Gr 12 APM MEMO PRELIM. 2019

Q1

Step 1: let $n = 1$

$\therefore 3 \mid 5^1 - 2^1 = 3$

the rule holds true for $n=1$

Step 2: let us assume the rule holds true for some $n = k$.

$\therefore 3r = 5^k - 2^k \quad ; r \in \mathbb{Z}$

Step 3: let $n = k+1$

$= 5^{k+1} - 2^{k+1}$

From our assumption,

$= (3r + 2^k)5 - 2^{k+1}$

$= 5 \times 3r + 5^1 \times 2^k - 2^1 \times 2^k$

$= 5 \times 3r + 3 \times 2^k$

$= 3(5r + 2^k)$

$\therefore 3 \mid 5^{k+1} - 2^{k+1}$

Step 4 By the principle of Mathematical induction the rule holds true $\forall\ n \in \mathbb{N}$.

Q2

2.1 (a)

$$\log 3x + \frac{\log(x-30)}{\log_{10} 0{,}1} - 1 = 0$$

$$\log 3x - \log(x-30) - \log_{10} 10 = 0$$

$$\log \frac{3x}{x-30} = \log 10$$

$$\frac{3x}{x-30} = 10$$

$$3x = 10x - 300$$

$$300 = 7x$$

$$\frac{300}{7} = x$$

6

(b) $\times e^x$

$$e^{2x} - 5e^x - 6 = 0$$

$$(e^x - 6)(e^x + 1) = 0$$

$e^x = 6$ and $e^x = -1$

NIV

$\ln 6 = x$

$x = 1{,}79175$

$\simeq 1{,}79$

6

(c)

let $k = |x|$

$k^2 - 4k - 12 = 0$

$(k-6)(k+2) = 0$

$k = 6$ or $k = -2$

$|x| = 6$ N/A

5

$\therefore x = \pm 6$

2.2

a. $x = e^{y+3} - 2$

$x + 2 = e^{y+3}$

$x + 2 = e^3 \times e^y$

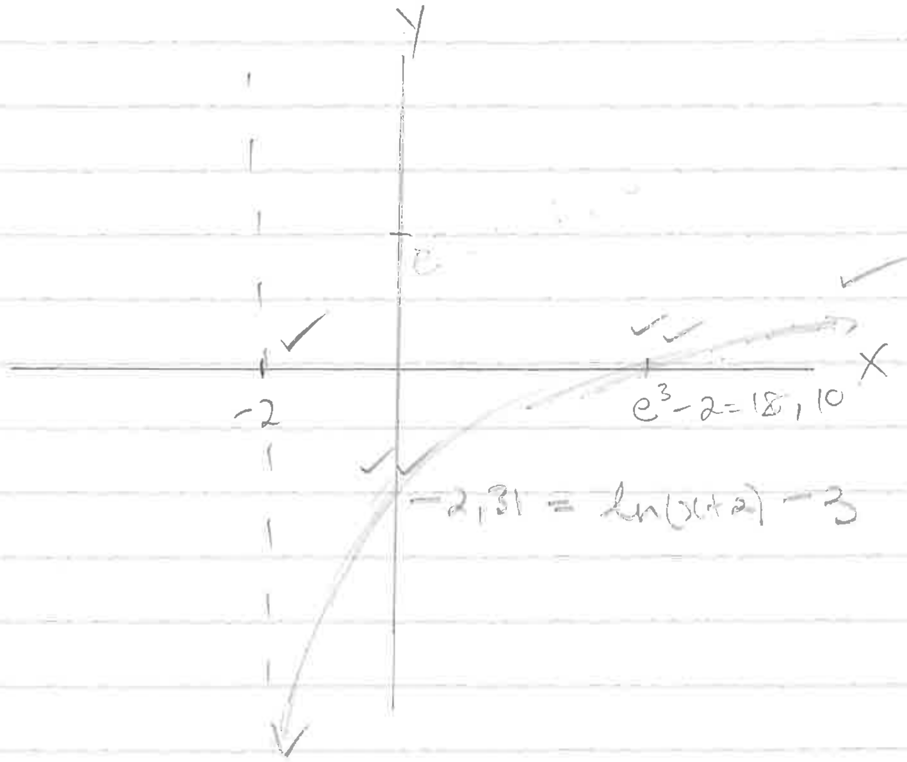
$\frac{1}{e^3}(x+2) = e^y$

$\ln\left(\frac{1}{e^3}(x+2)\right) = y$

$\ln(x+2) + \ln e^{-3} = y$

$\ln(x+2) - 3 = y$ 7

(b)




6

$\underline{\underline{30}}$

Q3

(a) $((x-1)+2i)((x-1)-2i)$

$= (x-1)^2 - (2i)^2$

$= x^2 - 2x + 1 - 4i^2$

$= x^2 - 2x + 1 + 4$

$= x^2 - 2x + 5$

5

$\therefore$ $x^2 - 2x + 5$ is a factor of $f(x)$.

(b) $\dfrac{x-10}{(2x+1)(x-3)}$

$= \dfrac{A}{2x+1} + \dfrac{B}{x-3}$

$B(2x+1) + A(x-3) = x - 10$

let $x = -1/2$

$-7/2\, A = -\frac{21}{2}$

$A = 3.$

let $x = 3$.

7

$7B = -7$

$B = -1$

$= \dfrac{3}{2x+1} - \dfrac{1}{x-3}$

12

Q4

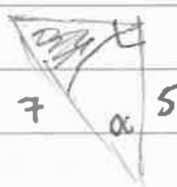
(a) $A = \frac{1}{2} r^2 (\theta - \sin\theta)$

$= \frac{1}{2}(7)^2(1,53 - \sin 1,53)$

$= 13,01 \text{ cm}^2$

5

(b)




$\alpha = 1,53/2 = 0,765$

$A = \frac{1}{2}(5)(7)\sin(0,765) - \frac{1}{2}(5)^2(0,765)$

$= 2,56 \text{ cm}^2$

8

13

Q5

To be differentiable the function is continuous at $x=1$

$$\frac{a}{1} = b - 2.$$

$$\therefore a = b - 2.$$

in addition

$$-\frac{a}{x^2} = -2 \quad \text{at } 1$$

$$\therefore -a = b - 2$$

$$-\therefore a = 2$$

$$\therefore b = 4$$

$$\therefore a = -2$$

[9]

Q6

$$f(x) = \frac{x(x+3)}{-6(x-2)}$$

(a) $x = 0$ and $x = -3$

$$y = \frac{0}{-6(-2)} = 0$$

4

(b) Horizontal / skewed.

$$x^2+3x = (-6x+12)\left(-\frac{1}{6}x + c\right) + R$$

$$-6xc - 2x = 3x$$

$$-6xc = 5x$$

$$c = \frac{5}{-6}$$

$$x^2+3x = (-6x+12)\left(-\frac{1}{6}x - \frac{5}{6}\right) + 10$$

$$\therefore \frac{x^2+3x}{-6x+12} = -\frac{1}{6}x + \frac{5}{6} + \frac{10}{-6x+12}$$

Oblique Asymptote $y = -\frac{1}{6}x - \frac{5}{6}$

Vertical

$$\lim_{x \to 2} f(x) = \text{DNE.}$$

at $x = 2$ jump discontinuity.

10

(c) $y' = \dfrac{(-6x+12)(2x+3) - (x^2+3x)(-6)}{(-6x+12)^2}$

at turning point:

$(-6x+12)(2x+3) - (-6x^2 - 18x) = 0$

$-12x^2 - 18x + 24x + 36 + 6x^2 + \cancel{18x} = 0$

$-6x^2 + 24x + 36 = 0$

$x = 2 + \sqrt{10} \approx 5{,}16$

$x = 2 - \sqrt{10} \approx -1{,}16$

$\frac{8}{22}$

Q7

7.1 $= \lim_{x \to \infty} \dfrac{x\left(\sqrt{9 - \frac{3}{x}} + \frac{2}{x}\right)}{x\left(9 - 5/x\right)}$

$= \lim_{x \to \infty} \dfrac{\sqrt{9 - 3/x} + \frac{2}{x}}{9 - 5/x}$

$= \dfrac{\sqrt{9}}{9} = \dfrac{1}{3}$ 7

7.2 $\lim_{h \to 0} \left( \dfrac{1}{\sqrt{x+h+2}} - \dfrac{1}{\sqrt{x+2}} \right) \times \dfrac{1}{h}$

$= \lim_{h \to 0} \dfrac{\sqrt{x+2} - \sqrt{x+h+2}}{(\sqrt{x+h+2})(\sqrt{x+2})} \times \dfrac{\sqrt{x+2} + \sqrt{x+h+2}}{\sqrt{x+2} + \sqrt{x+h+2}} \times \dfrac{1}{h}$

$= \lim_{h \to 0} \dfrac{x+2-x-h-2}{(\sqrt{x+h+2})(\sqrt{x+2})} \times \dfrac{1}{(\sqrt{x+2} + \sqrt{x+h+2})} \times \dfrac{1}{h}$

$= \lim_{h \to 0} \dfrac{-1}{(\quad)(\quad)(\quad)}$ 10

$= \dfrac{-1}{2(x+2)(\sqrt{x+2})} = \dfrac{1}{2} \dfrac{1}{(x+2)^{3/2}}$

7.3

(a) $f'(x) = 2\cos(\tan 2x)\sec^2(2x)$

$g(x) = x^{5/3} + x^{11/12}$

$g'(x) = \frac{5}{3}x^{2/3} + \frac{11}{12}x^{-1/12}$

$h'(x) = \frac{\cos x \times 2 - (-\sin x)(2x)}{\cos^2 x}$

$= \frac{2\cos x + 2x\sin x}{\cos^2 x}$ 12

(b) $f'(\pi) = 2$

$g'(1) = \frac{31}{12} \approx 2,58$

$h'(\pi) = -2$

$h'(\pi) < f'(\pi) < g'(1)$ 4

7.4 42

(a) $-\sin y \times y' = 1$

$\therefore \frac{dy}{dx} = \frac{1}{-\sin y}$ 5

(b) $0,5 = \cos y \quad \therefore y = \frac{\pi}{3} \quad \therefore \frac{dy}{dx} = -\frac{2\sqrt{3}}{3} \approx -1,15$ 4

Q8

a) let $f(x) = 3(x-2)^2 - 1 - \frac{4}{x}$

at $f(2) = -3$

at $f(3) = 2/3$

$\therefore$ there has to be a null value between 2 and 3. 4

b) $f'(x) = 6(x-2) + \frac{4}{x^2}$

let $a_0 = 2.5$

$\therefore a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)}$

$a_1 = 3.008241$

$a_n = 2.891337$

8
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12

Q9

$f(x) = 3x^2 + 2$

$x_i = a + \frac{b-a}{n} i$

$x_i = \frac{5}{n} i$

$f(x_i) = 3\left(\frac{5}{n} i\right)^2 + 2$

$\lim_{n \to \infty} \frac{5}{n} \sum_{i=1}^{n} \left(3\left(\frac{5}{n} i\right)^2 + 2\right)$

$= \lim_{n \to \infty} \frac{5}{n} \left(\sum \frac{75}{n^2} i^2 + 2\sum 1\right)$

$= \lim_{n \to \infty} \frac{5}{n} \left(\frac{75}{n^2} \sum i^2 + 2 \sum 1\right)$

$= \lim_{n \to \infty} \frac{5}{n} \left(\frac{75}{n^2}\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right) + 2n\right)$

$= \lim_{n \to \infty} \left(125 + \frac{375}{2n} + \frac{375}{6n^2} + 10\right)$

$= 125 + 10$

$= 135 \text{ unit}^2$

$=$ Area $= 135$ units$^2$

$\frac{}{15}$

Q10

10.1

(a) $\int (2x+1)\cos 2x \, dx$

let $u = 2x+1$ $\quad u' = 2$

$v' = \cos 2x$ $\quad v = \frac{1}{2}\sin 2x$

$\therefore \int (2x+1)\cos 2x \, dx$

$= \frac{1}{2}(2x+1)\sin 2x - 1\int \sin 2x \, dx + C$

$= \frac{1}{2}(2x+1)\sin 2x + \frac{1}{2}\cos 2x + C$

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(b) let $u = 2x^3$

$\frac{du}{dx} = 6x^2$

$\therefore du = 6x^2 \, dx$

$\int x^2 \sec^2(2x^3)\, dx = \frac{1}{6}\int \sec^2 u \, du$

$= \frac{1}{6}\tan u + C$

$= \frac{1}{6}\tan(2x^3) + C$

9

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2

10.2

Boundaries:

$$1 - x^2 = x^2 - 2x + 1$$

$$0 = 2x^2 - 2x$$

$$0 = 2x(x-1)$$

$\therefore$ $x = 0$ and $x = 1$

$$\therefore \quad V = \pi \int_a^b y^2 \, dx$$

$$\therefore \quad V = \pi \int_0^1 \left((1-x^2)\right)^2 dx - \pi \int_0^1 \left((x-1)^2\right)^2 dx.$$

$$= \tfrac{1}{3}\pi \text{ units}^3$$

10

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