Grade 12 AP Maths Prelim 2019

SECTION A ALGEBRA and CALCULUS (200 marks)

QUESTION 1

Jerry wants to prove the formula that is sometimes used in a Reimann sum:

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} \quad \text{and thus that,}$$

$$1+8+27+...+n^3 = \frac{n^2(n+1)^2}{4}, \quad \text{for all } n \in \mathrm{N}$$

He does the following steps:

Step	
1	Let $n=1$:
2	LHS $=1$ RHS $=1$
3	$\therefore$ statement is true for $n=1$
4	Let $n=k$:
5	$1+8+27+...+k^3 = \frac{k^2(k+1)^2}{4}$
6	Let $n=k+1$:
7	$RHS = \frac{(k+1)^2(k+2)^2}{4}$
8	$LHS =$
9	Conclusions

a) Jerry made an error in Step 4. Write the correct version of this step (1)

Assume true for $n=k$ ✓

b) Jerry could not finish Step 8. Write down this part of the proof

$$LHS = \frac{k^2(k+1)^2}{4} + (k+1)^3 \checkmark$$

$$= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4} \checkmark$$

$$= \frac{(k+1)^2\left[k^2+4(k+1)\right]}{4} \checkmark$$

$$= \frac{(k+1)^2\left[k^2+4k+4\right]}{4} \checkmark$$

$$=\frac{(k+1)^2(k+2)^2}{4}=RHS$$

Proved true for $n=1$, and that true for $n=k$ implies also true for $n=k+1$.

So, by Induction, $1+8+27+...+n^3=\frac{n^2(n+1)^2}{4}$, **for all** $n\in\mathbb{N}$ is true (8)

[9]

QUESTION 2

a) Solve for x, without the use of a calculator, if

$$\log_5(x-3)-\log_{\frac{1}{5}}(x-2)-\log_5 2=2$$

$$\therefore \log_5(x-3)+\log_5(x-2)-\log_5 2=\log_5 25$$

$$\Rightarrow \log_5\left[\frac{(x-3)(x-2)}{2}\right]=\log_5 25$$

$$\Rightarrow \frac{(x-3)(x-2)}{2}=25$$

$$\Rightarrow (x-3)(x-2)=50$$

$$\Rightarrow x^2-5x+6=50$$

$$\Rightarrow x^2-5x-44=0$$

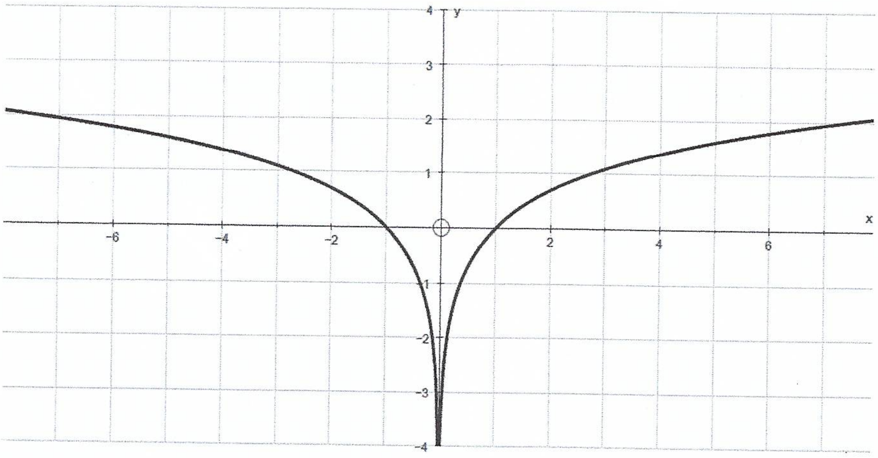
$$\Rightarrow x=\frac{-(-5)\pm\sqrt{25-4.1.(-44)}}{2}$$

$$\Rightarrow x=\frac{5\pm\sqrt{201}}{2}$$

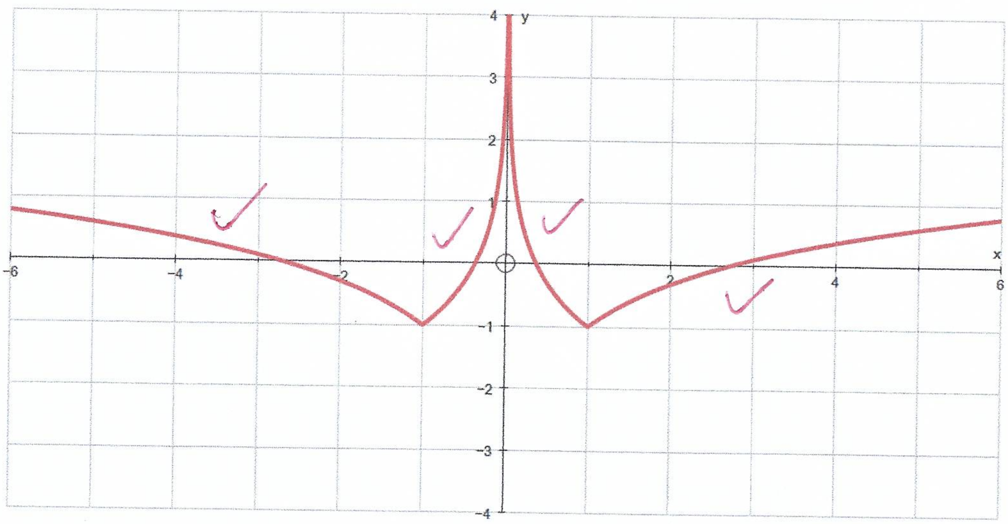
$\Rightarrow x=9{,}59$ or $\cancel{-4{,}59}$ (no logs of $-ve$ nos.) (8)

b) Below is the graph of $f(x) = \ln|x|$

(3)



On the Answer Sheet provided draw the graph of $g(x) = |f(x)| - 1$ clearly showing any intercepts with the axes.



(4)

Intercepts: $y-$**int.:** $x = 0$, **no int. as** $\ln 0$ **does not exist**

$x-$**int.:** $y = 0$, $\ln|x| = 1$ **or** $\ln|x| = -1$

$|x| = e \Rightarrow x = \pm 2{,}72$ **or** $|x| = e^{-1} \Rightarrow x = \pm \frac{1}{e} = \pm 0{,}37$ (6)

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QUESTION 3

a) Expand $\left(\sqrt{2016i^{2016}}-i\right)^2$, without the use of a calculator, leaving your answer in the form: $a+bi$ and including surds if necessary.

$$\left(\sqrt{2016i^{2016}}-i\right)^2 = 2016i^{2016} - 2i\sqrt{2016i^{2016}} + i^2$$

$$= 2016.1 - 2i.i^{1008}.6\sqrt{56} - 1$$

$$= 2015 - 24\sqrt{14}i$$

(6)

b) Given: $f(x) = x^4 + 4x^3 + 3x^2 + 4x + 2$

with $f\left(-2+\sqrt{2}\right)=0$ and $f\left(-2-\sqrt{2}\right)=0$

fully factorise $f(x)$ with $x \in C$

$f(x) = x^4 + ax^3 + 3x^2 + bx + 2.$

$x = -2 \pm \sqrt{2}.$

Add -4

$\times$ 2

$(x^2 + 4x + 2)(x^2 + 1) = 0$

$x^4 + 4a^3 + 3x^2 + 4x + 2 = 0$

$\therefore a = 4$

$b = 4$

(7)

c) (5)

Solve the following equations:

1) $|2x^2-3x|=1$

either $2x^2-3x=1$ or $2x^2-3x=-1$

either $2x^2-3x-1=0$ or $2x^2-3x+1=0$

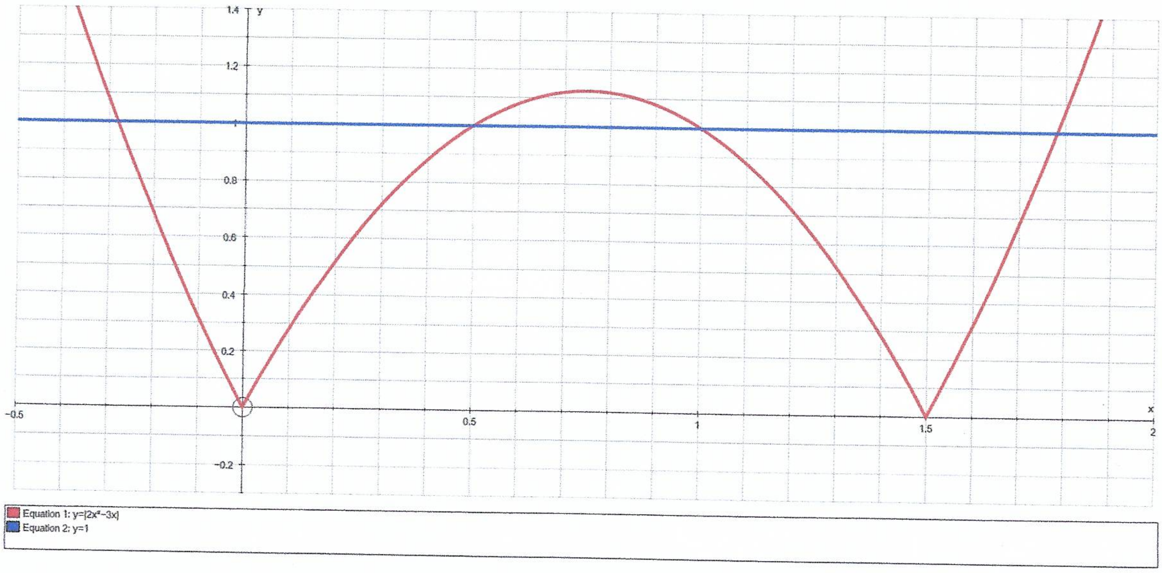
either $x=\dfrac{3\pm\sqrt{9-4.2.-1}}{4}$ or $(2x-1)(x-1)=0$

either $x=\dfrac{3\pm\sqrt{17}}{4}$ or $x=\tfrac{1}{2}$ or $x=1$

either $x=1,78$ or $x=-0,28$ or $x=\tfrac{1}{2}$ or $x=1$ (7)

Can sketch and solve or solve algebraically

restriction



2) $3e^x-\dfrac{2}{e^x}=1$

$\Rightarrow 3e^{2x}-e^x-2=0$

$\Rightarrow (3e^x+2)(e^x-1)=0$

$\Rightarrow \cancel{e^x=-\tfrac{2}{3}}$ or $e^x=1$

$\Rightarrow x=0$ (6)

.3) $\ln\left(e^{2x}-6\right)=x$

$\Rightarrow e^{2x}-6=e^{x}$ ✓

$\Rightarrow e^{2x}-e^{x}-6=0$ ✓

$\Rightarrow \left(e^{x}+2\right)\left(e^{x}-3\right)=0$ ✓

$\Rightarrow \cancel{e^{x}=-2}$ ✓ or $e^{x}=3$ ✓

$\Rightarrow x=\ln 3=1{,}1$ ✓✓ (7)

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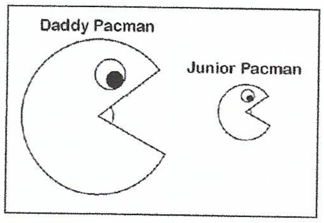
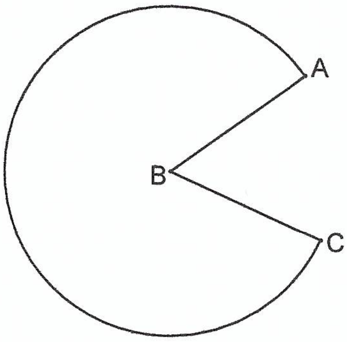
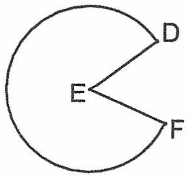
QUESTION 4

The dimensions of Junior Pacman are given below. Note that Junior Pacman has a radius **half** that of Daddy Pacman, and that $D\hat{E}F = A\hat{B}C$.







Length of **Major arc DF = 15,71 cm** and $D\hat{E}F = \frac{\pi}{3}$

a) Determine the perimeter of Daddy Pacman.

Junior major arc: $s = r\theta$

$$\Rightarrow 15,71 = r.\,{}^{5\pi}\!/_{3}$$

$$\Rightarrow r = 15,71.\,{}^{3}\!/_{5\pi} = 3cm$$

Daddy: $R = 2r \quad \Rightarrow R = 6cm$

Daddy: $S = R\theta \quad \Rightarrow S = 6\,{}^{5\pi}\!/_{3}\,cm = 10\pi cm$

$\therefore$ **Perimeter** $= 2R + 10\pi = 43,42cm$ (6)

b) Determine the area of Junior Pacman.

Junior: $a = \tfrac{1}{2} r^2\theta$

$$a = {}^{1}\!/_{2}\,3^2\,{}^{5\pi}\!/_{3} = 23,56cm^2$$ (3)

[9]

QUESTION 5

Consider the function: $f(x)=\begin{cases} -(x-1)^2+2 & \text{if} \quad x<1 \\ |x-3| & \text{if} \quad 1\le x<3 \\ -2^x+8 & \text{if} \quad 3\le x \end{cases}$

a) Discuss the continuity of $f(x)$ at $x=3$.

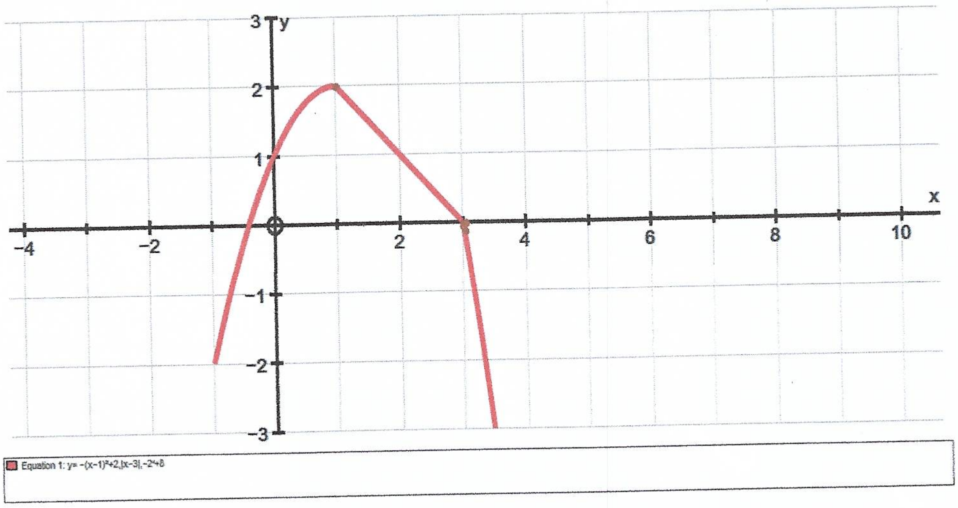
At $x=3$, $\lim\limits_{x\to 3^-}|x-3|=0$ and $\lim\limits_{x\to 3^+}(-2^x+8)=0$

$\therefore$ since $\lim\limits_{x\to 3^-}|x-3|=\lim\limits_{x\to 3^+}(-2^x+8)=0$

and since $f(x)$ **is defined at** $x=3$, $f(3)=-2^3+8=0$,

then $f(x)$ **is continuous at** $x=3$

(graph not required for solution, merely for illustration purposes)



(5)

b) Is $f(x)$ differentiable at $x=1$? Justify your answer algebraically, showing all of your working.

At $x=1$, $\lim_{x\to 1^-} -(x-1)^2+2=2$ and $\lim_{x\to 1^+}|x-3|=2$

and $f(x)$ is defined at $x=1$, then $f(x)$ is continuous at $x=1$

For differentiability at $x=1$, check if:- $\lim_{x\to 1^-} f'(x)=\lim_{x\to 1^+} f'(x)$

$\lim_{x\to 1^-} f'(x)=\lim_{x\to 1^-}(-2(x-1))=0$

at $x=1$, $f(x)=|x-3| = -x+3$

$\therefore \lim_{x\to 1^+} f'(x)=\lim_{x\to 1^+}(-1)=-1$

Since $\lim_{x\to 1^-} f'(x)\neq \lim_{x\to 1^+} f'(x)$

$f(x)$ is NOT differentiable at $x=1$ (5)

[10]

QUESTION 6

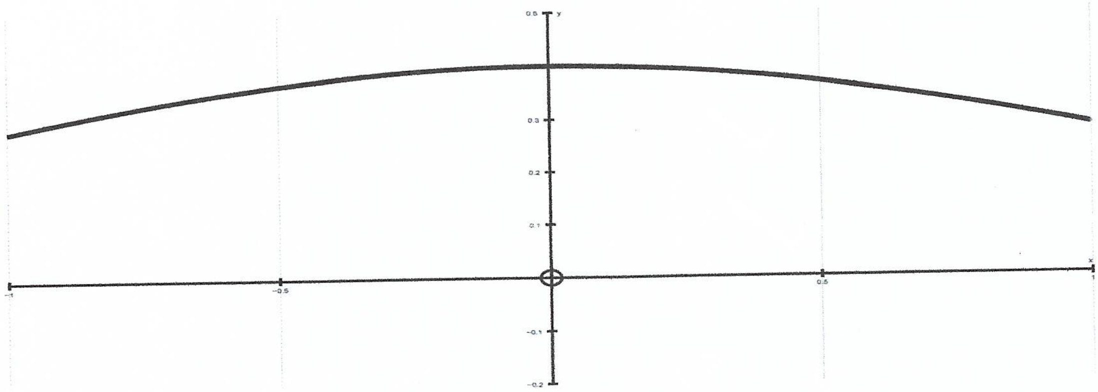
Determine the following limits, if they exist:

a) $\lim_{x\to 0}\frac{\sin 2x.\tan x}{5x^2}$

$=\lim_{x\to 0}\left(\frac{\sin 2x}{2x}\times\frac{\tan x}{x}\right).\frac{2}{5}$

$=\frac{2}{5}.\lim_{x\to 0}\left(\frac{\sin 2x}{2x}\right).\lim_{x\to 0}\left(\times\frac{\tan x}{x}\right)$

$=\frac{2}{5}\times 1\times 1=\frac{2}{5}$ (5)



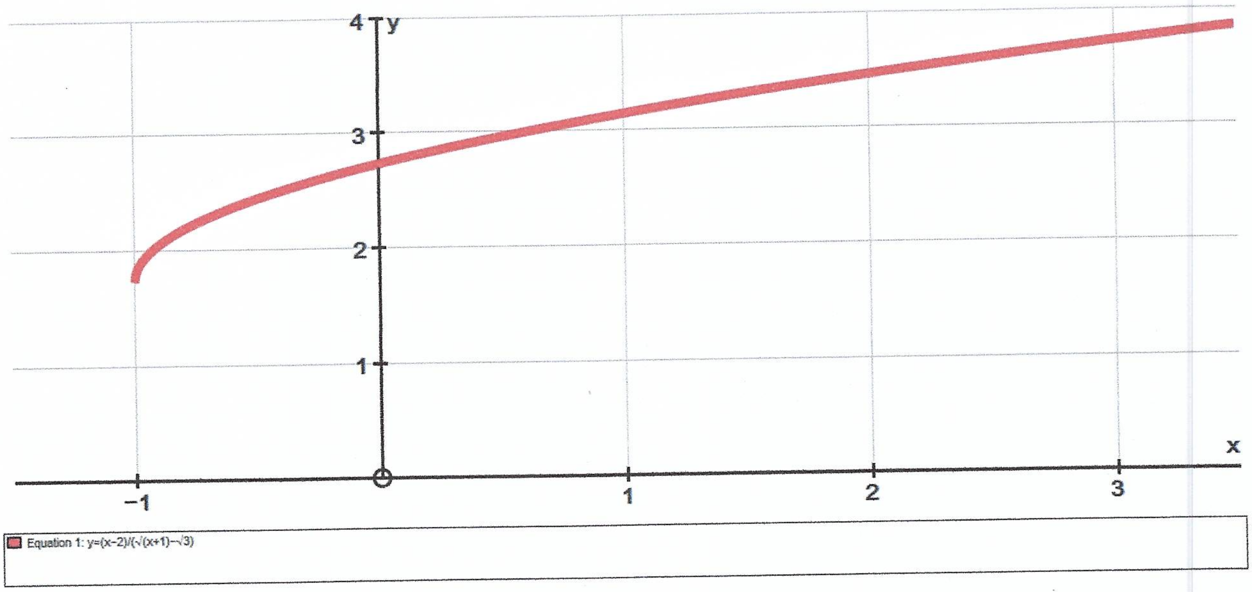
(graph not required for solution, merely for illustration purposes)

b) $\lim\limits_{x\to 2}\dfrac{x-2}{\sqrt{x+1}-\sqrt{3}}$

$$\lim_{x\to 2}\frac{x-2}{\sqrt{x+1}-\sqrt{3}}=\lim_{x\to 2}\frac{(x-2)}{(\sqrt{x+1}-\sqrt{3})}\times\frac{(\sqrt{x+1}+\sqrt{3})}{(\sqrt{x+1}+\sqrt{3})}$$

$$=\lim_{x\to 2}\frac{\cancel{(x-2)}(\sqrt{x+1}+\sqrt{3})}{\cancel{(x-2)}}=\underline{2\sqrt{3}} \qquad (7)$$

(graph not required for solution, merely for illustration purposes)



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c) $$\lim_{x\to\infty}\frac{5x^2-2x+3}{\sqrt{9x^4+5}}$$

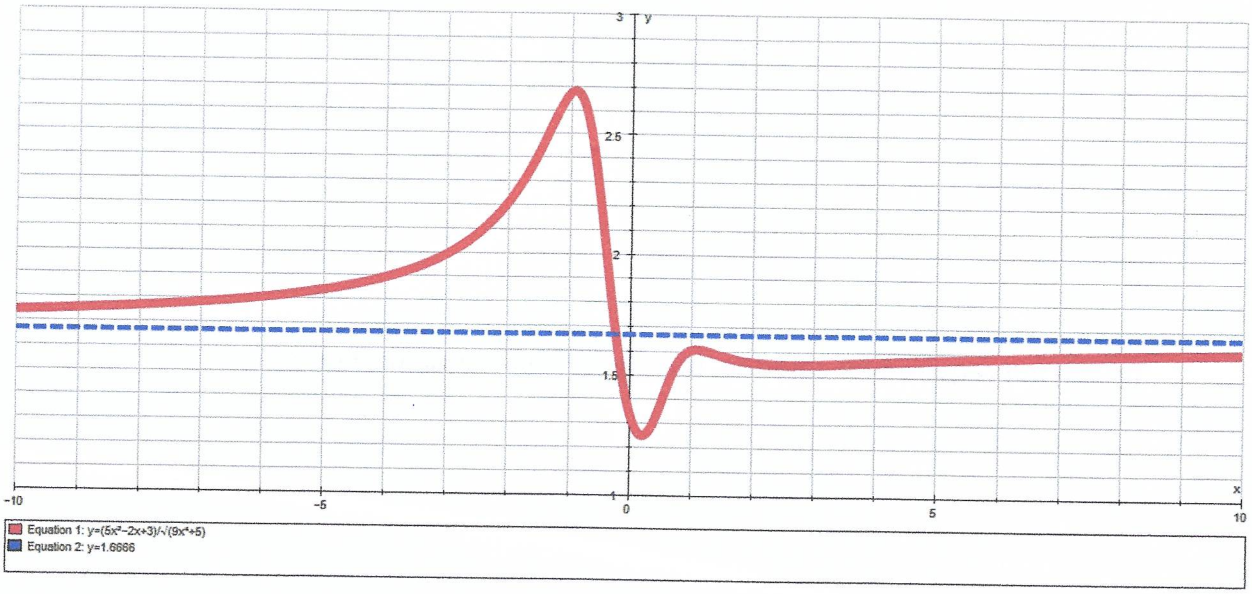
$$=\lim_{x\to\infty}\frac{\frac{5x^2}{x^2}-\frac{2x}{x^2}+\frac{3}{x^2}}{\sqrt{\frac{9x^4}{x^4}+\frac{5}{x^4}}}$$ ✓ ✓

(divide each term by highest power of x)

$$=\lim_{x\to\infty}\frac{5-\frac{2}{x}+\frac{3}{x^2}}{\sqrt{9+\frac{5}{x^4}}}=\frac{5}{3}$$ ✓ ✓ ✓ ✓ (6)

[18]

(graph not required for solution, merely for illustration purposes)



(12)

QUESTION 7

Show that the equation $x^3 - 3x + 1 = 0$ has a zero between -2 and -1 .

Hence use Newton's method to obtain a negative root of $x^3 - 3x + 1 = 0$

correct to 4 decimal places.

a) **For zero:** $f(-2) = -1$ **and** $f(-1) = 3$ (3)

$\therefore$ **Try** $x_1 = -1{,}8$

b) **Newton:** $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$ **where** $f'(x) = 3x^2 - 3$

When $x_1 = -1{,}8$ **:** $x_2 = -1{,}8 - \dfrac{(-1{,}8)^3 - 3(-1{,}8) + 1}{3(-1{,}8)^2 - 3}$

$x_2 = -1{,}88452381$
$x_3 = -1{,}879404727$
$x_4 = -1{,}879385242$

$x = -1{,}8794$ **(corr. to 4 d.p.)** (5)

[8]

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QUESTION 8

(1) Determine $f'(x)$ by first principles if $f(x)=\sqrt{2x+3}$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{\sqrt{2(x+h)+3}-\sqrt{2x+3}}{h}$$ ✓

$$f'(x)=\lim_{h\to 0}\frac{\left(\sqrt{2(x+h)+3}-\sqrt{2x+3}\right)}{h}\times\frac{\left(\sqrt{2(x+h)+3}+\sqrt{2x+3}\right)}{\left(\sqrt{2(x+h)+3}+\sqrt{2x+3}\right)}$$ ✓

$$f'(x)=\lim_{h\to 0}\frac{\left(\sqrt{2(x+h)+3}\right)^2-\left(\sqrt{2x+3}\right)^2}{h\left(\sqrt{2(x+h)+3}+\sqrt{2x+3}\right)}$$ ✓

$$f'(x)=\lim_{h\to 0}\frac{2(x+h)+3-2x-3}{h\left(\sqrt{2(x+h)+3}+\sqrt{2x+3}\right)}$$

$$f'(x)=\lim_{h\to 0}\frac{\cancel{2x}+2h\cancel{+3}\cancel{-2x}\cancel{-3}}{h\left(\sqrt{2(x+h)+3}+\sqrt{2x+3}\right)}$$ ✓

$$f'(x)=\lim_{h\to 0}\frac{2\cancel{h}}{\cancel{h}\left(\sqrt{2(x+h)+3}+\sqrt{2x+3}\right)}$$

$$f'(x)=\lim_{h\to 0}\frac{2}{\left(\sqrt{2(x+h)+3}+\sqrt{2x+3}\right)}$$ ✓

$$f'(x)=\frac{2}{\left(\sqrt{2x+3}+\sqrt{2x+3}\right)}$$ ✓

$$f'(x)=\frac{\cancel{2}}{\cancel{2}\sqrt{2x+3}}=\frac{1}{\sqrt{2x+3}}$$

(6)

b) Determine $f'(x)$ given that:

$f(x)=x^3\ln(3x)$ ~~and given that:~~ $\cancel{\frac{d}{dx}(\ln x)=\frac{1}{x}}$

$$f'(x)=x^{\cancel{3}2}\cdot\frac{1}{\cancel{3}\cancel{x}}\cdot\cancel{3}+3x^2.\ln(3x)$$

$$f'(x)=x^2+3x^2.\ln(3x)$$

(6)

c) Find the gradient of the graph of $y=2\tan^2\left(\frac{\pi}{2}-3\theta\right)$ when $\theta=\frac{\pi}{4}$

$$\frac{dy}{d\theta}=4\tan\left(\frac{\pi}{2}-3\theta\right).\sec^2\left(\frac{\pi}{2}-3\theta\right).-3$$

$$=-12\tan\left(\frac{\pi}{2}-3\theta\right).\sec^2\left(\frac{\pi}{2}-3\theta\right)$$

when $\theta=\frac{\pi}{4}$, gradient $=-12\tan\left(\frac{\pi}{2}-3.\frac{\pi}{4}\right).\sec^2\left(\frac{\pi}{2}-3.\frac{\pi}{4}\right)$

$$=-12\tan\left(-\frac{\pi}{4}\right).\sec^2\left(-\frac{\pi}{4}\right)$$

$$=-12(-1).\left(\frac{1}{\frac{1}{2}}\right)=24$$

(7)

d) Given $f(x)=\frac{1}{2x^2}$ determine a formula for $f^{(n)}(x)$

$$f(x)=\tfrac{1}{2}x^{-2}$$

$$f'(x)=\tfrac{1}{2}.(-2)x^{-3}$$

$$f''(x)=\tfrac{1}{2}.(-2).(-3)x^{-4}$$

$$f^{(3)}(x)=\tfrac{1}{2}.(-2).(-3)(-4)x^{-5}$$

$$f^{(n)}(x)=\tfrac{1}{2}.(-1)^n\,2.3.4...(n+1)x^{-n-2}$$

$$f^{(n)}(x)=\tfrac{1}{2}.(-1)^n(n+1)!\,x^{-n-2}$$

(7)[26]

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QUESTION 9

$$\text{Given } f(x) = \frac{x^2+5x+6}{x+1}.$$

a) Find the co-ordinates of:

.1) the intercepts with the axes

y − **axis:** $x = 0 \Rightarrow f(0) = 6 \quad (0;6)$

x − **axis:** $f(x) = 0$

$$\therefore \frac{x^2+5x+6}{x+1} = 0 \quad \Rightarrow x^2+5x+6=0$$

$$(x+2)(x+3) = 0 \quad \Rightarrow x = -2 \;\&\; x = -3 \quad (-2;0) \quad (-3;0)$$

(6)

.2) the stationary points (correct to 1 decimal digit)

$$f'(x) = \frac{(x+1)(2x+5)-(x^2+5x+6)}{(x+1)^2}$$

$$f'(x) = \frac{(2x^2+7x+5)-(x^2+5x+6)}{(x+1)^2}$$

$$f'(x) = \frac{x^2+2x-1}{(x+1)^2}$$

at stationary pt. $f'(x) = 0 \quad \therefore x^2+2x-1=0$

$$x = \frac{-2 \pm \sqrt{4-4.1.-1}}{2} = \frac{-2 \pm \sqrt{8}}{2}$$

$\Rightarrow x = 0{,}4$ **or** $-2{,}4$

stationary pts.: $(0{,}4;5{,}8) \quad (-2{,}4;0{,}2)$

(10)

b) Find the equations of:

.1 the vertical asymptote

For vert asymptote, $x+1=0 \Rightarrow x=-1$

(3)

.2) the oblique asymptote

$$f(x)=\frac{x^2+5x+6}{x+1}=\frac{x^2+x+4x+4+2}{x+1}$$

$$=\frac{x(x+1)+4(x+1)+2}{x+1}$$

✓ method

$$=x+4+\frac{2}{x+1}$$

Oblique asymptote: $y=x+4$ ✓ (3)

c) Discuss the concavity of $f(x)$ by means of suitable calculations, and hence classify the stationary points in 9.1.2.

$$f'(x)=\frac{x^2+2x-1}{(x+1)^2}$$

$$f''(x)=\frac{(x+1)^2(2x+2)-(x^2+2x-1).1}{(x+1)^4}$$!!! ✓✓

Try: $f(x)=x+4+\dfrac{2}{x+1}$

$$f'(x)=1+\frac{(x+1).0-2.1}{(x+1)^2}$$ ✓✓

$$f'(x)=1-\frac{2}{(x+1)^2}$$ ✓✓

$$f''(x)=-\frac{(x+1)^2 0-2.2(x+1)}{(x+1)^4}$$ ✓✓

$$=-\frac{-4(x+1)}{(x+1)^4}$$

$f''(0,4)=1,46$ $+ve$ $\therefore$ MIN ✓✓

$f''(-2,4)=-1,46$ $-ve$ $\therefore$ MAX ✓✓ (12)

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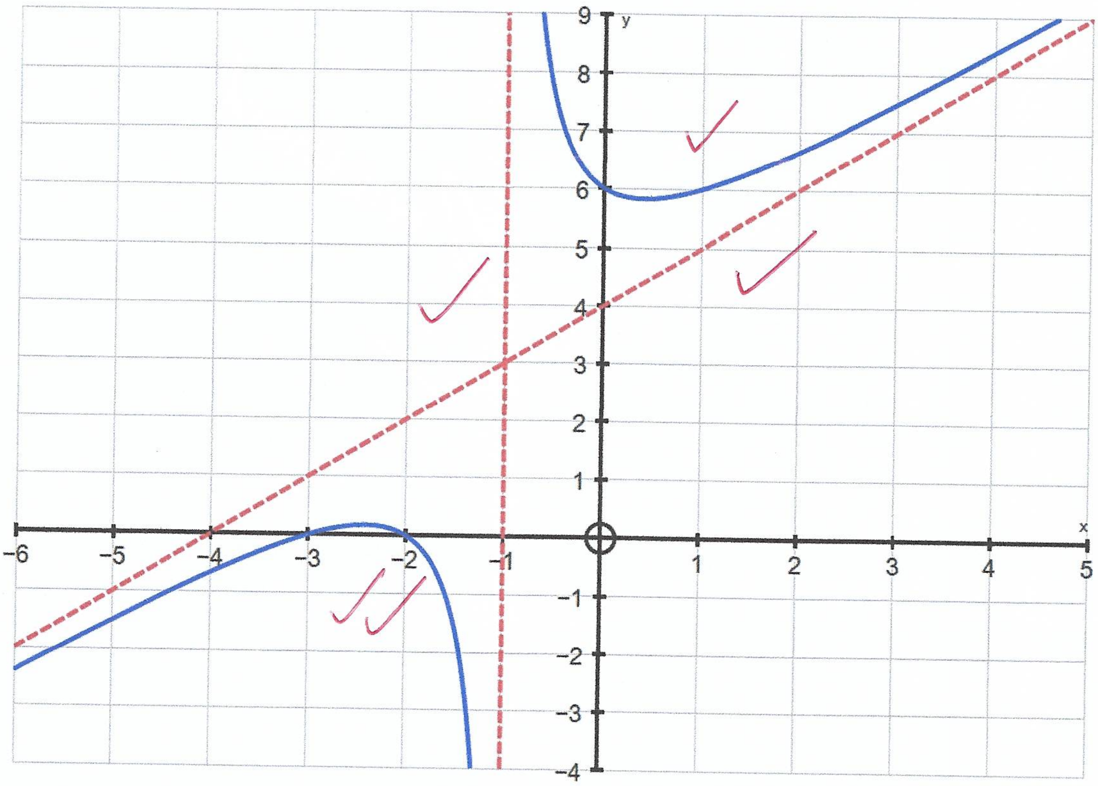
d) Use all the workings above to draw a neat sketch of $f(x)$



(5)

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QUESTION 10

a) Determine the following integrals:

.1)
$$\int \frac{3}{\sqrt{4x-1}}\,dx$$

$$= \int 3(4x-1)^{-\frac{1}{2}}\,dx$$

$$= 3(4x-1)^{\frac{1}{2}}.\frac{2}{4}+c \quad = \frac{3}{2}\sqrt{4x-1}+c$$ (5)

.2)
$$\int \frac{3x}{\sqrt{4x^2-1}}\,dx$$

let $u = 4x^2-1 \quad \Rightarrow \quad du = 8x\,dx \quad \Rightarrow \frac{3}{8}du = 3x\,dx$

$$\therefore I = \int \frac{3u^{-\frac{1}{2}}}{8}\,du$$

$$I = \frac{3}{4}u^{\frac{1}{2}}+c \quad = \frac{3}{4}\sqrt{4x^2-1}+c$$ (6)

3) $\int \frac{3x}{\sqrt{4x-1}}dx$ (Hint, use integration by parts)

let $u = x$ $\quad \frac{du}{dx} = 1$

$\frac{dv}{dx} = \frac{3}{\sqrt{4x-1}}$ $\quad v = \frac{3}{2}\sqrt{4x-1}$ (see 1)

$$I = uv - \int v du$$

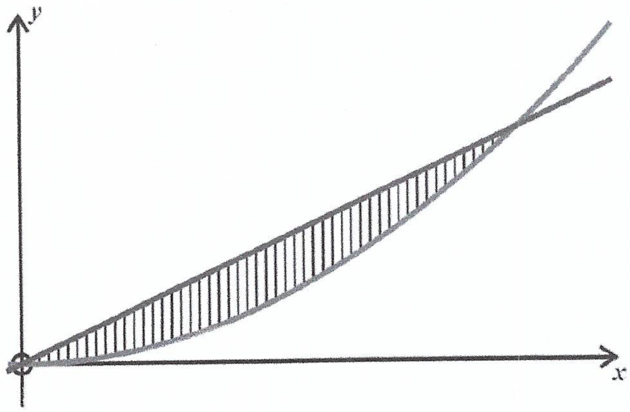
$$= \frac{3}{2}x\sqrt{4x-1} - \frac{3}{2}\int \sqrt{4x-1}\, dx$$

$$= \frac{3}{2}x\sqrt{4x-1} - \frac{3}{2}\int (4x-1)^{1/2}\, dx$$

$$= \frac{3}{2}x\sqrt{4x-1} - \frac{3}{2}(4x-1)^{3/2} \cdot \frac{2}{3.4} + c$$

$$= \frac{3}{2}x\sqrt{4x-1} - \frac{1}{4}\sqrt{(4x-1)^3} + c$$ (9)

b) Determine the volume of the solid generated by rotating the area bounded by the curves $y = x$ and $y = x^2$ about the x-axis, as indicated by the shaded regions below.



Vol $= \pi \int y^2 dx$

Pts of intersection: $y = x$ & $y = x^2$:

$x = x^2 \Rightarrow x^2 - x = 0$

$\Rightarrow x(x-1) = 0$

$\Rightarrow x = 0 \text{ or } x = 1$

$$\text{Vol} = \pi\int_0^1 (x)^2\,dx - \pi\int_0^1 (x^2)^2\,dx$$

$$= \pi\int_0^1 (x^2 - x^4)\,dx$$

$$= \pi\left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1$$

$$= \pi\left[\left(\frac{1}{3} - \frac{1}{5}\right) - (0)\right]$$

$$= \pi\left(\frac{1}{3} - \frac{1}{5}\right)$$

$$= \pi\left(\frac{5}{15} - \frac{3}{15}\right) = \frac{2}{15}\pi$$

(10)

[30]