Parklands College

PRELIMINARY EXAMINATIONS

ADVANCED PROGRAMME MATHEMATICS

GRADE 12

Paper: 1 Date: **12 September 2016**

SOLUTIONS

SECTION A (CALCULUS AND ALGEBRA)

QUESTION 1

Using Mathematical Induction, prove that $7^{2n}+3$ is divisible by 4 for all natural numbers n. [14]

$n=1$: $7^{2(1)}+3=52=4(13)$

Therefore the statement is true for $n=1$.

Assume that the statement is valid for $n=k$, therefore $7^{2k}+3$ is divisible by 4, therefore $7^{2k}+3=4p$, where $p\in N$, therefore $7^{2k}=4p-3$

Then $7^{2(k+1)}+3=7^{2k+2}+3$

$$=7^{2k}.7^2+3$$
$$=49.(4p-3)+3$$
$$=196p-147+3$$
$$=196p-144$$
$$=4(49p-36)\text{, where } 49p-36\in N$$

Therefor the statement is valid for $n=k+1$ if valid for $n=k$;
Therefore also for $n=1+1=2$, therefore also for $n=2+1=3$, etc.

Therefore, according to Mathematical Induction, for all natural numbers n.

QUESTION 2

Solve for x:

2.1 $e^x+6e^{-x}=5$ (6)

$$e^x+\frac{6}{e^x}=5$$

Let $k=e^x$

$$k+\frac{6}{k}=5$$

$$k^2+6=5k$$
$$k^2-5k+6=0$$
$$(k-2)(k-3)=0$$
$$k=2 \text{ or } k=3$$
$$e^x=2 \text{ or } e^x=3$$
$$x=\ln 2 \text{ or } x=\ln 3$$
$$x=0{,}69 \text{ or } x=1{,}10$$

2.2 $|2x-7|>3$ (4)

$$\pm(2x-7)>3$$
$$2x-7>3 \text{ or } -2x+7>3$$
$$2x>10 \text{ or } -2x>-4$$
$$x>5 \text{ or } x<2$$

2.3 $5^{|2x-3|}=125$ (6)

$$5^{|2x-3|}=5^3$$
$$|2x-3|=3$$
$$\pm(2x-3)=3$$
$$2x-3=3 \text{ or } -2x+3=3$$
$$2x=6 \text{ or } -2x=0$$
$$x=3 \text{ or } x=0$$

2.4 $\frac{1}{3x}\le\frac{3x}{4}$ (7)

$$\frac{1}{3x}-\frac{3x}{4}\le 0$$
$$\frac{4-9x^2}{12x}\le 0$$
$$\frac{9x^2-4}{x}\ge 0$$
$$\frac{(3x+2)(3x-2)}{x}\ge 0$$

$-\quad + \quad - \quad +$

$-\frac{2}{3} \quad 0 \quad \frac{2}{3}$

$$-\frac{2}{3}\le x<0 \text{ or } x\ge\frac{2}{3}$$

2.5 $\log x+2\log x=-6$ (4)

[27]

$$3\log x=-6$$
$$\log x=-2$$
$$x=10^{-2}=\frac{1}{100}$$

QUESTION 3

3.1 Simplify the following : $(i^2\cos\theta+i\sin\theta)(i^2\cos\theta-i\sin\theta)$ (5)

$$=i^4\cos^2\theta-i^2\sin^2\theta$$
$$=(-1)^2\cos^2\theta-(-1)\sin^2\theta$$
$$=\cos^2\theta+\sin^2\theta$$
$$=1$$

3.2 If $4i$ is a root of $f(x)=x^4-2x^3+33x^2-32x+272=0$, determine all the complex roots of the equation. (10)

[15]

$-4i$ is also a root ; therefore $(x+4i)(x-4i)$ is a factor of $f(x)$

therefore $x^2-16i^2=x^2-16(-1)=x^2+16$ is a factor of $f(x)$

$$\begin{array}{r|l}
 & x^2-2x+17 \\
x^2+16 & x^4-2x^3+33x^2-32x+272 \\
 & x^4 \qquad +16x^2 \\
\hline
 & -2x^3+17x^2-32x \\
 & -2x^3 \qquad -32x \\
\hline
 & 17x^2 \qquad +272 \\
 & 17x^2 \qquad +272 \\
\hline
\end{array}$$

$$f(x)=(x^2+16)(x^2-2x+17)=0$$

$$x=\pm 4i \text{ or } x=\frac{-(-2)\pm\sqrt{(-2)^2-4(1)(17)}}{2(1)}$$
$$x=\frac{2\pm\sqrt{-64}}{2}$$
$$x=\frac{2\pm 8i}{2}=1\pm 4i$$

QUESTION 4

4.1 $f(x)=\dfrac{3x^2-5x-2}{px^2+x}$ is given.

Determine the value(s) of p if:

4.1.1 $f(x)$ has two vertical asymptotes (4)

$$f(x)=\frac{(3x+1)(x-2)}{x(px+1)}$$

$$p\in R;\ p\neq 3$$

4.1.2 $\lim\limits_{x\to\infty} f(x)=\dfrac{1}{2}$ (4)

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{\frac{3x^2-5x-2}{x^2}}{\frac{px^2+x}{x^2}}$$

$$=\lim_{x\to\infty}\frac{3-\frac{5}{x}-\frac{2}{x^2}}{p+\frac{1}{x}}$$

$$=\frac{3}{p}=\frac{1}{2}$$

$$p=6$$

4.2 Determine the value of $\lim\limits_{x\to-\frac{2}{3}}\dfrac{3x^2+2x}{9x^2-4}$ (4)

$$=\lim_{x\to-\frac{2}{3}}\frac{x(3x+2)}{(3x+2)(3x-2)}$$

$$=\lim_{x\to-\frac{2}{3}}\frac{x}{(3x-2)}$$

$$=\frac{-\frac{2}{3}}{3\left(-\frac{2}{3}\right)-2}=\frac{-\frac{2}{3}}{-2-2}$$

$$=\frac{1}{6}$$

4.3 The function $f(x)$ is defined as follows :

$$f(x)=ax^2-1 \quad \text{if } x\leq -1$$

$$f(x)=\frac{b}{x}+6 \quad \text{if } x>-1$$

4.3.1 If f is continuous at $x=-1$, determine a in terms of b. (3)

$$\lim_{x\to-1^-} f(x)=\lim_{x\to-1^+} f(x)$$

$$a(-1)^2-1=\frac{b}{(-1)}+6$$

$$a-1=-b+6$$

$$a=-b+7$$

4.3.2 If f is also differentiable at $x=-1$, determine another expression for a in terms of b. (4)

$$\lim_{x\to-1^-} f'(x)=\lim_{x\to-1^+} f'(x)$$

$$\lim_{x\to-1^-} 2ax=\lim_{x\to-1^+}\left(-\frac{b}{x^2}\right)$$

$$2a(-1)=-\frac{b}{(-1)^2}$$

$$-2a=-b$$

$$a=\frac{b}{2}$$

4.3.3 Hence, solve for a and b. (4)

[23]

$$-b+7=\frac{b}{2}$$

$$-2b+14=b$$

$$-3b=-14$$

$$b=4\frac{2}{3}$$

$$a=\frac{14}{3}\times\frac{1}{2}=\frac{7}{3}$$

QUESTION 5

$f(x)=\dfrac{2x+1}{4-x}\ (x\neq 4)$; $g(x)=4-x^2$; $h(x)=3.2^x-1$

5.1 Determine in the simplest form:

5.1.1 $f(g(x))$ (3)

$$=\frac{2(4-x^2)+1}{4-(4-x^2)}$$
$$=\frac{8-2x^2+1}{4-4+x^2}$$
$$=\frac{9-2x^2}{x^2}$$

5.1.2 $f'(x)$ (5)

$$=\frac{2(4-x)-(2x+1)(-1)}{(4-x)^2}$$
$$=\frac{8-2x+2x+1}{(4-x)^2}$$
$$=\frac{9}{(4-x)^2}$$

5.1.3 The values of x for which f is increasing. (3)

$$f'(x)=\frac{9}{(4-x)^2}>0$$

$x\in R; x\neq 4$

5.1.4 $h^{-1}(x)$ in the form $y=...$ (4)

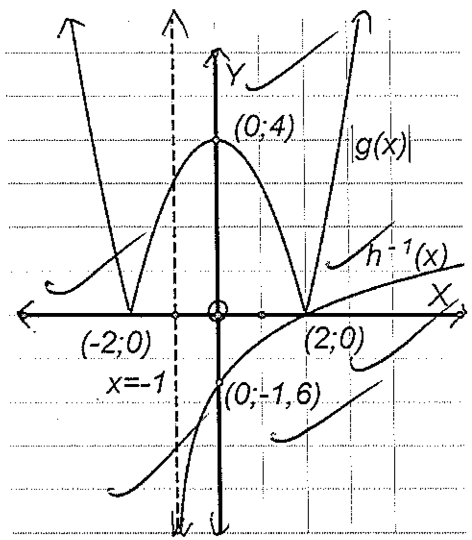
$$x=3.2^y-1$$
$$3.2^y=x+1$$
$$2^y=\frac{x+1}{3}$$
$$y=\log_2\left(\frac{x+1}{3}\right)$$

5.1.5 The Domain of h^{-1} (2)

$\dfrac{x+1}{3}>0$; $x\in R; x>-1$

5.2 Draw neat sketch graphs of $y=|g(x)|$ and $h^{-1}(x)$ on the same set of axes, using the DIAGRAM SHEET, and showing all asymptotes, intercepts with the axes and turning point(s). (6)

[23]



QUESTION 6

The following implicitly defined curve is given: $y^2 - yx = 7 - x \quad \left(y \neq \frac{x}{2}\right)$

6.1 Determine $\frac{dy}{dx}$. (7)

$$2y.y' - (y'x + y.1) = -1$$
$$2y.y' - y'x - y = -1$$
$$y'(2y - x) = y - 1$$
$$y' = \frac{y-1}{2y-x}$$

6.2 Determine the equation of the tangent to the curve at the point $(3;-1)$. (5)

[12]

$$m = \frac{-1-1}{2(-1)-3} = \frac{2}{5}$$
$$y = \frac{2}{5}x + c$$
$$(3;-1):\ -1 = \frac{2}{5}(3) + c = \frac{6}{5} + c$$
$$c = -2\frac{1}{5}$$
$$y = \frac{2}{5}x - 2\frac{1}{5}$$

QUESTION 7

7.1 Determine the following :

7.1.1 $f'(x)$ if $f(x) = \sin^2 2x.\sqrt{2x-1}$. (4)

$$f(x) = \sin^2 2x.(2x-1)^{\frac{1}{2}}$$
$$f'(x) = 2\sin 2x(\cos 2x)(2).\sqrt{2x-1} + \sin^2 2x.\left(\frac{1}{2}\right)(2x-1)^{-\frac{1}{2}}(2)$$
$$f'(x) = 4\sin 2x\cos 2x.\sqrt{2x-1} + \frac{\sin^2 2x}{\sqrt{2x-1}}$$

7.1.2 $f''(3)$ if $f(x) = (5-x)^5$ (5)

$$f'(x) = 5(5-x)^4.(-1)$$
$$= -5(5-x)^4$$
$$f''(x) = -5(4)(5-x)^3(-1)$$
$$= 20(5-x)^3$$
$$f''(3) = 20(5-3)^3$$
$$= 160$$

7.2 Determine the following:

7.2.1 The value of k, if $\int_1^4 (3x^2 + k)dx = 90$ (7)

$$\left[\frac{3x^3}{3} + kx\right]_1^4 = 90$$
$$[(4)^3 + k(4)] - [(1)^3 + k(1)] = 90$$
$$64 + 4k - 1 - k = 90$$
$$3k = 27$$
$$k = 9$$

7.2.2 $\int \sqrt{2+x}\,dx$ (4)

Let $u = 2 + x$; $\frac{du}{dx} = 1$

Then $du = dx$

$$Integral = \int u^{\frac{1}{2}} du = \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c$$
$$= \frac{2(2+x)^{\frac{3}{2}}}{3} + c = \frac{2\sqrt{(2+x)^3}}{3} + c$$

7.2.3 $\int \cos 5x.\cos 2x\,dx$ (6)

$$=\frac{1}{2}\int[\cos(5x-2x)+\cos(5x+2x)]dx$$

$$=\frac{1}{2}\int[\cos(3x)+\cos(7x)]dx$$

$$=\frac{1}{2}\left(\frac{1}{3}\sin 3x+\frac{1}{7}\sin 7x\right)+c$$

$$=\frac{1}{6}\sin 3x+\frac{1}{14}\sin 7x+c$$

7.2.4 $\int \cot^3 x.\operatorname{cosec}^2 x\,dx$ (5)

Let $u=\cot x$; $\frac{du}{dx}=-\operatorname{cosec}^2 x$

$du=-\operatorname{cosec}^2 x.dx$

$Integral=-\int u^3\,du$

$$=-\frac{u^4}{4}+c$$

$$=-\frac{\cot^4 x}{4}+c$$

7.2.5 $\int 3x.\sin x\,dx$ (using integration by parts here) (7)

[38]

Let $f(x)=3x$; $g'(x)=\sin x$

Then $f'(x)=3$; $g(x)=-\cos x$

Then $\int 3x.\sin x\,dx=3x(-\cos x)-\int(3)(-\cos x)dx$

$$=-3x\cos x+3\int \cos x\,dx$$

$$=-3x\cos x+3\sin x+c$$

QUESTION 8

The area enclosed by the curve of $f(x)=2x^2+1$, the $X-$axis and the lines $x=-1$ and $x=2$ must be determined by Riemann sums.

8.1 If 9 rectangles are used, show that the approximate area is $10\frac{1}{9}$. (6)

Width = $\frac{2-(-1)}{9}=\frac{1}{3}$

$x_k=-1+\frac{1}{3}k$

$$f(x_k)=f\left(-1+\frac{1}{3}k\right)$$

$$=2\left(-1+\frac{1}{3}k\right)^2+1$$

$$=2\left(1-\frac{2}{3}k+\frac{k^2}{9}\right)+1$$

$$=2-\frac{4}{3}k+\frac{2}{9}k^2+1$$

$$=3-\frac{4}{3}k+\frac{2}{9}k^2$$

$$A.Area=\frac{1}{3}\sum_{k=1}^{9}\left(3-\frac{4}{3}k+\frac{2}{9}k^2\right)$$

$$=\frac{1}{3}\left(3(9)-\frac{4}{3}.\frac{9(9+1)}{2}+\frac{2}{9}.\frac{9(9+1)(2(9)+1)}{6}\right)$$

$$=10\frac{1}{9}$$

8.2 For n rectangles, the area is given by $9+\frac{9}{n}+\frac{9}{n^2}$.

Determine the exact area. (2)

[8]

$$Area=\lim_{n\to\infty}\left(9+\frac{9}{n}+\frac{9}{n^2}\right)$$

$$=9$$

QUESTION 9

The equation $\sin 2x = x$ must be solved.

9.1 Show that a solution exists in the interval $\left(0{,}2;\frac{\pi}{2}\right)$. (3)

Let $f(x)=\sin 2x - x$

$f(0{,}2)=\sin 2(0{,}2)-0{,}2$
$=0{,}19>0$

$f\left(\frac{\pi}{2}\right)=\sin 2\left(\frac{\pi}{2}\right)-\frac{\pi}{2}$
$=-\frac{\pi}{2}<0$

Therefore a solution exists in the interval $\left(0{,}2;\frac{\pi}{2}\right)$.

9.2 Hence, determine this solution, correct to five decimal places, using the Newton-Raphson method, showing all the steps. (7)
[10]

$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$

$f'(x)=2\cos 2x-1$

$x_{n+1}=x_n-\frac{\sin(2x_n)-x_n}{2\cos(2x_n)-1}$

Let $x_1=1$

Therefore solution $x=0{,}94775$

QUESTION 10

The area enclosed by the graph of $f(x)=4-x$, the X-axis and the lines $x=1$ and $x=4$, is rotated about the X-axis. Determine the volume created by this rotation. [8]

$V=\pi\int_1^4 (4-x)^2\,dx$

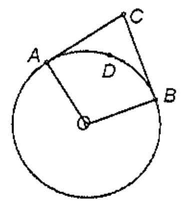
$=\pi\int_1^4 (16-8x+x^2)\,dx$

$=\pi\left[16x-4x^2+\frac{x^3}{3}\right]_1^4$

$=\pi\left(16(4)-4(4)^2+\frac{4^3}{3}\right)-\pi\left(16(1)-4(1)^2+\frac{1^3}{3}\right)$
$=9\pi$
$=28{,}27$

QUESTION 11

In the diagram, AC and BC are tangents to the circle with centre O.
$A\hat{O}B=\frac{2\pi}{3}$ *radians* and the radius is $5\,cm$.



11.1 Calculate the length of each tangent, giving reasons. (5)

$\Delta OAC \equiv \Delta OBC$ (rhs)
therefore $A\hat{O}C = B\hat{O}C$
$=\frac{\pi}{3}$

$\frac{AC}{5}=\tan\frac{\pi}{3}$

$AC=5\tan\frac{\pi}{3}$

$AC=5\sqrt{3}=8{,}66\ cm$
$BC=AC=8{,}66\ cm$ (tangents)

11.2 Hence, calculate the area enclosed by the tangents and the arc ADB. (7) [12]

$$\text{Area of } AOBC = 2\left(\frac{1}{2}OB.OC\right)$$
$$= 5(5\sqrt{3})\,cm^2$$
$$= 25\sqrt{3}\,cm^2$$
$$\text{Area of } ADBC = 25\sqrt{3} - \frac{1}{2}(5)^2\left(\frac{2\pi}{3}\right)\,cm^2$$
$$= 17{,}12\,cm^2$$

QUESTION 12

The base of a triangle has a length of $(4-x)$ units and the height is $\sqrt{x}$ units.

12.1 Calculate the value of x so that the area of the triangle has a maximum value. (8)

$$\text{Area} = \frac{1}{2}(4-x)\sqrt{x} = \frac{1}{2}(4-x)x^{\frac{1}{2}}$$
$$A(x) = 2x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{3}{2}}$$
$$A'(x) = x^{-\frac{1}{2}} - \frac{3}{4}x^{\frac{1}{2}} = 0$$
$$\frac{3}{4}x^{\frac{1}{2}} = \frac{1}{x^{\frac{1}{2}}}$$
$$\frac{3}{4}x = 1$$
$$x = \frac{4}{3} = 1\frac{1}{3}$$

12.2 Calculate this maximum area. (2) [10]

TOTAL : [200]

$$A_{max} = \frac{1}{2}\left(1-\frac{4}{3}\right)\sqrt{\frac{4}{3}}$$
$$A_{max} = \frac{8\sqrt{3}}{9} = 1{,}54\ units^2$$

SECTION B (TRANSFORMATION GEOMETRY AND MATRICES)

QUESTION 1

The matrices $A = \begin{pmatrix} 2 & -3 \\ 1 & 0 \\ -2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 1 & 0 \\ -1 & 2 & 3 \end{pmatrix}$ are given.

1.1 Calculate $A \times B$. (6)

$$A \times B = \begin{pmatrix} 8+3 & 2-6 & -9 \\ 4 & 1 & 0 \\ -8-1 & -2+2 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} 11 & -4 & -9 \\ 4 & 1 & 0 \\ -9 & 0 & 3 \end{pmatrix}$$

1.2 If $C = A \times B$, determine if C^{-1} is possible, showing the steps. (3) [9]

$$\det(C) = 11(3) - (-4)(12) - 9(9)$$
$$= 33 + 48 - 81$$
$$= 0$$
therefore C^{-1} is not possible

QUESTION 2

2.1 The matrix $A=\begin{pmatrix}3 & -2 & 4\\ -1 & 1 & -5\\ 1 & 1 & 0\end{pmatrix}$ is given.

2.1.1 Determine A^{-1}. (10)

$$"Minor"(A)=\begin{pmatrix}5 & 5 & -2\\ -4 & -4 & 5\\ 6 & -11 & 1\end{pmatrix} \quad CoFactor\ mat=\begin{pmatrix}5 & -5 & -2\\ 4 & -4 & -5\\ 6 & 11 & 1\end{pmatrix}$$

$$Adj(CoFactor\ mat)=\begin{pmatrix}5 & 4 & 6\\ -5 & -4 & 11\\ -2 & -5 & 1\end{pmatrix}$$

$$\det(A)=3(5)+2(5)+4(-2)$$
$$=17$$

$$A^{-1}=\frac{1}{17}\begin{pmatrix}5 & 4 & 6\\ -5 & -4 & 11\\ -2 & -5 & 1\end{pmatrix}$$

2.1.2 Hence, solve the system of equations: (6)

$$3x-2y+4z=-20$$
$$-x+y-5z=18$$
$$x+y=-1$$

$$\begin{pmatrix}3 & -2 & 4\\ -1 & 1 & -5\\ 1 & 1 & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ x\end{pmatrix}=\begin{pmatrix}-20\\ 18\\ -1\end{pmatrix}$$

$$AX=B$$
$$X=A^{-1}.B$$
$$=\frac{1}{17}\begin{pmatrix}5 & 4 & 6\\ -5 & -4 & 11\\ -2 & -5 & 1\end{pmatrix}\begin{pmatrix}-20\\ 18\\ -1\end{pmatrix}$$
$$=\frac{1}{17}\begin{pmatrix}-34\\ 17\\ -51\end{pmatrix}=\begin{pmatrix}-2\\ 1\\ -3\end{pmatrix}$$

Therefore $x=-2; y=1; z=-3$

2.2 The following equations are given:

$$2x+py-z=4$$
$$y-z=-5$$
$$x-y=3$$

Determine the value of p if there is no unique solution to the equations. (5)
[21]

$$\begin{vmatrix}2 & p & -1\\ 0 & 1 & -1\\ 1 & -1 & 0\end{vmatrix}=0$$

$$2(1)-p(1)-1(-1)=0$$
$$-p=-1-2=-3$$

Therefore $p=3$

QUESTION 3

The points $A(3;4)$, $B(-1;2)$, $C(2;-3)$, $D(6;-1)$ and $E(5;3)$ in the Cartesian plane are given.

3.1 Determine the coordinates of the images of each figure in each of the following, showing the relevant transformation matrices and using it in the transformations:

3.1.1 $ABCD$ after a rotation of $90°$ anticlockwise about the origin. (5)

$$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}3 & -1 & 2 & 6\\ 4 & 2 & -3 & -1\end{pmatrix}=\begin{pmatrix}-4 & -2 & 3 & 1\\ 3 & -1 & 2 & 6\end{pmatrix}$$

$A'=(-4;3)$, $B'=(-2;-1)$, $C'=(3;2)$, $D'=(1;6)$

3.1.2 $BCDE$ after a stretch with scale factor 3, parallel to the $Y-axis$, invariant line the $X-axis$. (5)

$$\begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix}\begin{pmatrix}-1 & 2 & 6 & 5\\ 2 & -3 & -1 & 3\end{pmatrix}=\begin{pmatrix}-1 & -2 & 6 & 5\\ 6 & -9 & -3 & 9\end{pmatrix}$$

$B''=(-1;6)$, $C''=(-2;-9)$, $D''=(6;-3)$, $E''=(5;9)$

3.2 The images after two consecutive transformations are $A'(4;-3)$, $B'(2;1)$, $C'(-3;-2)$, $D'(-1;-6)$ and $E'(3;-5)$.

Describe these transformations and write down the transformation matrices. (6)

Reflection in the line $y=x$; then reflection in the X-axis

$$T_1=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$T_2=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

3.3 If $\begin{pmatrix} \cos\theta & 1 \\ \sin\alpha & -1 \end{pmatrix} B = C$, determine the smallest positive values of θ and α.

(4)
[20]
TOTAL : [50]

$$\begin{pmatrix} \cos\theta & 1 \\ \sin\alpha & -1 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix}=\begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$-\cos\theta+2=2$ $\quad -\sin\alpha-2=-3$

$\cos\theta=0$ $\quad \sin\alpha=1$

$\theta=\frac{\pi}{2}$

$\alpha=\frac{\pi}{2}$