Form 5 AP prelim paper 1 2016

Question 1

$5^n - 2^n$ divisible by 3 for $n \in \mathbb{N}$

Let $n=1$: $\therefore 5^1 - 2^1$

$= 3$ which is divisible by 3

$\therefore$ True for $n=1$

Assume true for $n=k$: $\therefore 5^k - 2^k = 3.r$

$\therefore 5^k = 3.r + 2^k$

Let $n=k+1$: $5^{k+1} - 2^{k+1}$

$= 5^k.5^1 - 2^k.2^1$

$= (3r + 2^k).5^1 - 2^k.2^1$

$= 15r + 5.2^k - 2.2^k$

$= 15r + 3.2^k$

$= 3(5r + 2^k)$

which is divisible by 3

$\therefore$ True for $n=k+1$

$\therefore$ True for all $n \in \mathbb{N}$ (15)

Question 2

2.1 (a) $\log 2x + \log_{0,1}(x-20) - 1 = 0$

$\log 2x - \log(x-20) = 1$

$\log\left[\frac{2x}{x-20}\right] = 1$

$\frac{2x}{x-20} = 10$

$2x = 10x - 200$

$-8x = -200$

$x = 25$ (5)

(b) $e^x - \frac{6}{e^x} + 5 = 0$

$e^{2x} + 5.e^x - 6 = 0$

$(e^x + 6)(e^x - 1) = 0$

$e^x = -6$ or $e^x = 1$

N/A $\quad x = 0$ (6)

(c) $|x|^2 - 4|x| = 12$

$|x|^2 - 4|x| - 12 = 0$

$(|x| - 6)(|x| + 2) = 0$

$|x| = 6$ or $|x| = -2$

$x = \pm 6$ $\quad$ N/A. (6)

2.2. $f(x) = e^{x+2} - 1$

(a) Inverse: $x = e^{y+2} - 1$

$x + 1 = e^{y+2}$

$\ln(x+1) = y + 2$

$y = \ln(x+1) - 2$ (4)

(b) x-int $(y=0)$: $0 = \ln(x+1) - 2$ ✓

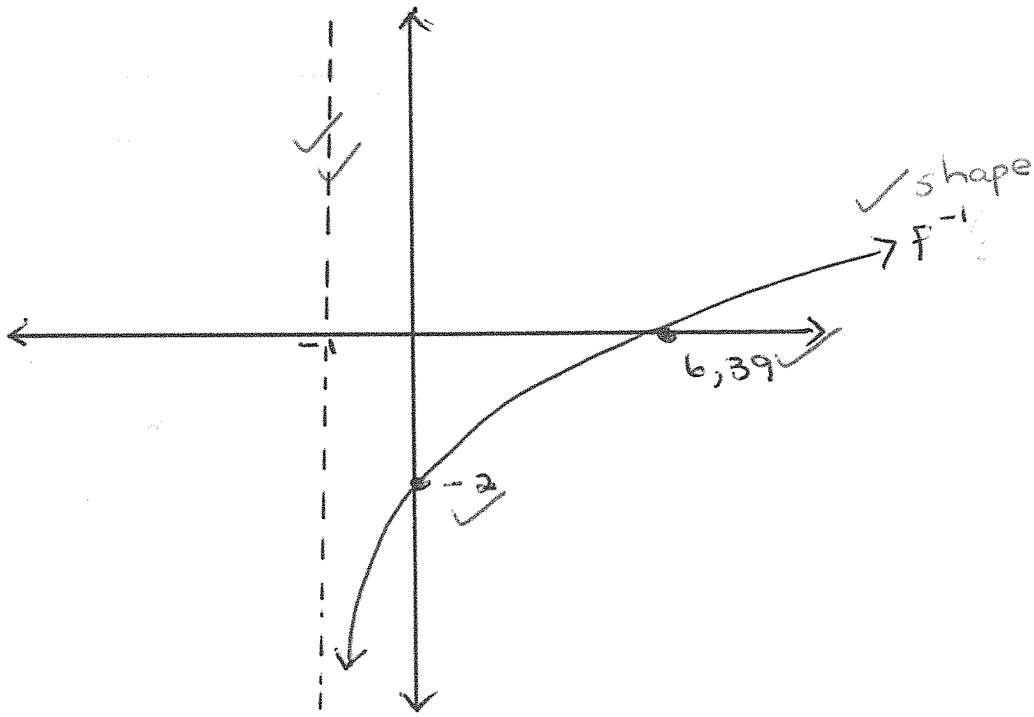
$2 = \ln(x+1)$

$e^2 - 1 = x$ ✓

$\underline{6{,}39 = x}$ ✓

y-int $(x=0)$: $y = \ln(0+1) - 2$ ✓

$\underline{y = -2}$ ✓



(9)

Question 3

(a) Factors: $(x-1+2i)$ and $(x-1-2i)$

$\therefore (x-1+2i)(x-1-2i)$

$= x^2 - x - 2xi - x + 1 + 2i + 2ix - 2i + 4$

$= x^2 - 2x + 5$

$\therefore x^2 - 2x + 5$ is a factor of $g(x)$

(4)

(b) $\dfrac{x-10}{2x^2+5x-3}$

$= \dfrac{x-10}{(2x-1)(x+3)}$

$\therefore \dfrac{A}{2x-1} + \dfrac{B}{x+3} = \dfrac{x-10}{(2x-1)(x+3)}$

$A(x+3) + B(2x-1) = x-10$

$Ax + 3A + 2Bx - B = x - 10$

$\therefore A + 2B = 1 \quad -- ①$

$3A - B = -10 \quad -- ②$

$A = 1 - 2B$

$\therefore 3(1-2B) - B = -10$

$3 - 6B - B = -10$

$-7B = -13$

$B = \dfrac{13}{7}$

$\therefore A = 1 - 2\left(\dfrac{13}{7}\right)$

$A = \dfrac{-19}{7}$

$\therefore \dfrac{\frac{-19}{7}}{2x-1} + \dfrac{\frac{13}{7}}{x+3}$

(10)

Question 4

(a) area sector OBD $= \frac{1}{2}(5)^2(\frac{5}{9}\pi)$

$= 21,82 \text{ cm}^2$

area $\triangle BOD = \frac{1}{2}(5)^2 \sin(\frac{5}{9}\pi)$

$= 12,31 \text{ cm}^2$

$\therefore$ Area segment $= 21,82 - 12,31$

$= \underline{9,51 \text{ cm}^2}$ (7)

(b) Area $\triangle BCO = \frac{1}{2}(4)(3)$

$= 6 \text{ cm}^2$

Area sector OAC $= \frac{1}{2}(3)^2(\frac{5}{18}\pi)$

$= 3,93 \text{ cm}^2$

$\therefore$ Area ACB $= 6 - 3,93$

$= \underline{2,07 \text{ cm}^2}$ (8)

Question 5

Continuous if $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$ ✓✓

$\therefore \quad a = b - 2$ ✓✓

Differentiable if $\lim_{x \to 1^-} f'(x) = \lim_{x \to 1^+} f'(x)$ ✓✓

$\therefore \quad -a = -2$ ✓✓

$\underline{a = 2.}$ ✓✓

$\therefore \quad 2 = b - 2$ ✓

$\underline{4 = b}$ ✓

(12)

Question 6

6.1. $f(x) = \dfrac{x^2+2x}{-4x+8}$

(a) x-int : $0 = \dfrac{x^2+2x}{-4x+8}$

$0 = x(x+2)$ ✓

$\underline{x=0}$ ✓ or $\underline{x=-2}$ ✓

y-int: $y = \dfrac{(0)^2+2(0)}{-4(0)+8}$

$\underline{y=0}$ ✓

(4)

(b) Vertical asymptotes : $x=2$ ✓

horizontal asymptotes : none. ✓

Oblique asymptotes :

$$m = \lim_{x \to \infty} \frac{f(x)}{x}$$

$$= \lim_{x \to \infty} \frac{x^2+2x}{-4x^2+8x} \checkmark$$

$$= \lim_{x \to \infty} \frac{1+\frac{2}{x}}{-4+\frac{8}{x}} \checkmark$$

$$= -\frac{1}{4} \checkmark$$

$$c = \lim_{x \to \infty} (f(x) - mx)$$

$$= \lim_{x \to \infty} \left(\frac{x^2+2x}{-4x+8} - \left(-\frac{1}{4}x\right) \right) \checkmark$$

$$c = \lim_{x \to \infty} \left(\frac{x^2+2x}{-4x+8} + \frac{x}{4} \right)$$

$$= \lim_{x \to \infty} \left(\frac{4x^2+8x+(-4x^2+8x)}{4(-4x+8)} \right)$$ ✓

$$= \lim_{x \to \infty} \frac{16x}{-16x+32}$$ ✓

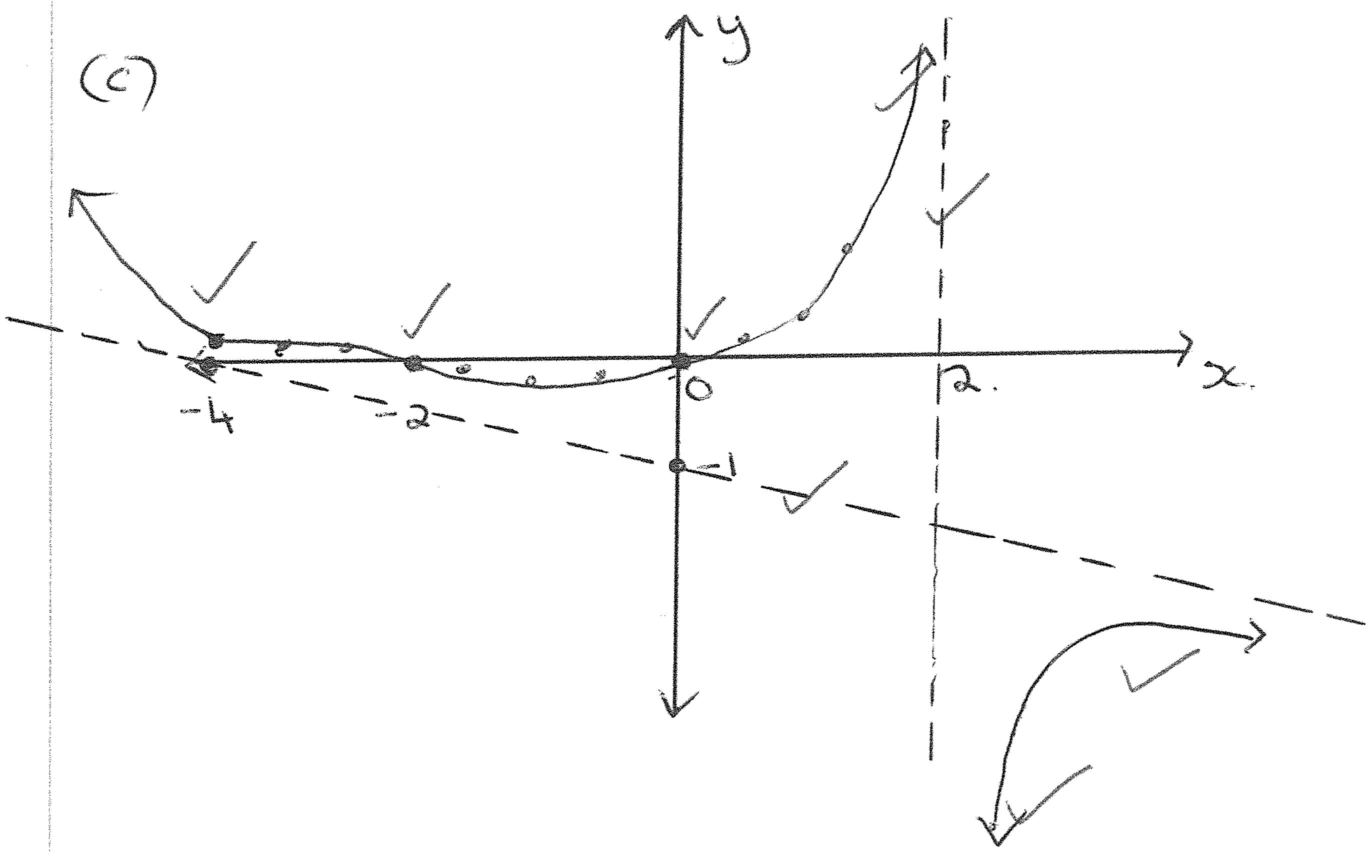
$$= \lim_{x \to \infty} \frac{16}{-16+32/x}$$ ✓

$$= -1$$

$\therefore \underline{y = -\frac{1}{4}x - 1}$ ✓

(10)

(c)



(8)

Question 7

7.1. $$\lim_{x \to \infty} \left(\frac{\sqrt{9x^2 - 3x} + 2}{9x - 5} \right) \cdot \left(\frac{\frac{1}{\sqrt{x^2}}}{\frac{1}{x}} \right)$$

$$= \lim_{x \to \infty} \frac{\sqrt{9 - 3/x} + 2/x}{9 - 5/x}$$

$$= \frac{\sqrt{9}}{9}$$

$$= \frac{1}{3}$$

(4)

7.2. $$f(x) = \frac{1}{\sqrt{x+1}}$$

$$f'(x) = \lim_{h \to 0} \left(\frac{\frac{1}{\sqrt{x+h+1}} - \frac{1}{\sqrt{x+1}}}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{\sqrt{x+1} - \sqrt{x+h+1}}{(\sqrt{x+h+1})(\sqrt{x+1})} \right) \cdot \frac{1}{h}$$

$$= \lim_{h \to 0} \left(\frac{\sqrt{x+1} - \sqrt{x+h+1}}{(\sqrt{x+h+1})(\sqrt{x+1})} \right) \left(\frac{\sqrt{x+1} + \sqrt{x+h+1}}{\sqrt{x+1} + \sqrt{x+h+1}} \right) \cdot \frac{1}{h}$$

$$= \lim_{h \to 0} \left(\frac{x+1 - (x+h+1)}{(\sqrt{x+h+1})(\sqrt{x+1})(\sqrt{x+1} + \sqrt{x+h+1})} \right) \cdot \frac{1}{h}$$

$$= \lim_{h \to 0} \left(\frac{-h}{(\sqrt{x+h+1})(\sqrt{x+1})(\sqrt{x+1} + \sqrt{x+h+1})} \right) \cdot \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{-1}{(\sqrt{x+h+1})(\sqrt{x+1})(\sqrt{x+1} + \sqrt{x+h+1})}$$

$$= \frac{-1}{(x+1)(2\sqrt{x+1})}$$

$$= \frac{-1}{2(x+1)^{3/2}}$$

(10)

7.3 $f(x) = \sin(\tan(2x))$
$g(x) = x^{2/3}(x + \sqrt[4]{x})$
$h(x) = \dfrac{2x}{\cos x}$

(a) $f'(x) = \cos(\tan(2x)) \cdot \sec^2(2x) \cdot 2$

$g'(x) = \frac{2}{3}x^{-1/3}(x + \sqrt[4]{x}) + x^{2/3}(1 + \frac{1}{4}x^{-3/4})$

$h'(x) = \dfrac{2 \cdot \cos x - 2x \cdot (-\sin x)}{\cos^2 x}$ (10)

(b) $f'(\pi) = 2$

$g'(1) = 2{,}58$

$h'(\pi) = -2$

$\therefore \underrightarrow{g'(1) \; ; \; f'(\pi) \; ; \; h'(\pi)}$ (4)

7.4. $\cos y = x \qquad 0 < y < \pi/2.$

(a) $-\sin y \cdot \dfrac{dy}{dx} = 1$

$\dfrac{dy}{dx} = \dfrac{-1}{\sin y}$ (5)

(b) $\dfrac{dy}{dx} = \dfrac{-1}{\sin y}$

$\cos y = 0{,}5$
$\therefore y = \cos^{-1}(0{,}5)$
$y = \pi/3$

$$\therefore \frac{dy}{dx} = \frac{-1}{\sin \pi/3}$$

$$= -\frac{2\sqrt{3}}{3}.$$

(5)

Question 8

(a) $f(x) = 3(x-2)^2 - 1 - \frac{4}{x}$

$f(2) = 3(2-2)^2 - 1 - \frac{4}{2}$
$= -3$

$f(3) = 3(3-2)^2 - 1 - \frac{4}{3}$
$= 2/3$

(3)

$\therefore$ Change of sign $\therefore$ solution on $x \in (2,3)$

(b) $x_{r+1} = x_r - \dfrac{3(x-2)^2 - 1 - \frac{4}{x}}{6(x-2) + \frac{4}{x^2}}$

$x_1 = 2{,}5$
$x_2 = 3{,}008241758\ldots$
$x_3 = 2{,}897330865\ldots$
$x_4 = 2{,}891354053\ldots$
$x_n = 2{,}891337\ldots$

(7)

Question 9

$$\text{Width} = \frac{2}{n}$$

$$x_i = 0 + i\left(\frac{2}{n}\right)$$

$$= \frac{2i}{n}$$

$$\therefore f(x_i) = 2\left(\frac{2i}{n}\right)^2 + 1$$

$$= \frac{8i^2}{n^2} + 1$$

$$\text{Area} = \lim_{n \to \infty} \left(\frac{2}{n}\right) \sum_{i=1}^{n} \left(\frac{8i^2}{n^2} + 1\right)$$

$$= \lim_{n \to \infty} \left(\frac{2}{n}\right)\left(\frac{8}{n^2} \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} 1\right)$$

$$= \lim_{n \to \infty} \left(\frac{2}{n}\right)\left(\frac{8}{n^2}\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right) + n\right)$$

$$= \lim_{n \to \infty} \left(\frac{2}{n}\right)\left(\frac{8n}{3} + 4 + \frac{4}{3n} + n\right)$$

$$= \lim_{n \to \infty} \left(\frac{16}{3} + \frac{8}{n} + \frac{8}{3n^2} + 2\right)$$

$$= \frac{22}{3} \text{ units}^2$$

(15)

Question 10.

10.1. (a) $\int (2x+1) \cdot \cos 2x \, dx$

Let $u = 2x+1$ ✓ and $dv = \cos 2x \, dx$ ✓

$\therefore du = 2 \, dx$ ✓ $\quad \int dv = \int \cos 2x \, dx$

$v = \frac{\sin 2x}{2}$ ✓

$\therefore \int (2x+1) \cdot \cos 2x \, dx$

$= (2x+1)\left(\frac{\sin 2x}{2}\right)$ ✓ $- \int \left(\frac{\sin 2x}{2}\right) \cdot 2 \, dx$ ✓

✓

$= (2x+1)\left(\frac{\sin 2x}{2}\right) - \left(-\frac{\cos 2x}{4} \cdot 2\right) + c$

$= (2x+1)\left(\frac{\sin 2x}{2}\right) + \frac{\cos 2x}{2} + c$. ✓ (8)

(b) $\int x^2 \cdot \sec^2(2x^3) \, dx$

Let $u = 2x^3$ ✓

$du = 6x^2 \, dx$ ✓

$\frac{1}{6x^2} \cdot du = dx$. ✓

$\therefore \int x^2 \cdot \sec^2 u \cdot \frac{1}{6x^2} \, du$ ✓

✓

$= \frac{1}{6} \int \sec^2 u \, du$ ✓

$= \frac{1}{6} \cdot \tan u + c$ ✓

$= \frac{1}{6} \tan(2x^3) + c$. ✓ (8)

10.2. $V_1 = \pi \int_0^1 (1-x^2)^2 \, dx$

$= \pi \int_0^1 (1 - 2x^2 + x^4) \, dx$

$= \pi \left[x - \frac{2x^3}{3} + \frac{x^5}{5} \right]_0^1$

$= \pi \left[1 - 2/3 + \frac{1}{5} - 0 \right]$

$= \frac{8}{15} \pi$

$V_2 = \pi \int_0^1 ((x-1)^2)^2 \, dx$

$= \pi \int_0^1 (x-1)^4 \, dx$

$= \pi \left[\frac{(x-1)^5}{5} \right]_0^1$

$= \pi \left[0 - \left(\frac{-1}{5} \right)^5 \right]$

$= \pi \cdot \frac{1}{3125}$

$\therefore V = \frac{8}{15}\pi - \frac{1}{3125}\pi$

$= \frac{4997}{9375} \pi$

(18)