$$

$(\cos 2\theta = 1 - 2\sin^2\theta)$

$$= -r^2 \int \frac{1}{2}\left[1 - \cos 2\theta\right] d\theta$$

$$= \frac{r^2}{2} \int (\cos 2\theta - 1) \, d\theta$$

$$= \frac{r^2}{2}\left(\frac{1}{2}\sin 2\theta - \theta\right) + C$$

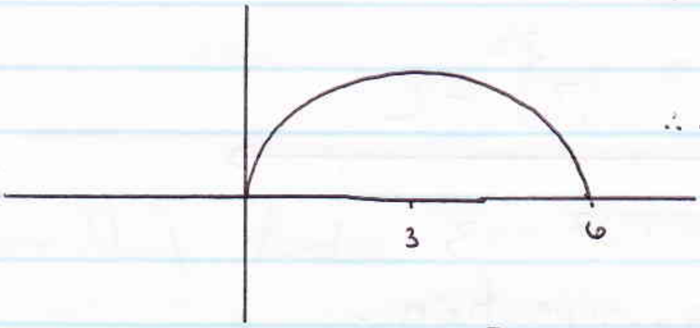
$$= \frac{r^2}{2}\left\{\frac{2}{2}\sqrt{1 - \frac{(x-\mu)^2}{r}} \cdot \frac{x-\mu}{r} - \arccos\left(\frac{x-\mu}{r}\right)\right\}$$

With a little algebra ....

$$= \frac{(x-\mu)\sqrt{r^2-(x-\mu)^2} - r^2 \arccos\frac{x-\mu}{r}}{2} + C$$

9.3. $\sqrt{9-(x-3)^2}$ = semi-circle radius = 3

centre = $(3;0)$



$\therefore$ area $= \frac{\pi r^2}{2} = \frac{9\pi}{2}$

given pdf: $f_x(x) = \begin{cases} k\sqrt{9-(x-3)^2} & \text{if } x \in [0;6] \\ 0 & \text{elsewhere} \end{cases}$

prob. density function

$\therefore \int_0^6 k\sqrt{9-(x-3)^2} \cdot dx = 1$ ($\because$ area under pdf $= 1$)

$\therefore k\int_0^6 \sqrt{9-(x-3)^2} = 1$

$\therefore k\left(\frac{9\pi}{2}\right) = 1 \quad \therefore k = \frac{2}{9\pi}$

$k\int_2^4 \sqrt{9-(x-3)^2}\, dx = \left(F(4) - F(2)\right) k$

where $F(x) = \dfrac{(x-\mu)\sqrt{r^2-(x-\mu)^2} - r^2 \arccos\left(\frac{x-\mu}{r}\right)}{2}$

* From given anti-derivative in 9.1 !!
with $r = 3$ and $\mu = 3$

$\therefore k\int_2^4 \sqrt{9-(x-3)^2}\, dx = k\left[\dfrac{(x-3)\sqrt{3^2-(x-3)^2} - (3)^2\arccos\left(\frac{x-3}{3}\right)}{2}\right]_2^4$

$= \frac{2}{9\pi}\left[5{,}89\right]$

$= 0{,}42$

10.1.1) $\int 7x^2\sqrt{5x^3-13}\,dx$

$$= \frac{14}{45}(5x^3-13)^{3/2} + C$$

could use $u = 5x^3 - 13$ but full marks for doing it by inspection.

10.1.2) on next page.

10.2) $= \frac{1}{2}\int_{\frac{\pi}{6}}^{\pi/3}\left[\cos(5\theta-2\theta) - \cos(5\theta+2\theta)\right] d\theta$

$$= \frac{1}{2}\int_{\pi/6}^{\pi/3} (\cos 3\theta - \cos 7\theta)\,d\theta$$

$$= \frac{1}{2}\left[\frac{1}{3}\sin 3\theta - \frac{1}{7}\sin 7\theta\right]_{\pi/6}^{\pi/3}$$

$$= \frac{-17+3\sqrt{3}}{84} = -0,26$$

10.3 let $f(x) = x$ $\therefore f'(x) = 1$

and $g'(x) = \cos 2x$ $\therefore g(x) = \frac{1}{2}\sin 2x$

$$\therefore \int = \left[\frac{1}{2}x\sin 2x\right]_{\frac{3\pi}{4}}^{5\pi/4} - \int_{\frac{3\pi}{4}}^{5\pi/4} \frac{1}{2}\sin 2x\,dx$$

$$= \pi + \left[\frac{1}{4}\cos 2x\right]_{3\pi/4}^{5\pi/4}$$

$$= \pi + 0$$

$$= \pi$$

10.1.2.

$$\boxed{\begin{gathered}\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x \\ \frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}\end{gathered}}$$

$$\int \frac{8}{9\sqrt{9+16x-4x^2}} \cdot dx$$

$$= \int \frac{8}{9\sqrt{9-(8x^2-16x)}} \cdot dx$$

$$= \int \frac{8}{9\sqrt{9-4(x^2-4x)}} \cdot dx$$

$$= \frac{8}{9}\int \frac{1}{\sqrt{9-4\left[(x^2-4x+4)-4\right]}} \cdot dx$$

$$= \frac{8}{9}\int \frac{1}{\sqrt{9+16-4(x^2-4x+4)}} \cdot dx$$

$$= \frac{8}{9}\int \frac{1}{\sqrt{25-4(x-2)^2}} \cdot dx$$

$$= \frac{8}{9}\int \frac{1}{5\sqrt{1-\frac{4(x-2)^2}{25}}} \cdot dx$$

$$= \frac{8}{45}\int \frac{1}{\sqrt{1-\left[\frac{2(x-2)}{5}\right]^2}} \cdot dx$$

$$= \frac{4}{9}\arcsin\left[\frac{2}{5}(x-2)\right] + c$$

Bonus: (a) $2-i$ root $\therefore$ $2+i$ also (comp. ~~conj~~ conj ~~surd~~ thm)

$\therefore (x-(2-i))(x-(2+i))$ is a factor

$= [(x-2)+i][(x-2)-i]$

$= (x-2)^2 - i^2 \quad = x^2-4x+5$

$1+\sqrt{2}$ root $\therefore$ $1-\sqrt{2}$ (conj. surd thm)

$\therefore [x-(1+\sqrt{2})][x-(1-\sqrt{2})]$ is a factor

$= (x-1)^2 - (\sqrt{2})^2$

$= x^2-2x-1$

(b) $(x^2-4x+5)(x^2-2x-1) = x^4-6x^3+12x^2-6x-5$

$$
\begin{array}{r|l}
 & 4x^2 + 1 \\
\hline
x^4-6x^3+12x^2-6x-5 & 4x^6-24x^5+49x^4-30x^3-8x^2-6x-5 \\
 & 4x^6-24x^5+48x^4-24x^3-20x^2 \\
\hline
 & x^4-6x^3+12x^2-6x-5 \\
 & x^4-6x^3+12x^2-6x-5 \\
\hline
\end{array}
$$

$4x^2+1 = (2x+i)(2x-i)$

$\therefore f(x) = (2x+i)(2x-i)(x-(2-i))(x-(2+i))(x-(1+\sqrt{2}))(x-(1-\sqrt{2}))$