ADVANCED PROGRAMME MATHEMATICS

GRADE 12

SEPTEMBER 2019

PAPER 1 ALGEBRA AND CALCULUS

TIME: 2 HOURS **MARKS: 200**

PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY

1. This question paper consists of 9 pages and an Information Booklet of 4 pages (i-iv).
2. Non-programmable and non-graphical calculators may be used, unless otherwise indicated.
3. All necessary calculations must be clearly shown, and writing should be legible.
4. Diagrams have not been drawn to scale.
5. All angles must be expressed in **RADIANS** rather than degrees.
6. Round off your answers to **TWO** decimal digits, unless otherwise indicated.

QUESTION 1

1.1 Solve for $x \in \mathbb{R}$ without using a calculator:

$$|x|^2 - 4|x| - 12 = 0 \quad (6)$$

1.2 By using the laws of logarithms, prove that:

$$\frac{3 + \ln x^3}{\ln(ex)} = 3 \quad (3)$$

1.3 Newton's law of cooling for a liquid, in this case a cup of soup, is given by the equation:

$$T = T_s + (T_0 - T_s)e^{-kt}$$

where

t	is the time in minutes
k	is a constant for the specified fluid
T	is the temperature in °C at any given time t
T_0	is the initial temperature in °C (the value of T at $t = 0$)
T_s	is the surrounding temperature in °C

(a) A cup of soup cooled from 90°C to 60°C in 10 minutes in a room where the temperature was 20°C. Show, by solving for k and showing all working, that

$$k = -\frac{1}{10}\ln\left(\frac{4}{7}\right) \quad (8)$$

(b) Determine the temperature of the soup after 15 minutes (from the beginning). Round your answer off to the nearest whole number. (3)

[20]

QUESTION 2

Given an equation of the form $x^2 + bx + c = 0$, with one of its roots $x = 2 - 3i$.

2.1 If b and c are real numbers, find their values. (6)

2.2 If $c = -4 + 19i$, determine the value of b, in the form $p + qi$. (10)

[16]

QUESTION 3

Use Mathematical Induction to prove that $\sum_{p=1}^{n} \frac{1}{(2p-1)(2p+1)} = \frac{n}{2n+1}$ **[12]**

QUESTION 4

4.1 Given: $f(x) = \begin{cases} e^x & \text{if } x < 0 \\ x^2 + 1 & \text{if } 0 \le x < 2 \\ \ln(x-1) & \text{if } x \ge 2 \end{cases}$

(a) Determine whether *f* is differentiable at $x = 0$. Show full workings. (6)

(b) Determine whether *f* is differentiable at $x = 2$. Show full workings. (4)

4.2 Given: $f(x) = \ln|x|$ and $g(x) = |\ln x|$

(a) Sketch the graph of *f* and *g* on the same set of axes. (6)

(b) For which values of *x* is $f(x) = g(x)$? (1)

[17]

QUESTION 5

5.1 Determine $\frac{dy}{dx}$ if $y = e^{2x}.\ln 2x \qquad (x > 0)$ (4)

5.2 Given: $f(x) = \cot x$

(a) Determine $f'(x)$ and hence show that $f''(x) = \frac{2\cos x}{\sin^3 x}$. (4)

(b) Show that *f* has a non-stationary point of inflection at $x = \frac{\pi}{2}$. (5)

5.3 Given: $g(x) = x^4 - 4x^3 + 6x^2 - 4x + 1$

Consider the following statement:

If the second derivative of a function at $x = a$ is equal to zero, then the graph of the function will always have a point of inflection at $x = a$.

Determine whether this statement is true for the graph of *g*. (6)

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QUESTION 6

6.1 Determine the equations of the asymptotes of the graphs of the following:

(a) $f(x)=\dfrac{3x^2-2x+1}{4x^2+3}$ (1)

(b) $f(x)=\dfrac{3x^2-2x+1}{4x^3+3}$ (1)

(c) $f(x)=\dfrac{3x^2-2x+1}{x+1}$ (3)

6.2 Given: $f(x)=\dfrac{(x+3)(x-4)}{(2x-1)(x-4)}$

Explain why the graph of *f* only has one vertical asymptote. (2)

6.3 Given: $f(x)=\dfrac{x}{\ln x}$

Determine the coordinates of the stationary point on the graph of *f.* (8)

[15]

QUESTION 7

7.1 Given: $2x + 3y - xy^2 + 4 = 0$

(a) Show that $\frac{dy}{dx} = \frac{y^2 - 2}{3 - 2xy}$ (5)

(b) Determine the equation of the tangent at $(-1; -2)$. (3)

7.2 Given: $f(x) = 3\sin 4x$ and $g(x) = \frac{1}{2}(x+1)^2 - 6$

(a) Consider the interval $x \in [0; \frac{\pi}{2}]$ in which $f(x) > g(x)$.
Show that the maximum distance between the two graphs in the interval can be obtained by solving the equation

$12\cos 4x = x + 1$. (4)

(b) Use Newton-Raphson iteration to write down a recursive equation that can be used to solve the equation in (a). (4)

(c) Hence, taking $x = 0{,}5$ radians as an initial approximation, calculate the solution to the equation, correct to five decimal places. (5)

[21]

QUESTION 8

8.1 Determine:

(a) $\int e^{2x+1}\,dx$ (3)

(b) $\int \frac{4x}{\sqrt{x^2-1}}\,dx$ (6)

(c) $\int \frac{4x}{x^2-1}\,dx$ (4)

8.2 (a) Express $\frac{1}{x^3+x^2}$ in partial fractions in the form $\frac{a}{x}+\frac{b}{x^2}+\frac{c}{x+1}$. (6)

(b) Hence determine $\int \frac{1}{x^3+x^2}\,dx$.

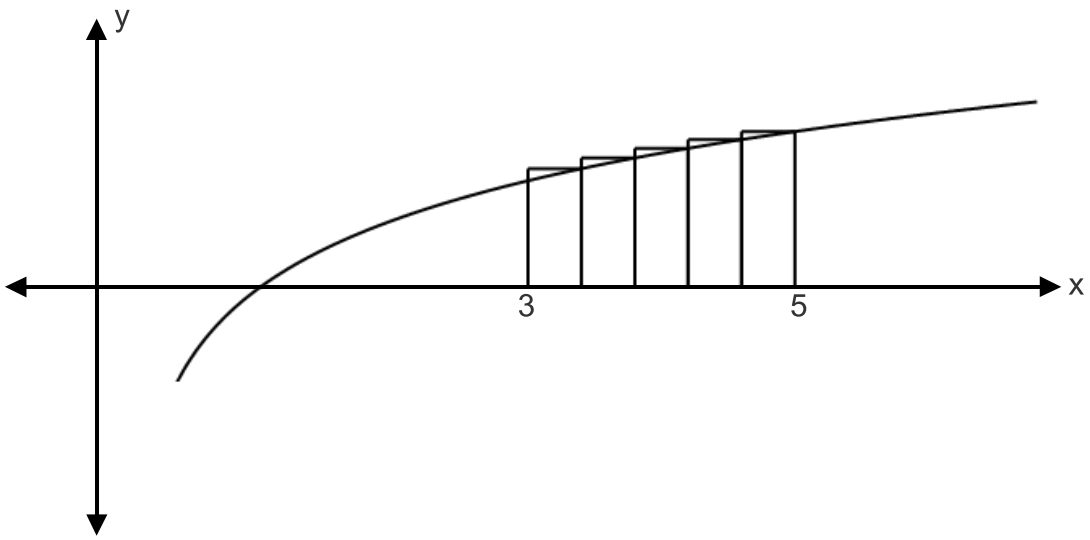
You may assume that $a=-1$; $b=1$ and $c=1$. (5)

8.3 Use integration by parts to determine $\int 2x.e^{3x}\,dx$. (6)

[30]

QUESTION 9

The graph of $y = f(x) = \ln x$ is shown below.



The area enclosed by the graph, the x-axis and the lines $x = 3$ and $x = 5$, can be determined by the Riemann sum formula:

$$A = \lim_{n\to\infty} \Delta x_i \sum_{i=1}^{n} f(x_i)$$

9.1 Write down an expression for Δx_i in terms of n. What does this symbol represent? (2)

9.2 What does $f(x_i)$ represent? (1)

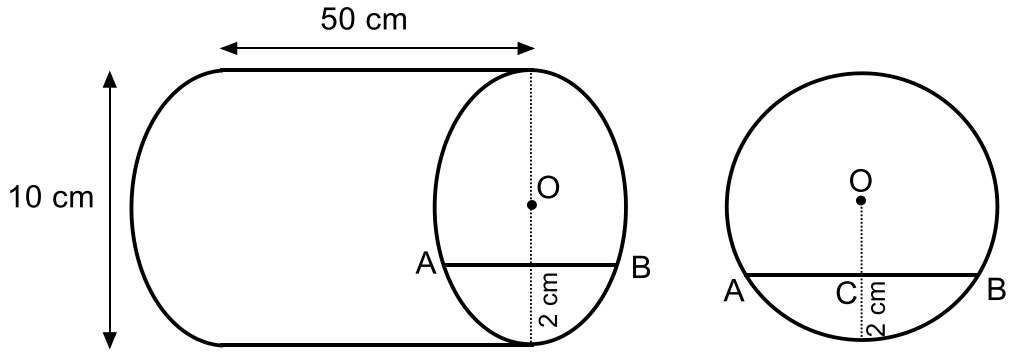
9.3 Determine an approximation for the area enclosed by the graph, the x-axis and the lines $x = 3$ and $x = 5$, by working with the 5 rectangles shown in the diagram. (10)

9.4 Calculate, rounded off to three decimal places, the exact enclosed area by using integration. (2)

9.5 Is the approximation for the area calculated using the rectangles an over or an under-approximation of the exact area. Explain your answer. (2)

[17]

QUESTION 10



A cylindrical pipe with a diameter of 10 cm and length of 50 cm, is lying on its side and there is water inside to a depth of 2 cm.

10.1 Show that the width of the surface of the water (the length of AB) is 8 cm. (4)

10.2 Hence find the volume of water in the pipe. (9)

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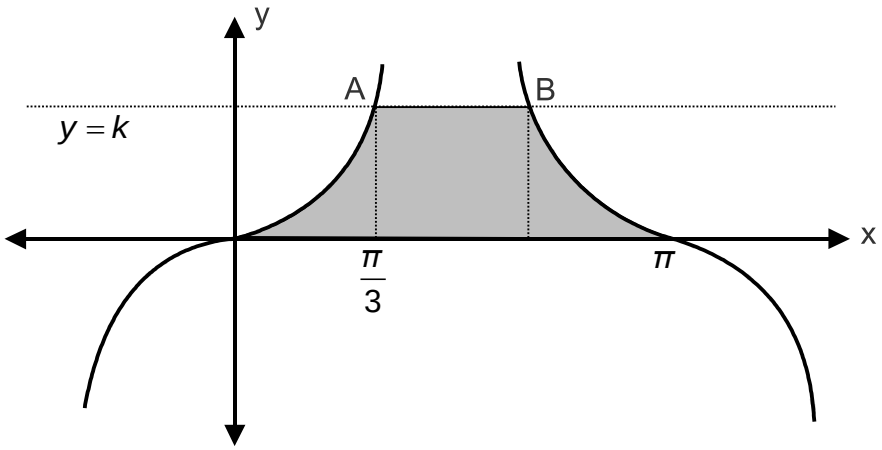
QUESTION 11

11.1 The sketch below shows the graph of $f(x) = \tan x.\sec x$, which cuts the x-axis at 0 and π, and the line $y = k$, which cuts f at $A\left(\frac{\pi}{3}; k\right)$ and at B.



(a) Determine the coordinates of A and B. Give the answer in surd form where necessary. (4)

(b) Write down an expression that can be used to obtain the area of the shaded region, which lies between f, the line $y = k$ and the x-axis. (4)

(c) Hence, calculate the value of this area. (4)

11.2 The graph of f rotates around the x-axis. The volume of the solid of revolution formed between $x = 0$ and $x = a$ is equal to $\frac{\pi}{3}$.

Determine the value of a. (8)

[20]

Total: 200 marks