Memo Gr 12 AP Math Prelim 2019 P1 200 marks:

Q1.

1.1 $\ln(1+\sqrt{x}) = 3$

$e^3 = 1 + \sqrt{x}$ ✓

$e^3 - 1 = \sqrt{x}$ ✓

$x = (e^3 - 1)^2$ ✓ (3)

1.2 $\dfrac{-2}{x-2} = |2x-4|$

If $2x - 4 \geq 0$ then

$\dfrac{-2}{x-2} = 2x - 4$ ✓

$-2 = (2x-4)(x-2)$

$0 = 2x^2 - 8x + 10$

$0 = x^2 - 4x + 5$ ✓

$x = \dfrac{4 \pm \sqrt{16-20}}{2}$

x undefined ✓

If $2x - 4 < 0$ then

$\dfrac{-2}{x-2} = -2x + 4$ ✓

$-2 = (x-2)(-2x+4)$

$-2 = -2x^2 + 8x - 8$

$0 = 2x^2 - 8x + 6$

$0 = x^2 - 4x + 3$ ✓

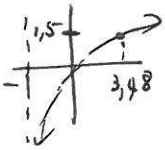
$0 = (x-3)(x-1)$ ✓

$x = 3$ or $x = 1$ ✓

But $x < 2$

$\therefore \quad x = 1$ ✓ (8)

1.3 $f(x) = \ln(x+1)$



1.3.1 $x = -1$ (1)

1.3.2 $y \in \mathbb{R}$ (1)

1.3.3 $x = \ln(y+1)$ swap

$e^{x} = y + 1$

$f^{-1}(x) = e^{x} - 1$ (2)

1.3.4 $y = 1$ (1)

1.3.5 $y = 1,5$

$1,5 = \ln(x+1)$

$e^{1,5} - 1 = x$

$x = 3,48$

$x > 3,48$ (1)

[7]

Q2 2.1 $f(x) = \begin{cases} 3a + 3ax & \text{if } x < -3 \\ 3a^2 - x^2 & \text{if } -3 \leq x \leq 2 \\ 4x - 4 & \text{if } x > 2 \end{cases}$

2.1.1 $\lim_{x \to -3^-} (3a + 3ax) = \lim_{x \to -3^+} (3a^2 - x^2)$ Continious

subst

$3a - 9a = 3a^2 - (-3)^2$

$-6a = 3a^2 - 9$

$0 = a^2 + 2a - 3$

$0 = (a+3)(a-1)$

$a = -3$ or $a = 1$

$\underline{a=-3}$: $f(x) = 3(-3) + 3(-3)x = -9 - 9x$ decreasing $m = -9$

$\underline{a=1}$: $f(x) = 3(1) + 3(1)x = 3x + 3$ increasing $m = 3$

$\therefore a = -3$ (6)

2.1.2 IF $\lim_{x \to 2^-} f'(x) = \lim_{x \to 2^+} f'(x)$

LHS: $\lim_{x \to 2^-} (-2x) = -4$

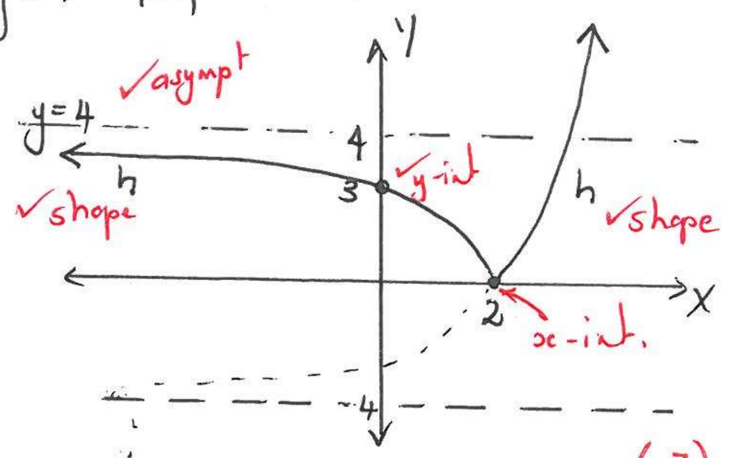
RHS: $\lim_{x \to 2^+} 4 = 4$

LHS $\neq$ RHS

$\therefore$ f will not be diff at $x = 2$ (6)

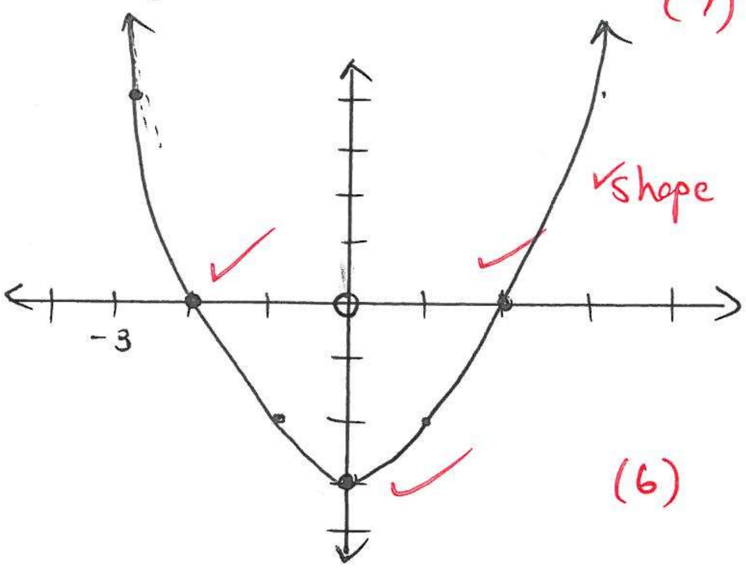
2.2 $f(x) = 2^x - 4$ $g(x) = |x|$

2.2.1 $h(x) = g(f(x))$ ✓
$= g(2^x - 4)$
$= |2^x - 4|$ ✓



(7)

2.2.2 $k(x) = f(g(x))$
$= f(|x|)$ ✓
$= 2^{|x|} - 4$ ✓



(6)

Q3: $2(x^2+y^2)^2 = 25(x^2-y^2)$ $P(-3;-1)$

$2[x^4 + 2x^2y^2 + y^4] - 25x^2 + 25y^2 = 0$

$2x^4 + 4x^2y^2 + 2y^4 - 25x^2 + 25y^2 = 0$

$8x^3 + 8xy^2 + 8x^2y\frac{dy}{dx} + 8y^3\frac{dy}{dx} - 50x + 50y\frac{dy}{dx} = 0$ ✓ ✓ ✓ ✓ ✓

$\frac{dy}{dx}\left[8x^2y + 8y^3 + 50y\right] = 50x - 8x^3 - 8xy^2$

$\frac{dy}{dx} = \frac{50x - 8x^3 - 8xy^2}{50y + 8y^3 + 8x^2y}$ at $P(-3;-1)$

$m = \frac{50(-3) - 8(-3)^3 - 8(-3)(-1)^2}{50(-1) + 8(-1)^3 + 8(-3)^2(-1)}$ ✓subst

$= -\frac{9}{13}$ ✓

[8]

Equation: $y + 1 = -\frac{9}{13}(x+3) \Rightarrow y = -\frac{9}{13}x - 3\frac{1}{13}$ ✓
$-3{,}07 / -\frac{40}{13}$

Q3: ALT:

$$2(x^2+y^2)^2 = 25(x^2-y^2)$$

$$2(x^2+y^2)^2 - 25(x^2-y^2) = 0$$

$$2(x^2+y^2)^2 - 25x^2 + 25y^2 = 0$$

$$4(x^2+y^2)\left(2x + 2y\frac{dy}{dx}\right) - 50x + 50y\frac{dy}{dx} = 0$$

$$4\left[2x^3 + 2x^2y\frac{dy}{dx} + 2xy^2 + 2y^3\frac{dy}{dx}\right] + 50y\frac{dy}{dx} = 50x$$

$$8x^3 + 8x^2y\frac{dy}{dx} + 8xy^2 + 8y^3\frac{dy}{dx} + 50y\frac{dy}{dx} = 50x$$

$$\frac{dy}{dx}\left[8x^2y + 8y^3 + 50y\right] = 50x - 8xy^2 - 8x^3$$

$$\frac{dy}{dx} = \frac{50x - 8xy^2 - 8x^3}{8x^2y + 8y^3 + 50y} \qquad P(-3;-1)$$

$$m = \frac{50(-3) - 8(-3)(-1)^2 - 8(-3)^3}{8(-3)^2(-1) + 8(-1)^3 + 50(-1)}$$

$$m = \frac{-9}{13} \quad \Rightarrow \quad y+1 = \frac{-9}{13}(x+3)$$

$$y = -\frac{9}{13}x - 3\frac{1}{13} \,/\, -3{,}07 \,/\, \frac{-40}{13}$$

Q4

4.1.1

$$y = \left(\frac{2x+1}{3x-1}\right)^4$$

$$\frac{dy}{dx} = 4\left(\frac{2x+1}{3x-1}\right)^3 \left[\frac{2(3x-1) - 3(2x+1)}{(3x-1)^2}\right]$$

$$= 4\left(\frac{2x+1}{3x-1}\right)^3 \left(\frac{6x-2-6x-3}{(3x-1)^2}\right)$$

$$= \frac{-20(2x+1)^3}{(3x-1)^5}$$

(6)

4.1.2

$$y = \sin^2(3+2x)$$

$$\frac{dy}{dx} = 2\sin(3+2x)\cos(3+2x)\cdot 2$$

$$= 2\left[2\sin(2x+3)\cos(2x+3)\right]$$ Dbl Angle

$$= 2\sin(4x+6)$$

(5)

4.2

$$f(x) = \frac{e^{5x}}{\ln x}$$

$$f'(x) = \frac{5e^{5x}\ln x - e^{5x}\left(\frac{1}{x}\right)}{(\ln x)^2}$$

(5)

[16]

Q5:

(5.1.1) $\int \frac{x}{\sqrt{x^2-9}}\,dx$

$= \int x(x^2-9)^{-\frac{1}{2}}\,dx$

$= \sqrt{x^2-9} + k$

OR

$\int x(x^2-9)^{-\frac{1}{2}}\,dx$

$= \frac{1}{2}\int 2x(x^2-9)^{-\frac{1}{2}}\,dx$

$= \sqrt{x^2-9} + k$

OR $\int x(x^2-9)^{-\frac{1}{2}}\,dx$

Let $u = x^2-9$ $\therefore$ $\frac{du}{dx} = 2x$ $\therefore$ $\frac{du}{2dx} = x$

$= \int u^{-\frac{1}{2}} \frac{du}{2\cancel{dx}} \cdot \cancel{dx}$

$= \frac{1}{2}\int u^{-\frac{1}{2}}\,du$

$= \frac{1}{2}\left[2u^{\frac{1}{2}}\right] + k$

$= \sqrt{x^2-9} + k$ subst back

(6)

(5.1.2) $\int \cos^2\left(\frac{1}{2}x\right)dx$

$= \int \frac{\cos x + 1}{2}\,dx$

$= \frac{1}{2}\int \cos x + 1\,dx$

$= \frac{1}{2}\sin x + \frac{1}{2}x + k.$

Dbl Angle

$\cos x = 2\cos^2\frac{x}{2} - 1$

$\cos x + 1 = 2\cos^2\frac{x}{2}$

$\cos^2\frac{x}{2} = \frac{\cos x + 1}{2}$

(6)

(5.1.3) $\int 3x\sin 3x\,dx = \int f(x)g'(x)\,dx$ Int by Parts

Let $f(x) = 3x$ $\quad g'(x) = \sin 3x$

$f'(x) = 3$ $\quad g(x) = -\frac{1}{3}\cos 3x$

$= 3x\left(-\frac{1}{3}\cos 3x\right) - \int 3\left(-\frac{\cos 3x}{3}\right)dx$ sub in formula

$= -x\cos 3x + \int \cos 3x\,dx$

$= -x\cos 3x + \frac{1}{3}\sin 3x + c$

(6)

(5.1.4) $\int \frac{e^{3x}+1}{e^x}\,dx$

$= \int\left(e^{2x} + e^{-x}\right)dx$

$= \frac{1}{2}e^{2x} - e^{-x} + c$

(5)

Q5: (Conti)

LHS: RHS:

5.2 (5.2.1.)

$$\frac{\cos(-2x)+\cos^2 x+3\sin^2 x}{2-2\sin^2 x} \qquad \sec^2 x$$

$$= \frac{\cos 2x+\cos^2 x+3\sin^2 x}{2(1-\sin^2 x)}$$

Dbl Angle

$$= \frac{\cos^2 x-\sin^2 x+\cos^2 x+3\sin^2 x}{2\cos^2 x} \quad \text{Id.}$$

$$= \frac{2\cos^2 x+2\sin^2 x}{2\cos^2 x} \quad \text{simplify}$$

$$= \frac{2(1)}{2\cos^2 x} = \sec^2 x$$

LHS = RHS. (6)

(5.2.2.)

$$\int \frac{\cos(-2x)+\cos^2 x+3\sin^2 x}{2-2\sin^2 x}\,dx$$

$$= \int \sec^2 x \, dx \quad \text{subst}$$

$$= \tan x + k$$

(3)

[32]

Q6:

$$\ln x + 2\ln x + 3\ln x + \dots + n\ln x = \frac{n}{2}\ln x^{n+1} \quad \forall\, n \in \mathbb{N}$$

1) Prove sm true for n=1:

LHS: $\ln x$

RHS: $\frac{1}{2}\ln x^2 = \ln x$ ✓ n=1

LHS = RHS

∴ sm true ∀ n = 1. ✓

2) Assume sm true for n=k:

$$\ln x + 2\ln x + 3\ln x + \dots + k\ln x = \frac{k}{2}\ln x^{k+1}$$ ✓ Assumpt.

3) Prove sm true for n=k+1:

LHS:

$$\underbrace{\ln x + 2\ln x + 3\ln x + \dots + k\ln x}_{\text{Assumption}} + (k+1)\ln x$$ ✓

$$= \frac{k}{2}\ln x^{k+1} + (k+1)\ln x$$ ✓

$$= \frac{k(k+1)}{2}\ln x + (k+1)\ln x$$

$$= \left[\frac{k(k+1)}{2} + k+1\right]\ln x$$

$$= \left[\frac{k^2+k+2k+2}{2}\right]\ln x$$ ✓

$$= \left[\frac{k^2+3k+2}{2}\right]\ln x$$

$$= \frac{(k+2)(k+1)}{2}\ln x$$ ✓

RHS:

$$\frac{k+1}{2}\ln x^{k+2}$$ ✓

$$= \frac{(k+1)(k+2)}{2}\ln x$$ ✓

LHS = RHS

∴ sm is true ∀ n = k+1

4) ∴ If the sm is true for n=k ∴ also true for n=1; it is true for n=k+1 ∴ sm is true ∀ n ∈ ℕ.

By mathematical induction is true ∀ n ∈ ℕ

Conclusion ✓

[10]

Q7:

$$x^4 + x^3 - 15x^2 - 23x + 12 = 0 \qquad (-1-\sqrt{2}) \text{ is a root}$$

$$\therefore \quad x = -1-\sqrt{2} \; / \; x = -1+\sqrt{2}$$

$$(x+1+\sqrt{2})(x+1-\sqrt{2}) = x^2 + 2x - \cancel{\sqrt{2}x} + 1 - \cancel{\sqrt{2}} + \cancel{\sqrt{2}x} + \cancel{\sqrt{2}} - 2$$

$$= x^2 + 2x - 1$$

OR

$$x + 1 = \pm\sqrt{2}$$

$$(x+1)^2 = 2$$

$$x^2 + 2x + 1 = 2$$

$$x^2 + 2x - 1 = 0$$

$$\begin{array}{r|l}
 & x^2 - x - 12 \\
x^2 + 2x - 1 & x^4 + x^3 - 15x^2 - 23x + 12 \\
 & x^4 + 2x^3 - x^2 \\
\hline
 & -x^3 - 14x^2 - 23x \\
 & -x^3 - 2x^2 + x \\
\hline
 & -12x^2 - 24x + 12 \\
 & -12x^2 - 24x + 12 \\
\hline
\end{array}$$

$$(x^2 + 2x - 1)(x^2 - x - 12) = 0$$

$$x = -1-\sqrt{2} \; / \; x = -1+\sqrt{2} \; / \; (x-4)(x+3) = 0$$

$$x = 4 \; / \; x = -3$$

[14]

Q8:

(8.1)

$$\frac{6x^2 - x + 11}{(x^2+3)(x-1)} = \frac{Ax+B}{(x^2+3)} + \frac{C}{(x-1)}$$ Partial Fract.

(8.1.1)

$$\frac{6x^2 - x + 11}{(x^2+3)(x-1)} \equiv \frac{(Ax+B)(x-1) + C(x^2+3)}{(x^2+3)(x-1)}$$

$$6x^2 - x + 11 \equiv (x-1)(Ax+B) + C(x^2+3)$$ lcd.

Let $x=1$:

$$6 - 1 + 11 = 4C$$

$$16 = 4C$$

$$\therefore 4 = C$$

Let $x=0$:

$$11 = -1B + 3C$$

$$11 = -B + 12$$

$$B = 1$$

Let $x=-1$:

$$6 + 1 + 11 = -2(-A+1) + 4(1+3)$$

$$18 = -2(1-A) + 16$$

$$2 = -2(1-A)$$

$$-1 = 1 - A$$

$$A = 2$$

$$\therefore \frac{6x^2 - x + 11}{(x^2+3)(x-1)} = \frac{2x+1}{(x^2+3)} + \frac{4}{x-1}$$

(6)

8.1.2 $$\int \frac{6x^2 - x + 11}{(x^2+3)(x-1)}\, dx$$

$$= \int \frac{2x+1}{(x^2+3)}\, dx + \int \frac{4}{x-1}\, dx$$

$$= \int \frac{2x}{x^2+3}\, dx + 4\int \frac{1}{x-1}\, dx$$

$$= \ln(x^2+3) \quad + 4\ln(x-1) + c$$

(4)

8.2 $$\lim_{n \to \infty} \frac{2}{n} \sum \left(\left(\frac{2i}{n}\right)^3 + \frac{2i}{n} \right)$$

$$\frac{b-a}{n} = \frac{2}{n} \qquad x_i = a + i\left(\frac{2}{n}\right)$$

$$\therefore b - a = 2 \qquad \therefore a + \frac{2i}{n} = \frac{2i}{n}$$

$$b = 2 \qquad \therefore a = 0$$

$$f(x) = x^3 + x$$

$$\therefore \int_0^2 (x^3 + x)\, dx$$

(5)

[15]

Q 9:

$\left[0 ; \frac{2\pi}{3}\right]$

9.1 $f(x) = 2\sin 3x$ $\qquad g(x) = \frac{1}{2}(x+1)^2 - 7$

$\text{Distance} = 2\sin 3x - \frac{1}{2}(x+1)^2 + 7$ ✓ subtraction (f − g)

$\text{Max Distance} \Rightarrow D'(x) = 0$

$0 = 2\cos 3x \cdot 3 - (x+1) + 0$

$0 = 6\cos 3x - (x+1)$

$x + 1 = 6\cos 3x$ $\qquad$ (6)

9.2 $k(x) = 6\cos 3x - x - 1$ $\qquad$ (5 dec)

$k'(x) = -18\sin 3x - 1$

✓ Formula.

$x_0 = 0{,}8$

$x_1 = 0{,}8 - \dfrac{6\cos 3x - x - 1}{-18\sin 3x - 1}$

OR

$x_1 = 0{,}8 - \dfrac{-(x+1) - 6\cos 3x}{-(18\sin 3x + 1)}$

$= 0{,}8 - \dfrac{x+1 - 6\cos 3x}{18\sin 3x + 1}$

$= 0{,}8 + \dfrac{6\cos 3(0{,}8) - 0{,}8 - 1}{18\sin 3(0{,}8) + 1}$

$= 0{,}3191851\ldots$

Not Radian $= 0{,}326964\ldots$

$x_2 = 0{,}452965\ldots$

$x_3 = 0{,}442692\ldots$

$x_4 = 0{,}442658\ldots$

$x_5 = 0{,}442658\ldots$

$\therefore x = 0{,}44266$ $\qquad$ (9)

[15]

Q 10: $f(x) = \dfrac{2x^2+9x-5}{x-2}$

10.1 VA: $x = 2$ ✓

HA: None ✓

OA:

$$\begin{array}{r|l} & 2x + 13 \\ \hline x-2 & 2x^2 + 9x - 5 \\ & 2x^2 - 4x \\ \hline & \quad 13x - 5 \\ & \quad 13x - 26 \\ \hline & \qquad 21 \end{array}$$ ✓

$$\frac{2x^2+9x-5}{x-2} = 2x+13 + \frac{21}{x-2}$$

$\therefore$ OA: $y = 2x+13$ ✓

(4)

10.2 TP: $f'(x) = 0$ ✓

$$0 = \frac{(4x+9)(x-2) - (2x^2+9x-5)}{(x-2)^2}$$ ✓ ✓ ✓ Q-Rule

$0 = 4x^2 + x - 18 - 2x^2 - 9x + 5$ ✓ $\times (x-2)^2$

$0 = 2x^2 - 8x - 13$ ✓ simplify

$x = \dfrac{4 \pm \sqrt{42}}{2}$

$\therefore x = 5{,}24$ ✓ / $x = -1{,}24$ ✓

$y = 29{,}96$ ✓ $y = 4{,}04$ ✓

TP: $(5{,}24 ; 29{,}96)$ Min TP ✓ any method

$(-1{,}24 ; 4{,}04)$ Max TP ✓ any method

(12)

10.3

y-int: $x = 0$

$y = \dfrac{-5}{-2} = \dfrac{5}{2} \,/\, 2\tfrac{1}{2}$ ✓

x-int: $y = 0$

$0 = \dfrac{2x^2+9x-5}{x-2}$ ✓

$0 = 2x^2 + 9x - 5$

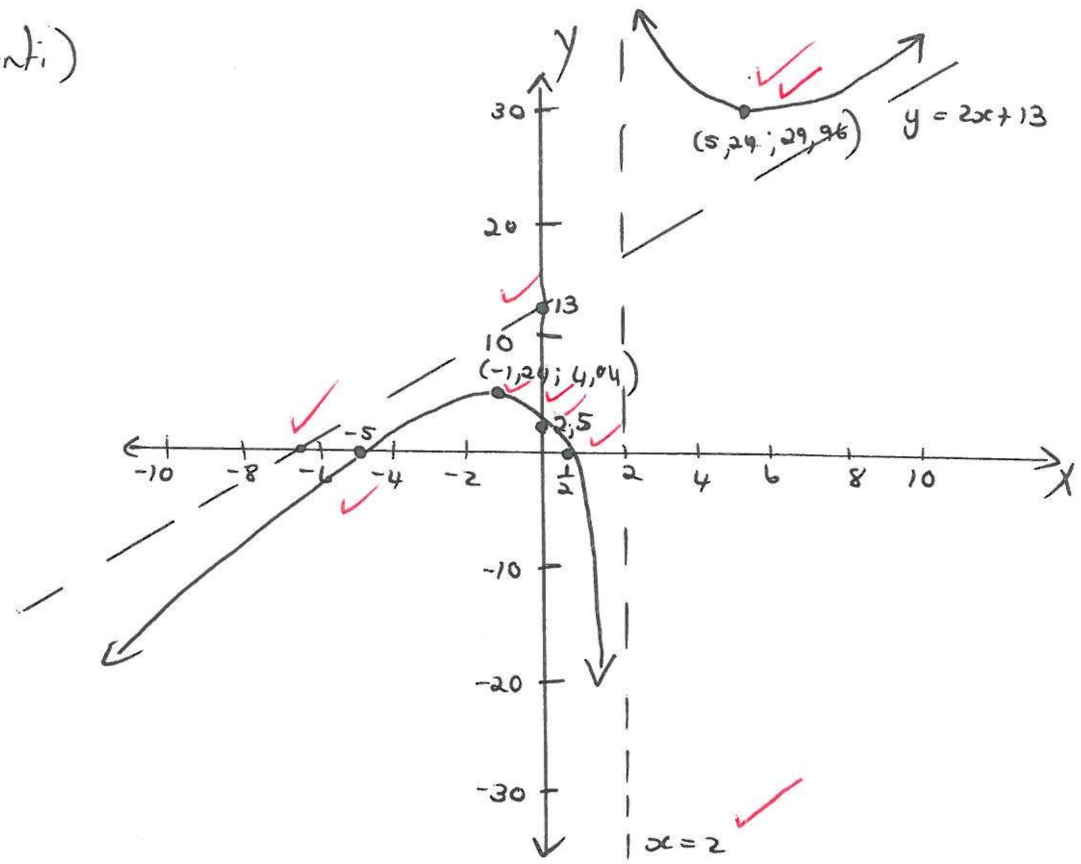
$= (2x-1)(x+5)$ ✓

$\therefore x = \frac{1}{2}$ / $x = -5$ ✓

(4)

Q : 10 (Conti)

10.4



(10)

[30]

Q 11 :

$y = x^2$ and $y = 8 - x^2$

$V = \pi \int_a^b y^2 \, dx$ $x = 0$ and point of intersection.

$x^2 = 8 - x^2$

$2x^2 = 8$

$x^2 = 4$

$x = \pm 2$

$\therefore x = 2 \; ; \; y = 4$

$x = -2 \quad y = 4$ N/A

✓Subst in Formula

$V = \pi \left\{ \int_0^2 (8 - x^2)^2 \, dx - \int_0^2 x^4 \, dx \right\}$

$= \pi \left\{ \int_0^2 (64 - 16x^2 + x^4) \, dx - \int_0^2 x^4 \, dx \right\}$

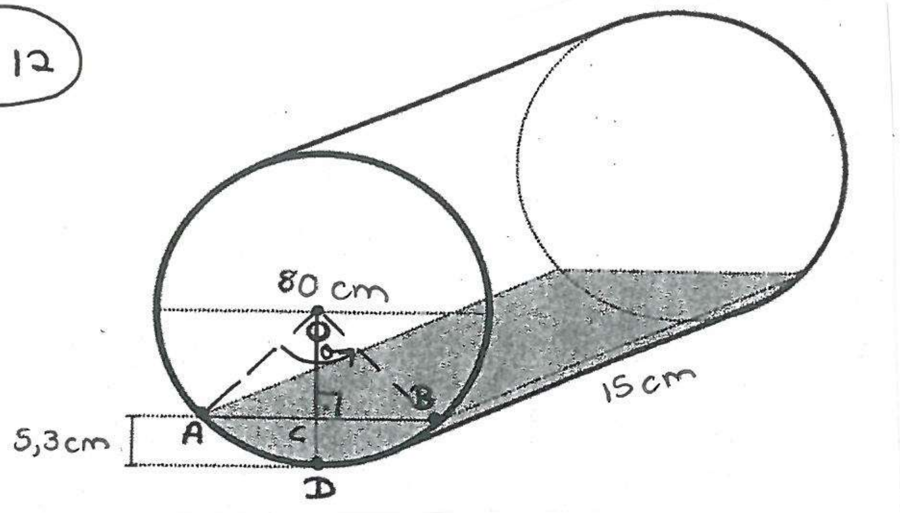
$= \pi \left(91{,}7\dot{3} - 6{,}4 \right)$

$= \pi \left(85\frac{1}{3} \right)$ units^3 / $\frac{256}{3}\pi$ units^3

(8)

[8]

Q 12



12.1

$OA = OB = \frac{80}{2} = 40cm$

$OC = 40 - 5{,}3 = 34{,}7\ cm$

$AC^2 = OA^2 - OC^2$ Pyth

$= (40)^2 - (34{,}7)^2$

$AC^2 = 395{,}91$

$AC = 19{,}897$

$AC = 20cm$

$\therefore\ AB = 40\ cm$ $(AC = CB)$

$AB^2 = AO^2 + OB^2 - 2AO \cdot BO \cos\theta$

$40^2 = 40^2 + 40^2 - 2(40)(40)\cos\theta$

$\frac{1600 - 3200}{-3200} = \cos\theta$

$\cos\theta = -\frac{1}{2}$

$\theta = \frac{2}{3}\pi / 2{,}0944\ Rad / 2{,}094395102.$

(7)

Q 12.2.

Area of segment ABD $= \frac{1}{2} r^2 (\theta - \sin\theta)$

$= \frac{1}{2} (OA)^2 (\theta - \sin\theta)$ subst

$= \frac{1}{2} (40)^2 (2{,}0944 - \sin 2{,}0944)$

$= 983 \text{ cm}^2$

Vol of water = Area × length (height)

$= 983 \times 15$

$= 14\ 745 \text{ cm}^3$

(3)

[10]

[TOTAL 200]