MARKING GUIDELINE

A.P. MATHEMATICS

PRELIM 2016

PAPER 1

QUESTION 1

Prove true for $n=1$. ✓

$LHS = 1^2+2^2 = 5$ ✓

$RHS = \frac{1(2+1)(4+1)}{3} = 5$ ✓

$\therefore LHS = RHS$.

Assume true for $n=k$. ✓

$1^2+2^2+\ldots+(2k)^2 = \frac{k(2k+1)(4k+1)}{3}$ ✓ ✓

Prove true for $k+1$ ✓

$LHS = 1+2^2+\ldots\ldots+(2(k+1))^2$ ✓

$= 1^2+2^2+\ldots+(2k)^2+(2k+1)^2+(2k+2)^2$ ✓

$= \frac{k(2k+1)(4k+1)}{3} + (2k+1)^2+(2k+2)^2$

$= \frac{k(8k^2+6k+1)+3(4k^2+4k+1)+3(4k^2+8k+4)}{3}$

$= \frac{8k^3+30k^2+37k+15}{3}$ ✓

$RHS = \frac{(k+1)((2(k+1)+1)(4(k+1)+1)}{3}$ ✓

$= \frac{(k+1)(2k+3)(4k+5)}{3}$

$= \frac{8k^3+30k^2+37k+15}{3}$ ✓ (12)

By PMI, it is true for $n \geq 1$ for $n \in \mathbb{N}$.

QUESTION 2

a) 1. $|4x-2| \geq |2-3x|$

$4x-2 = 2-3x$

$7x = 4$ ✓

$x = 4/7$.

$4x-2 = -2+3x$

$x = 0$ ✓ ✓

$x \leq 0$ or $x \geq 4/7$ (5)

2. $z^3+xz+10=0$ ✓

$z = 1-2i$, $z = 1+2i$ ✓ ✓

$(z^2-2z+5)(z+2)=0$

$\therefore -4z+5z = xz$ ✓

$+1 = x$ ✓ (6)

b) $\ln\left(\frac{a+b}{3}\right) = \frac{1}{2}(\ln a + \ln b)$ ✓

$\ln\left(\frac{a+b}{3}\right) = \ln(ab)^{1/2}$ ✓

$\frac{a+b}{3} = (ab)^{1/2}$ ✓ (6)

$\left(\frac{a+b}{3}\right)^2 = ab$ ✓ ✓

$a^2 + 2ab + b^2 = 9ab$

$a^2 + b^2 - 7ab = 0$

c) $f(x) = \sqrt{x+2}$ $g(x) = \ln(1-x^2)$

$g(f(x)) = \ln(1-(\sqrt{x+2})^2)$ ✓

$= \ln(-x-1)$ ✓✓

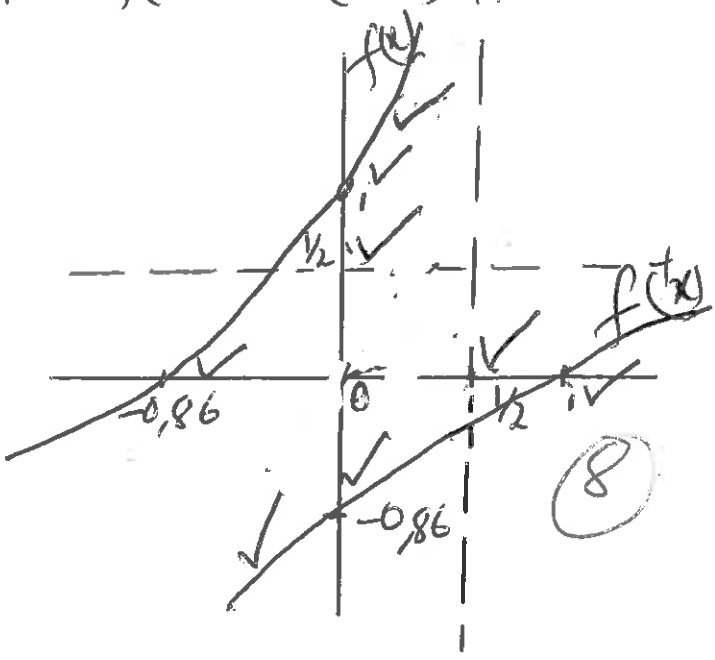
$-x-1 > 0$

$x < -1$ ✓ (6)

$x > -2$ ✓✓

$-2 < x < -1$

d. $f(x) = -\ln(1-2x) + 7$



(8)

2. $x = -\ln(1-2y) + 7$ ✓

$1 - x = \ln(1-2y)$

$e^{1-x} = 1-2y$ ✓ (4)

$f^{-1}(x) = \frac{1}{2} - \frac{e^{1-x}}{2}$ ✓✓

QUESTION 3:

a) $\sin\theta = \frac{r}{R-r}$ ✓✓

$R - r = \frac{r}{\sin\theta}$ ✓✓ (4)

$R = r\left(\frac{1}{\sin\theta} + 1\right)$

b) $R = r\left(\frac{1}{\sin \pi/6} + 1\right)$

$= 3r$ ✓

Area of $\odot = \pi r^2$ ✓

Area of sector $= \frac{2\theta \cdot R^2}{2}$

$= \pi/6 \cdot (3r)^2$ ✓

$= \frac{3\pi r^2}{2}$ ✓ (6)

∴ fraction of the garden

$= \frac{\pi r^2}{\frac{3\pi r^2}{2}} = 2/3$ ✓✓

c) $R = 15$ ✓

$r = 5$ ✓ $\quad S = \sqrt{10^2 - 5^2}$

$= 5\sqrt{3}$ ✓

Length of sector $= \frac{2\pi}{3} \cdot 5 = 10\pi/3$ ✓

∴ Perimeter of shaded region $= 2(5\sqrt{3}) + 10\pi/3$ (5)

$= 27{,}8$ ✓✓

QUESTION 4

Q1. $\lim_{x\to 0} \frac{1-\cos 2x}{5x^2}$

$= \lim_{x\to 0} \frac{1-(1-2\sin^2 x)}{5x^2}$ ✓✓

$= \lim_{x\to 0} \frac{2\sin^2 x}{5x^2}$ ✓ (6)

$= \lim_{x\to 0} \frac{2}{5}\left(\frac{\sin^2 x}{x^2}\right) = 2/5$ ✓✓

2. $\lim_{x\to -\infty} \frac{x-2}{\sqrt{x^2+1}}$

$= \lim_{x\to -\infty} \frac{1 - 2/x}{\sqrt{1 + 1/x^2}}$ ✓✓ $= -1$ ✓

(6)

b) $\frac{8x+13}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$ ✓✓

$8x+13 = A(x+2)^2 + B(x+2)(x-1) + C(x-1)$

Sub $x=1$: $21 = 9A$ ✓

$7/3 = A$ ✓

Sub $x=-2$ $-3 = -3C$ ✓ (8)

$C = 1$ ✓

Sub $x=0$: $13 = 4A - 2B - C$

$7/3 = B$ ✓

∴ $= \frac{7}{3(x-1)} + \frac{7}{3(x+2)} + \frac{1}{(x+2)^2}$

QUESTION 5:

$$g(x) = \begin{cases} -2|x+1| & x \leq 1 \\ 3 & 1 < x < 3 \\ 6-2x & x \geq 3 \end{cases}$$

a) $g(x)$ continuous and differentiable.

$x \in \mathbb{R}$ $x \neq -1$ $x \neq 1$: $x \neq 3$.

or $x < -1$ or $-1 < x < 1$ or $1 < x < 3$ or $x > 3$ (6)

b) $\lim_{x \to 1} -2|x+1| = -4$.

$\lim_{x \to 1} 3 = 3$

At $x = 1$ jump discontinuity

$\lim_{x \to 3} 3 = 3$

$\lim_{x \to 3} (6-2x) = 0$

∴ At $x = 3$ jump discontinuity (6)

QUESTION 6

a) $y = \frac{2x-6}{x+2}$

$\frac{dy}{dx} = \frac{(x+2)2 - (2x-6)1}{(x+2)^2}$

$= \frac{2x+4-2x+6}{(x+2)^2}$

$= \frac{10}{(x+2)^2}$ (5)

$\therefore k = 10$

b) $Q(0:-3)$

$P(3:0)$

$\frac{dy}{dx} = \frac{10}{(5)^2} = 2/5$

$\therefore m_{PR} = -5/2$

$y = -5/2 x + c$

$0 = -5/2(3) + c$ (6)

$y = -5/2 x + \frac{15}{2}$

$\therefore R(0 : 15/2)$

$\therefore$ length of $QR = \frac{21}{2}$ (10,5)

QUESTION 7

a) $f(x) = \dfrac{4}{(\frac{1}{2}x-3)^{50}} = 4(\frac{1}{2}x-3)^{-50}$

$f'(x) = 4(-50)(\frac{1}{2}x-3)^{-51}(\frac{1}{2})$

$f''(x) = 4(-50)(-51)(\frac{1}{2}x-3)^{-52}(\frac{1}{2})^2$

$f^{n}(x) = 4(-1)^n \dfrac{(49+n)!}{49!}(\frac{1}{2}x-3)^{-50-n}(\frac{1}{2})^n$ (8)

b) $x^3 - 6xy - ky^3 = a$

Point $(0;1)$ $\quad 0 - 0 - k = a$

$a = -k$

$3x^2 - 6x\dfrac{dy}{dx} - 6y - 3ky^2\dfrac{dy}{dx} = 0$

$3x^2 - 6y = 6x\dfrac{dy}{dx} + 3ky^2\dfrac{dy}{dx}$

$\dfrac{3x^2-6y}{6x+3ky^2} = \dfrac{dy}{dx}$

$\dfrac{3(0)-6(1)}{6(0)+3k(1)} = -1$

$-6 = -3k$

$k = 2$

$a = -2$ (12)

QUESTION 8

$f(x) = \dfrac{x^2 - 3/2\,x}{2x+1}$

a) $f(0) = 0$

$f(x) = 0$

$x = 0 \quad x = 3/2$

$f'(x) = \dfrac{(2x+1)(2x-3/2) - (x^2-3/2x)2}{(2x+1)^2}$

$4x^2 - x - 3/2 - 2x^2 + 3x = 0$ (4)

$2x^2 + 2x - 3/2 = 0$

$x = 1/2 \quad x = -3/2$

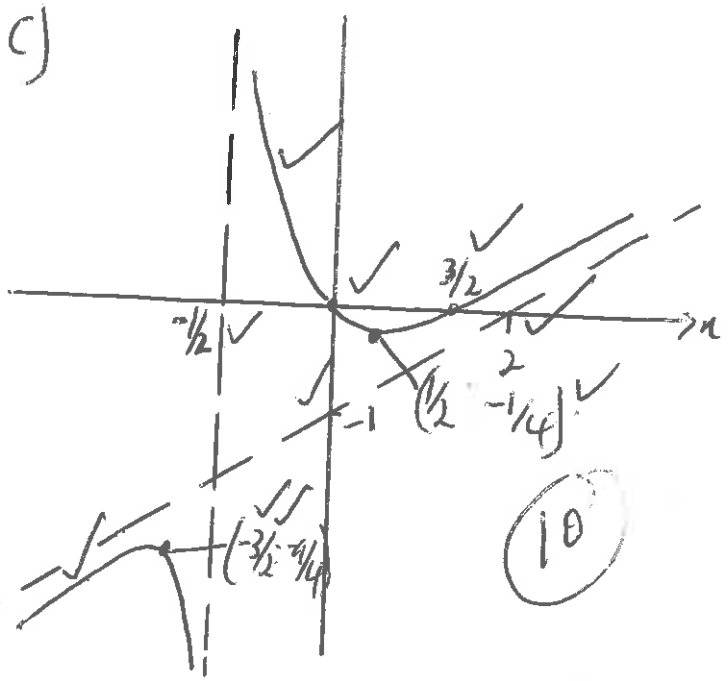
$y = -1/4 \quad y = -9/4$

b) Asymptotes: $x = -1/2$ (VA)

OA: $(2x+1)(\frac{1}{2}x-1) + R = x^2 - 3/2\,x$ (6)

$y = \frac{1}{2}x - 1$

c)



(10)

QUESTION 4

a) $\int_0^2 (x\sqrt{x^2+1})\,dx$

$\text{Area} = \frac{1}{2}\left[f(0)+2f(\tfrac{2}{5})+2f(\tfrac{4}{5})+2f(\tfrac{6}{5})\right.$

$\left.+2f(\tfrac{8}{5})+f(2)\right]\tfrac{2}{5}$

$= \frac{1}{2}\left[0+2(0,431)+2(1,024)+\right.$

$\left.2(1,874)+2(3,014)+4,472\right](\tfrac{2}{5})$

$= 3,434$ (8)

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b1. $\int \sin(\tfrac{1}{2}x-3)\cdot\cos(3x+2)\,dx$ (7)

$= \frac{1}{2}\int \sin(\tfrac{7}{2}x-1)+\sin(-\tfrac{5}{2}x-5)\,dx$

$= \frac{1}{2}\left[\frac{-\cos(\frac{7}{2}x-1)}{7/2} - \frac{\cos(-\frac{5}{2}x-5)}{-5/2}\right]+c$

$= -\frac{1}{7}\cos(\tfrac{7}{2}x-1)+\frac{1}{5}\cos(-\tfrac{5}{2}x-5)+c$

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2. $\int x\tan(x)^2\sec(x^2)\,dx$

let $u = x^2$ $du = 2x\,dx$

$= \frac{1}{2}\int \tan u \sec u\,du$ (7)

$= \frac{1}{2}\sec u + c$

$= \frac{1}{2}\sec(x)^2 + c$

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3. $\int_{-1}^{1} \frac{x+1}{(x^2+2x+2)^3}\,dx$

$u = x^2+2x+2$

$du = (2x+2)\,dx$

$\frac{du}{2} = (x+1)\,dx$ (8)

$\int \frac{du}{2u^3}$

$= \frac{1}{2}\int u^{-3}\,du = \frac{1}{2}\frac{u^{-2}}{-2}+c$

$= -\frac{u^{-2}}{4}\Big|_{-1}^{5}$

$= -\frac{1}{100}+\frac{1}{4} = \frac{6}{25}$

c) $y = 6-2x-x^2$ $y = x+6$

$6-2x-x^2 = x+6$

$-x^2-3x = 0$

$x = 0$ $x = -3$

$V = \pi\int_{-3}^{0}(6-2x-x^2)^2\,dx - \pi\int_{-3}^{0}(x+6)^2\,dx$

$= \pi\int_{-3}^{0}(36-24x-8x^2+4x^3+x^4)\,dx$

$-\pi\left[\frac{(x+6)^3}{3}\right]_{-3}^{0}$

$= \pi\left[36x-12x^2-\frac{8x^3}{3}+x^4+\frac{x^5}{5}\right]_{-3}^{0}$

$-\pi\frac{(x+6)^3}{3}\Big|_{-3}^{0}$ (8)

$= \frac{243\pi}{5}$ (152,68)

9d.

$y = 2\sin x + 4\cos x$

$A(0;4)$ ✓

$\frac{dy}{dx} = 2\cos x - 4\sin x = 0$ ✓ ✓

$2\cos x = 4\sin x$

$\frac{1}{2} = \tan x$

$x = 0,46$ ✓ ✓

$\text{Area} = \int_0^{0,46} (2\sin x + 4\cos x - 4)\,dx$ ✓ ✓

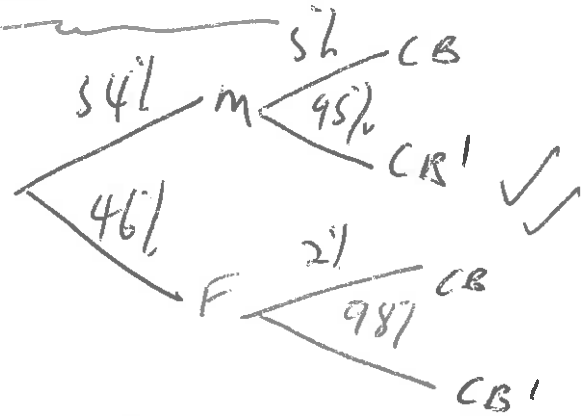
$= -2\cos x + 4\sin x - 4x \Big|_0^{0,46}$ ✓ ✓ ✓

$= 0,144$ ✓ (12)

PAPER 2 – STATISTICS

QUESTION 1

a)



$P(M/CB) = \frac{P(M \cap CB)}{P(CB)}$ (6)

$= \frac{54\% \times 5\%}{54\% \times 5\% + 46\% \times 2\%}$ ✓ ✓

$= 0,7459$ ✓ ✓

b) $^{2}C_0\,{}^{16}C_6 + {}^{2}C_2\,{}^{16}C_4$ ✓ ✓

$= 9828$ ✓ ✓ (6)

or $^{18}C_6 - {}^{2}C_1\,{}^{16}C_5$

$= 9828$

QUESTION 2

a1. $3c + 4c + 5c + 16c = 1$ ✓

$c = \frac{1}{18}$ ✓ (3)

2. Mode = 4. ✓ (2)

b. $c\int_{-1}^{1}(1-x^2)\,dx = 1$ ✓

$c\left[x - \frac{x^3}{3}\right]_{-1}^{1} = 1$ ✓ (6)

$c\left[\frac{2}{3} - \left(-\frac{2}{3}\right)\right] = 1$

$\frac{4}{3}c = 1$ ✓

$c = 3/4$ ✓

b. $n = 24$.

1. $P(X=2) = {}^{24}C_2 (3\%)^2 (97\%)^{22}$ ✓✓

$= 0,1271$ ✓ (6)

2. $P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$

$= {}^{24}C_1 (0,03)^1 (0,97)^{23} + (0,97)^{24} + {}^{24}C_2 (0,03)^2 (0,97)^{22} + {}^{24}C_3 (0,03)^3 (0,97)^{21}$

$= 0,9947$ ✓ (8)

QUESTION 3:

a1. $P(X=0) = \frac{{}^{50}C_0 \; {}^{450}C_{12}}{{}^{500}C_{12}}$ ✓ (6)

$= 0,2783$ ✓

2. $P(X \geq 2) = 1 - [P(X=0) + P(X=1)]$

$= 1 - \left[\frac{{}^{100}C_0 \; {}^{400}C_{12} + {}^{100}C_1 \; {}^{400}C_{11}}{{}^{500}C_{12}}\right]$ ✓

$= 0,7286$ ✓ (8)

QUESTION 4:

a) $X \sim N(2; 3^2)$

$P(2 < X < a) = 0,4115$

$P\left(0 < X < \frac{a-2}{3}\right) = 0,4115$

$\Phi(0,4115) = 1,35$

$\frac{a-2}{3} = 1,35$ (6)

$a = 6,05$

b) $X \sim N(500; 100^2)$

$P(X > 585)$

$= P\left(Z > \frac{585-500}{100}\right)$

$= P(Z > 0,85)$

$= 0,5 - 0,3023$ (8)

$= 0,1977$

$= 19,77\%$

He will be admitted since he is in the last 30th percentile

c) $P(X < 28) = 0,69$

$P\left(Z < \frac{28-\mu}{\delta}\right) = 0,69$

$z = 0,5$

$28 - \mu = 0,5\delta$

$P(X < 35) = 0,9$

$P\left(z < \frac{35-\mu}{\delta}\right) = 0,9$

$z = 1,28$

$35 - \mu = 1,28\delta$

$7 = 0,78\delta$ (10)

$8,974 = \delta$

$8,974 = \delta$

$\mu = 23,513$

QUESTION 5

a. $X \sim N(1400; 220)$

1. $\mu = \bar{x} \pm z \frac{\delta}{\sqrt{n}}$

$1400 \pm 2,58 \cdot \frac{220}{\sqrt{32}}$ ✓✓ ✓✓ ✓✓ ✓✓ (8)

$[1299,66 ; 1500,34]$

2. Parameter ✓ (1)

b. $\hat{p} \pm z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

$= \frac{692}{1048} \pm 1,96\sqrt{\frac{\frac{692}{1048} \cdot \frac{356}{1048}}{1048}}$ ✓✓ ✓✓ ✓✓ ✓✓

$= [0,6316 ; 0,6890]$ ✓✓ (9)

QUESTION 6

$nPr = 6720$

$\frac{n!}{(n-r)!} = 6720$ ✓✓ (1)

$nCr = 56$

$\frac{n!}{(n-r)!\,r!} = 56$ ✓✓ (2)

(1) ÷ (2) (7)

$r! = 120$ ✓

$r = 5$ ✓✓