HeronBridge COLLEGE

Grade 12: Advanced Programme Mathematics Preliml

Paper 1: Calculus & Algebra

2 Hour

200 Marks

September, 2019

Examiner: M Klein

Name: Memo

Teacher: _____

1. This paper consists of 6 pages and an information sheet.
2. Show ALL calculations, answers only will NOT be awarded full marks.
3. Approved non-programmable calculators are permissible unless stated otherwise. Ensure your calculator is set to RADIANS
4. Round off answers to TWO decimal places, unless stated otherwise.
5. Diagrams are NOT necessarily drawn to scale.

Question	1	2	3	4	5	6	7	8	9	10	11	12	Total
Mark Achieved													
Possible Mark													
Marker													

Question 1.

(a) -3 ; $2-i$; $2+i$

Sum: 4

Prod: $4-i^2 = 5$

$(x+3)(x^2-4x+5)=0$.

$x^3-4x^2+5x+3x^2-12x+15=0$.

$x^3-x^2-7x+15=0$. (4)

b) $\ln(e^{2x}-6)=x$

$e^{2x}-6=e^x$.

$k=e^x$

$k^2-6=k$

$k^2-k-6=0$.

$(k-3)(k+2)=0$

$k=3 \quad k=-2$.

$e^x=3 \quad e^x=-2$. (7)

$x=\ln 3 \quad x=\ln -2$ N/A.

c) $\log_{1/10}(x-20)+\log(2x)=1$.

$-\log(x-20)+\log 2x = \log 10$.

$\log \frac{2x}{(x-20)} = \log 10$.

$2x = 10(x-20)$

$2x = 10x-200$.

$200 = 8x$

$25 = x$ (6)

d) $\frac{x+4}{|x+1|} < x$

$x+4 < x|x+1|$ ✓

$x+1 \geq 0$ ✓ $x+1 < 0$

$x \geq -1$ $x < -1$

✓ ✓

$x+4 < x(x+1)$ $x+4 < x(-x-1)$

$x+4 < x^2+x$ $x+4 < -x^2-x$

$4 < x^2$ $x^2+2x+4 < 0$

$0 < x^2-4$ ✓ $(x-1)^2-1+4 < 0$

$0 < (x+2)(x-2)$ ✓ Not possible

$x \in (2, \infty)$ ✓ (9)

e) $(x+2i)(4-i) = 14+iy$

$4x - xi + 8i - 2i^2 = 14+iy$ ✓

$4x+2+i(-x+8) = 14+iy$ ✓

$4x+2 = 14$ ✓ $-x+8 = y$

$4x = 12$ $-3+8 = y$

$x = 3$ $5 = y$ ✓ (6)

Question 2.

$8^n - 4n + 6$. div by 4.

For $n=1$

$8^1 - 4(1) + 6 = 4$. div by 4.

∴ True for $n=1$. ✓ (2)

Assume ✓ true for $n=k$

i.e. assume $8^k - 4k + 6 = 4x$. ✓

Prove true for $n=k+1$

ie Prove $8^{k+1} - 4(k+1) + 6 = 4y$. 3

LHS $= 8^{k+1} - 4(k+1) + 6$.

$= 8^k \cdot 8 - 4k - 4 + 6$.

$= 8(4k + 4x - 6) - 4k - 1$ $\quad 8^k = 4k + 4x - 6$.

$= 56k + 56x - 48 - 4k - 1$.

$= 49k + 56x - 49$ ✓

$= 4(7k + 8x - 7)$ ✓ 5

∴ True for $n=k+1$ ✓ 2.

∴ True $\forall n \in \mathbb{N}$ by mathematical induction ✓

(12)

Question 3

$$f(x) = \begin{cases} ax - b - 1, & x < 2 \\ bx^2 - ax + 5, & x \geq 2 \end{cases}$$

Continuous.

$$\therefore \lim_{x \to 2^-} f = \lim_{x \to 2^+} f = f(2).$$

$$\therefore \lim_{x \to 2^-} ax - b - 1 = \lim_{x \to 2^+} bx^2 - ax + 5.$$

$$2a - b - 1 = 4b - 2a + 5$$

$$4a - 5b = 6$$ ✓ 4.

Differentiable:

$$\lim_{h \to 2^-} a = \lim_{h \to 2^+} 2bx - a$$

$$a = 4b - a$$

$$2a = 4b$$

$$a = 2b.$$ ✓ 4

$$4(2b) - 5b = 6$$

$$8b - 5b = 6.$$

$$3b = 6$$

$$b = 2.$$ ✓

$$a = 4.$$ ✓ 2.

Question 4

(a) $D_x\left[\cos^3(2x+1) \times \tan(x^2+2)^3\right]$

$= \left[3\cos^2(2x+1)(-\sin(2x+1))(2)\right] + \tan(x^2+2)^3 + \cos^3(2x+1)\left[\sec^2(x^2+2)^3 \times 3(x^2+2)^2 \times 2x\right]$

(9)

(b) $D_x\left[\dfrac{\operatorname{cosec} x^2}{\cot^2 x}\right]$

$= \dfrac{\left[-\operatorname{cosec} x^2 \cot x^2 \times 2x\right]\cot^2 x - \operatorname{cosec} x^2\left[2\cot x(-\operatorname{cosec}^2 x)\right]}{\cot^4 x}$

(8)

Question 5

$$f(x) = \frac{1}{3x+1} = (3x+1)^{-1}$$ ✓

$$f'(x) = -1(3x+1)^{-2} \times 3$$

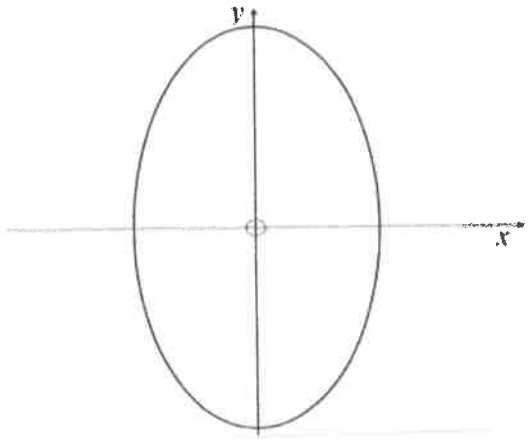
$$f''(x) = (-1)(-2)(3x+1)^{-3} \times 3 \times 3$$

$$f'''(x) = (-1)(-2)(-3)(3x+1)^{-4} \times 3^3$$

$$f^{(n)}(x) = (-1)^n n! (3x+1)^{-(n+1)} \times 3^n$$ (8)

Question 6.

$4x^2 + y^2 = 72.$



$x + 2y + 3 = 0.$

$2y = -x - 3.$ ✓

$y = -\frac{1}{2}x - \frac{3}{2}.$

$m_{tan} = 2.$ ✓

$\therefore dy/dx = 2.$

$8x + 2y \cdot dy/dx = 0.$ ✓ ✓ ✓ ✓

$8x + 2y(2) = 0.$ ✓

$4y = -8x.$

$y = -2x.$ ✓

$4x^2 + (-2x)^2 = 72.$ ✓

$4x^2 + 4x^2 = 72.$

$8x^2 = 72.$ ✓

$x^2 = 9$

$x = \pm 3.$ ✓

$(3; -6).$ ✓

$y = 2x + c.$

$-6 = 2(3) + c$

$-6 = 6 + c$

$-12. = c$

$y = 2x - 12.$ ✓

$(-3; 6).$ ✓

$y = 2x + c$

$6 = 2(-3) + c.$

$6 = -6 + c$

$12. = c$

$y = 2x + 12.$ ✓

15.

Question 7

$f(x) = 2x\sin x \qquad g(x) = x.$

(a) $f'(x) = 2\sin x + 2x\cos x.$ (4)

(b) $f(x) = g(x).$

$2x\sin x = x$

$h(x) = 2x\sin x - x = 0.$

$$x_{n+1} = x_n - \frac{h(x)}{h'(x)}$$

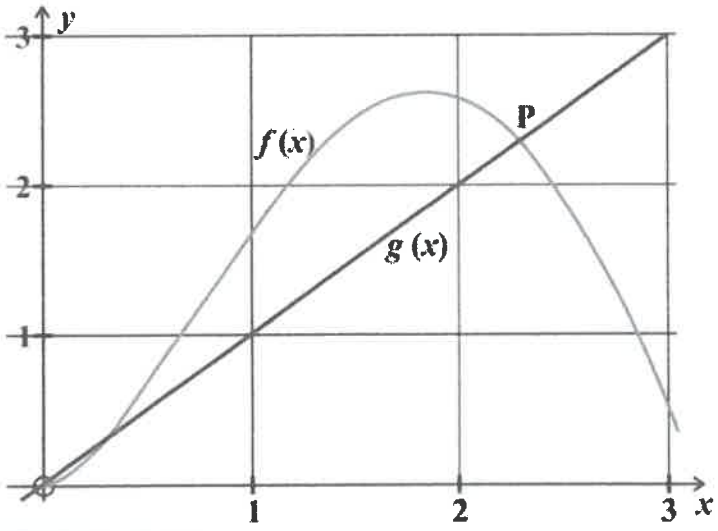
$$= x - \frac{(2x\sin x - x)}{(2\sin x + 2x\cos x - 1)}$$

$x_1 = 2,5$

$\vdots$

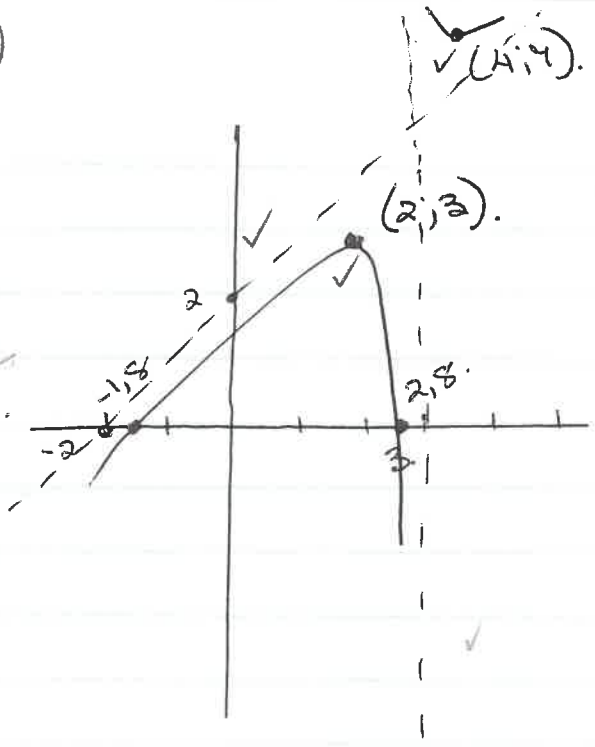
$x = +2,61799$

$= 2,6180$ $\left(\frac{5\pi}{6}\right)$ (6)



Question 8

$$f(x) = \frac{x^2 - x - 5}{x - 3}$$



(a) $f'(x) = 0$.

$$\frac{(2x-1)(x-3) - (x^2 - x - 5)(1)}{(x-3)^2} = 0$$

$$2x^2 - 7x + 3 - x^2 + x + 5 = 0$$

$$x^2 - 6x + 8 = 0$$

$$(x-4)(x-2) = 0$$

$x = 4$ $\quad x = 2$.

$f(4) = \frac{16 - 4 - 5}{4 - 3}$ $\quad f(2) = \frac{4 - 2 - 5}{2 - 3}$.

$= 7$. $\quad = \frac{-3}{-1} = 3$.

$(4;7)$ $\quad (2;3)$. 12.

$$f'(x) = \frac{x^2 - 6x + 8}{(x-3)^2}$$

$$f''(x) = \frac{(2x-6)(x-3)^2 - (x^2 - 6x + 8)\,2(x-3)}{(x-3)^4}$$

$f''(4) = 2 - (16 - 24 + 8) = 2 > 0$ $\cup$ Local Min

$f''(2) = -2 < 0$ $\quad \cap$ Local Max. 6.

(b) Vert $x = 3$.

Oblique

$$\begin{array}{r|l} & x + 2 \\ \hline x - 3 & x^2 - x - 5 \\ & -x^2 + 3x \\ \hline & 2x - 5 \\ & -2x + 6 \\ \hline & 1. \end{array}$$

3.

$y = x + 2$.

(c) Y-int $(0, -\frac{5}{-3})$

X-int $0 = \frac{x^2 - x - 5}{x - 3}$

$x^2 - x - 5 = 0$.

$x = \frac{-(-1) \pm \sqrt{1 - 4(-5)}}{2}$.

$= \frac{1 \pm \sqrt{21}}{2}$. $\quad x = 2,8$ $\quad x = -1,8$. (4) + 4

Question 9

(a) $\int (2x+1)\sqrt{3x^2-3x+3}\,dx$.

Let $K = 3x^2-3x+3$. ✓

$\frac{dK}{dx} = 6x-3$ ✓

$\frac{dK}{dx} = 3(2x-1)$. ✓

$\frac{dK}{3} = (2x-1)dx$ ✓

$\int K^{\frac{1}{2}} \cdot \frac{dK}{3}$ ✓

$\frac{1}{3}\int K^{\frac{1}{2}}\,dK$.

$\frac{1}{3}\,\frac{K^{\frac{3}{2}}}{\frac{3}{2}}$ ✓

$\frac{2}{9}K^{\frac{3}{2}}$

$\frac{2}{9}(3x^2-3x+3)^{3/2} + C$. ✓ ✓ (4)

(b) $\int \sin 5x \cos 3x\,dx$.

$= \frac{1}{2}\int \sin(5x+3x) + \sin(5x-3x)\,dx$.

$= \frac{1}{2}\int \sin 8x + \sin 2x\,dx$ ✓ ✓

$= \frac{1}{2}\left[-\frac{\cos 8x}{8} - \frac{\cos 2x}{2}\right] + C$. ✓ ✓

$= -\frac{\cos 8x}{16} - \frac{\cos 2x}{4} + C$. ✓ (5)

(c) (i) $\operatorname{cosec}^4 x$. $\operatorname{cosec}^2 x \cot^2 x + \operatorname{cosec}^2 x$

$\operatorname{cosec}^2 x (\cot^2 x + 1)$. ✓

$\operatorname{cosec}^2 x \times \operatorname{cosec}^2 x$ ✓

$\operatorname{cosec}^4 x$ ✓

LHS = RHS. (3)

(ii) $\int \operatorname{cosec}^4 x \, dx = \int \operatorname{cosec}^2 x \cot^2 x + \operatorname{cosec}^2 x \, dx$.

$\int \operatorname{cosec}^2 x \, dx = -\cot x$.

$\int \operatorname{cosec}^2 x \times \cot^2 x \, dx$.

$k = \cot x$ ✓

$dk/dx = -\operatorname{cosec}^2 x$. ✓

$-dk = \operatorname{cosec}^2 x \, dx$ ✓

$\int k^2 (-dk) = -\int k^2 dk$ ✓

$= -\frac{k^3}{3}$ ✓

$= -\frac{\cot^3 x}{3}$ ✓

$\int \operatorname{cosec}^4 x \, dx = -\frac{\cot^3 x}{3} - \cot x + C$. ✓ (9)

Question 10.

$$f(x) = \frac{x^3}{x^2-4}.$$

(a)
$$\begin{array}{r|l} x^2-4 & x \\ \hline x^3. & \\ -x^3+4x. & \\ \hline 4x. & \end{array}$$

$$\frac{x^3}{(x+2)(x-2)} = x + \frac{4x}{(x+2)(x-2)}.$$

$$\frac{4x}{(x+2)(x-2)} = \frac{A}{(x+2)} + \frac{B}{(x-2)}$$

$$4x = A(x-2) + B(x+2)$$

$x=2.$ $\quad 8 = 4B$

$$2 = B$$

$x=-2.$ $\quad -8 = -4A$

$$2. = A$$

7.

$$\frac{x^3}{x^2-4} = x + \frac{2}{x+2} + \frac{2}{x-2}.$$

(b)
$$\int \frac{x^3}{x^2-4}\,dx = \int x + \frac{2}{x+2} + \frac{2}{x-2}\,dx.$$

$$= \frac{x^2}{2} + 2\ln|x+2| + 2\ln|x-2| + C.$$

4

Question 11

$f(x) = e^x - x$

$g(x) = -e^x + 2$.

(a) Area $= \int_{-1}^{0} g(x) - f(x)\,dx$

$= \int_{-1}^{0} (-e^x + 2) - (e^x - x)\,dx$.

$= \int_{-1}^{0} -2e^x + 2 + x\,dx$.

$= \left[-2e^x + 2x + \frac{x^2}{2}\right]_{-1}^{0}$

$= (-2e^0) - (-2e^{-1} - 2 + \frac{1}{2})$

$= \frac{2}{e} + \frac{1}{2}$. (8)

$= 0,235$

(b) Vol $= \pi \int_{-1}^{0} g(x)^2\,dx$.

$= \pi \int_{-1}^{0} (-e^x + 2)^2\,dx$.

$= \pi \int_{-1}^{0} e^{2x} - 4e^x + 4\,dx$.

$= \pi \left[\frac{e^{2x}}{2} - 4e^x + 4x\right]_{-1}^{0}$.

$= \pi\left[\frac{1}{2} - 4\right] - \pi\left[\frac{e^{-2}}{2} - 4e^{-1} - 4\right]$

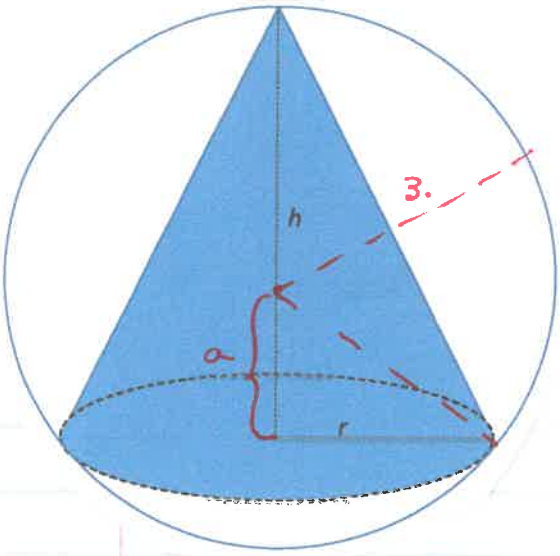
$= \frac{\pi}{2} - \frac{\pi}{2e^2} + \frac{4\pi}{e}$.

$= 1,903\pi$. (7)

$= 5,981$.

Question 12

$$V = \frac{1}{3}\pi r^2 h$$



h = height of cone.

$a = h - 3$. ✓

$a^2 + r^2 = 9$. ✓

$r^2 = 9 - a^2$

$= 9 - (h-3)^2$. ✓

$V = \frac{\pi}{3} \times \left(9 - (h^2 - 6h + 9)\right) h$. ✓

$= \frac{\pi}{3} \times h(6h - h^2)$ ✓

$= \frac{\pi}{3} \times h^2(6-h)$ (5)

$dV/dh = 0$. ✓

$0 = \frac{\pi}{3} 2h(6-h) + \frac{\pi}{3} h^2(-1)$. ✓ ✓ ✓

$0 = 2h(6-h) - h^2$.

$= 12h - 2h^2 - h^2$.

$= -3h^2 + 12h$. ✓

$0 = -3h(h-4)$. ✓

$h = 0 \quad h = 4$. ✓

Min vol Max vol

$V = \frac{\pi}{3} \times 4^2(6-4)$. ✓

(10)

$= \frac{\pi}{3} \times 32$. ✓