Grade 12 – APM Trials Exam 2015 – P1 memo

1.1 A. Prove true for $n = 1$

$(1)^3 + 2(1) = 3$ which is divisible by 3.
$\therefore$ True for $n = 1$

B. Assume true for $n = k$ where $k \in \mathbb{N}$

$k^3 + 2k$ is divisible by 3

C. Prove true for $n = k + 1$

$(k + 1)^3 + 2(k + 1)$
$= k^3 + 3k^2 + 3k + 1 + 2k + 2$
$= (k^3 + 2k) + 3k^2 + 3k + 3$ since $k^3 + 2k$ is divisible by 3

$\therefore (k + 1)^3 + 2(k + 1)$ is divisible by 3
$\therefore$ the statement is true for $n = k + 1$

By the principle of Mathematical Induction $n^3 + 2n$ is divisible by 3 for $n \in \mathbb{N}$

(10)

1.2 $$\frac{x - 10}{2x^2 + 5x - 3} = \frac{x - 10}{(2x - 1)(x + 3)}$$

$$\frac{x - 10}{(2x - 1)(x + 3)} = \frac{A}{2x - 1} + \frac{B}{x + 3}$$

$$\frac{x - 10}{(2x - 1)(x + 3)} = \frac{A(x + 3)}{2x - 1} + \frac{B(2x - 1)}{x + 3}$$

$A + 2B = 1$ $3A - B = -10$
$A = -\frac{19}{7}$ $B = \frac{13}{7}$

$$\therefore \frac{x - 10}{(2x - 1)(x + 3)} = \frac{-19}{7(2x - 1)} + \frac{13}{7(x + 3)}$$

(8)

1.3 $x = 1 + i \quad \therefore \quad x = 1 - i$

$sum = 2 \quad product = 1 - i^2 = 2$

$\therefore x^2 - 2x + 2$ is a factor of $x^3 + ax^2 + bx - 6$

$(x^2 - 2x + 2)(x - 3) = x^3 - 5x^2 + 8x + 6$

$\therefore a = -5$ and $b = 8$ (8)

2.1 $\dfrac{e^x}{e^x - 1} = 5$

$e^x = 5.e^x - 5$

$4.e^x - 5 = 0$

$e^x = \dfrac{5}{4}$

$x = \ln\dfrac{5}{4} = 0{,}22$ (4)

2.2 $\dfrac{-6}{1-x} \leq x$

$\dfrac{-6}{1-x} - x \leq 0$

$\dfrac{-6 - x(1-x)}{1-x} \leq 0$

$\dfrac{x^2 - x - 6}{1-x} \leq 0$

$\dfrac{(x-3)(x+2)}{x-1} \geq 0$

$-2 \leq x < 1 \quad or \quad x > 3$ (6)

2.3 $|x|^2 - 5|x| - 14 = 0$

$|x| = 7 \qquad |x| \neq -2$

$\therefore x = \pm 7$

(6)

2.4 $3(lnx)^2 + lnx - 1 + \dfrac{1}{3(lnx)^2 + lnx - 3} = 0$

Let $k = 3(\ln x)^2 + \ln x$

$k - 1 + \dfrac{1}{k-3} = 0$

$k^2 - 4k + 3 + 1 = 0$

$k^2 - 4k + 4 = 0$

$k = 2$

$3(\ln x)^2 + \ln x - 2 = 0$

$(3\ln x - 2)(\ln x + 1) = 0$

$\ln x = \dfrac{2}{3} \quad \ln x = -1$

$x = e^{\frac{2}{3}} \quad x = e^{-1}$ (8)

3.1 $\lim_{x\to\infty} \dfrac{x^2 - x + 6}{4x^3 + 3x^2 + x}$

$= 0$

(3)

3.2 $\lim_{x\to\frac{\pi}{4}} \dfrac{\cos 2}{\cos\theta - \sin\theta}$

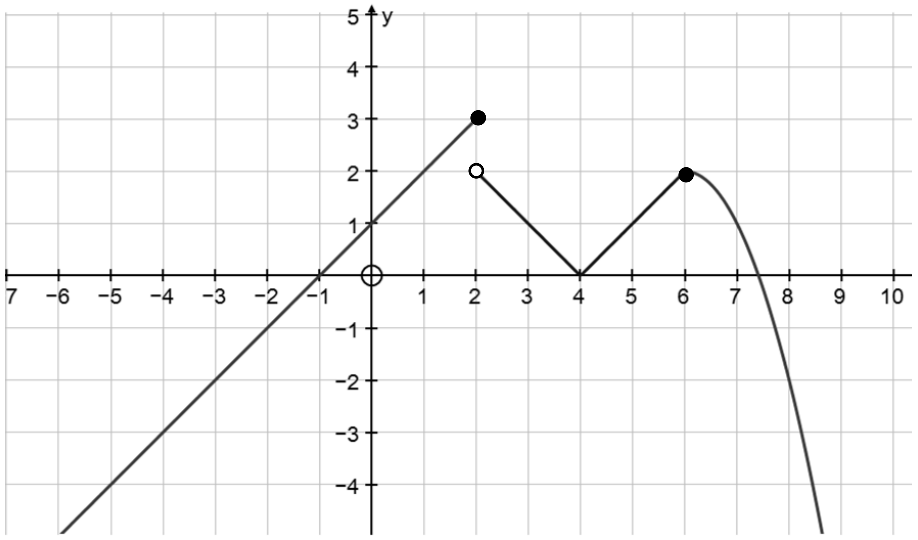
$= \lim_{x\to\frac{\pi}{4}} \dfrac{\cos^2\theta - \sin^2\theta}{\cos\theta - \sin\theta}$

$= \lim_{x\to\frac{\pi}{4}} \dfrac{(\cos\theta + \sin\theta)(\cos\theta - \sin\theta)}{\cos\theta - \sin\theta}$

$= \lim_{x\to\frac{\pi}{4}} (\cos\theta + \sin\theta)$

$= \sqrt{2}$ (5)

4.1.1



(8)

4.1.2 $f(2) = 2 + 1 = 3$

$\lim_{x\to 2^-} f(x) = 2 + 1 = 3$

$\lim_{x\to 2^+} f(x) = |2 - 4| = 2$

$\therefore \lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$

Not continuous at $x = 2$

Jump discontinuity. (4)

4.2 $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x) = f(x)$

$-a + b + 1 = -1 - 3b$

$a = 4b + 2 \ldots(1)$

$ab = -1 - 3b$

$b(4b + 2) = -1 - 3b$

$4b^2 + 5b + 1 = 0$

$b = -\frac{1}{4}$ or $b = -1$

$a = 1$ or $a = -2$ (6)

5.1.1 $fog(x) = \sqrt{(x^2+1)^2 - 1}$ (4)

5.1.2 $(x^2+1)^2 - 1 \geq 0$

$x^4 + 2x^2 \geq 0$

$x^2(x^2+2) \geq 0$

$x \in \mathbb{R}$ (2)

5.1.3 $D_x\left[\sqrt{x+1}\ .(x^2+1)^2\right]$

$= \frac{1}{2}(x+1)^{-\frac{1}{2}}(x^2+1)^2 + 2(x^2+1)(2x)(x+1)^{\frac{1}{2}}$ (6)

5.2.1 $f'(x) = 4(2x^2+x-1)^3(4x+1)$ (5)

5.2.2 $D_x\left[\frac{\cos 3x}{\tan 5x}\right]$

$= \frac{(-3\sin 3x\,.\tan 5x - 5\sec^2 5x\,.\cos 3x)}{\tan^2 5x}$

(7)

6.1 $f(x) = (1-x)^{-5}$

$f'(x) = -5(1-x)^{-6}(-1)$

$f''(x) = (-5)(-6)(1-x)^{-7}(-1)(-1)$

$f^n(x) = \frac{(n+4)!}{4!}\ (1-x)^{-(n+5)}$

(6)

6.2 $xy^3 + 3x^2 = xy + 12$

$y^3 + 3xy^2\ \frac{dy}{dx} + 6x = y + x\frac{dy}{dx}$

$3xy^2\ \frac{dy}{dx} - x\frac{dy}{dx} = -6x - y^3 + y$

$\frac{dy}{dx}(3xy^2\ - x) = -6x - y^3 + y$

$\frac{dy}{dx} = \frac{-6x - y^3 + y}{3xy^2\ - x} = -3$

$1 = -3(2) + c$

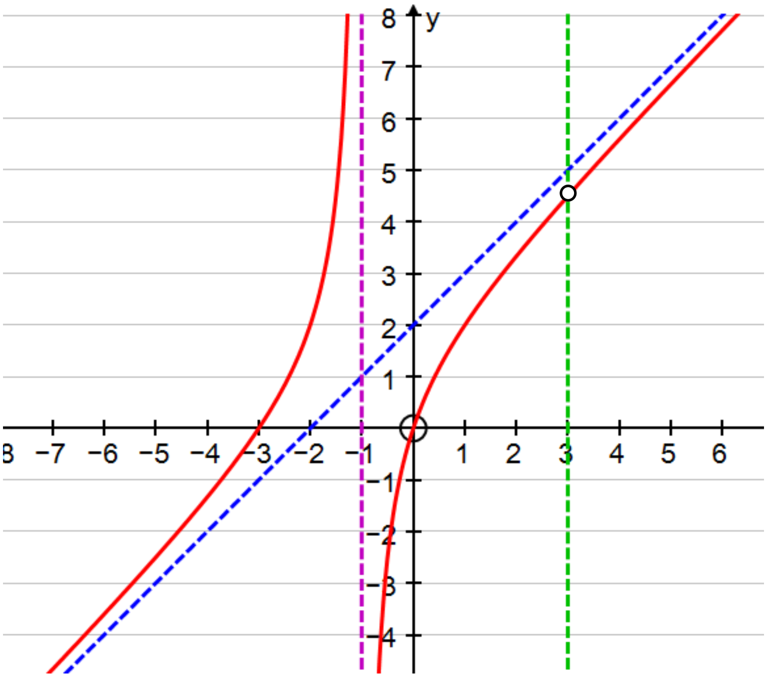
$y = -3x + 7$ (8)

7.1 $x^2 - 2x - 3 = 0$

$x = -1 \quad x = 3$ (4)

7.2 $y = \left(\frac{1}{3}x + \frac{2}{3}\right)$ by long division or other method. (6)

7.3



(8)

8.1 $2\cos\theta + \theta - 2 = 0$

$\theta = 3{:}\ 2\cos 3 + 3 - 2 = -0{,}98$

$\theta = 5{:}\ 2\cos 5 + 5 - 2 = 3{,}57$

$\therefore$ *Graph passes through the x − axis between* 3 *and* 5 (4)

8.2 $f(x) = 2\cos\theta + \theta - 2 = 0$
$f'(x) = -2\sin\theta + 1$

$x_1 = 4$

$$x_2 = 4 - \frac{f(4)}{f'(4)} = 3{,}72441$$
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 3{,}69843$$
$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 3{,}69815$$
$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = 3{,}69815$$

$\therefore x = 3{,}69815$

(6)

9.1 $\int x\sqrt{x^2+3}\ dx$

$= \int u^{\frac{1}{2}} \frac{du}{2}$ Let $u = x^2 + 3$ $du = 2x\ dx$

$= \frac{1}{2} \times \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c$

$= \frac{(x^2+3)^{\frac{3}{2}}}{3} + c$

(6)

9.2 $\int \sin 4x \cos 3x\ \ dx$

$= \frac{1}{2}\int (\sin 7x - \sin x)\, dx$

$= \frac{1}{2}(-\frac{\cos 7x}{7} + (-\cos x) + c)$

$= -\frac{\cos 7x}{14} - \frac{\cos x}{2} + c$

(6)

9.3 $\int x \sin^2 x\ \ dx$

$= x\left(\frac{x}{2} - \frac{\sin 2x}{4}\right) - \int\left(\frac{x}{2} - \frac{\sin 2x}{4}\right) dx$ Let $u = x$ $dv = \sin^2 x\, dx = \frac{1}{2}(1 - \cos 2x)dx$

$= x\left(\frac{x}{2} - \frac{\sin 2x}{4}\right) - \left(\frac{x^2}{4} + \frac{\cos 2x}{8}\right) + c$ $du = 1dx$ $v = \frac{x}{2} - \frac{\sin 2x}{4}$

(9)

9.4 $\int \frac{x}{\sqrt{x-1}}\ dx$ Let $u = x - 1$ $du = dx$

$= \int u^{-\frac{1}{2}}(u+1)du$ $x = u + 1$

$= \int \left(u^{\frac{1}{2}} + u^{-\frac{1}{2}}\right) du$

$= \frac{2(x-1)^{\frac{3}{2}}}{3} + 2(x-1)^{\frac{1}{2}} + c$

(9)

10.1 $(x+2)^2-3=2x+4$

$x=-3 \qquad x=1$
$y=-2 \qquad y=6$ (4)

10.2 Distance $= 2x+4-x^2-4x-4+3=-x^2-2x+3$

$D_x[-x^2-2x+3]=0$

$-2x-2=0$

$x=-1$

Max distance $= -(-1)^2-2(-1)+3=4units$

(6)

10.3

$$Area = \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n} f\left(\frac{4i}{n}\right)$$

$$= \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n}\left(-\left(-3+\frac{4i}{n}\right)^2-2\left(-3+\frac{4i}{n}\right)+3\right)$$

$$= \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n}\left(-9+\frac{24i}{n}-\frac{16i^2}{n^2}+6-\frac{8i}{n}+3\right)$$

$$= \lim_{n\to\infty}\frac{4}{n}\left(\frac{16i}{n}-\frac{16i^2}{n^2}\right)$$

$$= \lim_{n\to\infty}\frac{4}{n}\left(\frac{16}{n}\left(\frac{n^2}{2}+\frac{n}{2}\right)-\frac{16}{n^2}\left(\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{2}\right)\right)$$

$$= \lim_{n\to\infty}\left(32+\frac{32}{n}-\frac{64}{3}-\frac{32}{n}-\frac{32}{n^2}\right)=\frac{32}{3}$$

(10)

10.4

$$\pi\int_0^1(-x^2-2x+3)^2dx$$

$$=\pi\int_0^1(x^4+4x^3-2x^2-12x+9)dx$$

$$=\pi\left(\frac{x^5}{5}+x^4-\frac{2x^3}{3}-6x^2+9x+c\right]_0^1\Bigg)$$

$$=\pi\left(\frac{1^5}{5}+(1)^4-\frac{2(1)^3}{3}-6(1)^2+9(1)\right)$$

$$=\frac{53}{15}\pi$$ (8)