MEMO.

St Mary's

DIOCESAN SCHOOL FOR GIRLS

GRADE 12 EXAMINATION

JULY 2019

ADVANCED PROGRAMME MATHEMATICS

PAPER 1: CALCULUS AND ALGEBRA

Examiner: Mrs Eiselen

Moderators: Mrs A van den Berg; Mrs R Narsai

Time: 2 hours **Marks: 200**

PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY

1. This question paper consists of **7** questions and 7 pages.
2. **ANSWER QUESTIONS 3(e), 4(a) and 4(b) ON THE QUESTION PAPER ON THE AXES PROVIDED.**
3. Read and answer all the questions carefully.
4. Number your answers exactly as the questions are numbered.
5. You may use an approved, non-programmable, and non-graphical calculator, unless otherwise stated.
6. Round off your answers to **one decimal digit** where necessary, **UNLESS STATED OTHERWISE**.
7. All the necessary working details must be clearly shown.
8. It is in your own interest to write legibly and to present your work neatly.

Name:____

Marking Grid (for Educators' use only)

Question Number	1	2	3	4	5	6	7	Total
Marks Earned								
Total for Question	74	53	42	4	7	10	10	200

1a(1) $f(x) = \dfrac{x^2 - 4x + 4}{x^2 + x - 6}$

$f(x) = \dfrac{(x-2)^2}{(x+3)(x-2)}$

$\therefore f(x) = \dfrac{(x-2)}{(x+3)}$

CNS: $x = -3; 2$

$+ \quad | \quad - \quad | \quad +$

$\quad\;\; -3 \qquad\;\; 2$

$f(x) > 0$ if $x \in (-\infty; -3) \cup (2; \infty)$ (7)

(2) $f'(x) = \dfrac{(1)(x+3) - (1)(x-2)}{(x+3)^2}$

$f'(x) = \dfrac{5}{(x+3)^2}$

CNS: $x = -3$

$+ \quad \phi \quad +$

$\qquad -3$

$f'(x) > 0$ if $x \in \mathbb{R}; x \neq -3.$ (10)

1b. $f(x) = \begin{cases} 2-x^2 & x \le a \\ x-4 & x > a. \end{cases}$

If continuous

$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

$\therefore \lim_{x \to a^-} 2-x^2 = \lim_{x \to a^+} x-4$

$\therefore 2-a^2 = a-4$

$a^2+a-6=0$

$(a+3)(a-2)=0$

$a=-3$ OR $a=2.$ (5)

c. $h(x) = 2x^4 + x^3 - 7x^2 + 21x - 9$

(1) if $x = 1+\sqrt{2}i$; $x = 1-\sqrt{2}i$

$\therefore$ polynomial factor of degree 2:

$a = 1$

$b = -(1+\sqrt{2}i + 1 - \sqrt{2}i) = -2$

$c = (1+\sqrt{2}i)(1-\sqrt{2}i) = 3$

$\therefore$ the factor is $x^2 - 2x + 3$ (3)

(2) $h(x) = (x^2 - 2x + 3)(2x^2 + kx - 3)$

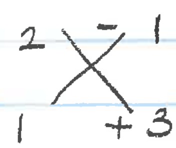
$= (x^2 - 2x + 3)(2x^2 + 5x - 3)$ if $6 + 3k = 21$

$3k = 15$

$k = 5.$

$= (x^2 - 2x + 3)(2x-1)(x+3)$

(7)



d. $f(x) = px + q$

$f(f(x)) = p(px+q) + q$

$\therefore \; p^2x + pq + q = 9x + 20$

$\therefore \; p^2 = 9$

$p = \pm 3$

and $pq + q = 20$

if $p = 3$

$4q = 20$

$q = 5$

if $p = -3$

$-2q = 20$

$q = -10.$

(12)

e. $50 = e^{50a} - 1$

$e^{50a} = 51$

$\log e^{50a} = \log 51$

$50a = \ln 51$

$a = \dfrac{\ln 51}{50}.$

(3)

$\approx 0,0786$

f. $f(x) = \sqrt{1-x}$

$f(x+h) = \sqrt{1-x-h}$

$$f'(x) = \lim_{h\to 0} \frac{\left(\sqrt{1-x-h} - \sqrt{1-x}\right)}{h} \cdot \frac{\left(\sqrt{1-x-h} + \sqrt{1-x}\right)}{\left(\sqrt{1-x-h} + \sqrt{1-x}\right)}$$

$$= \lim_{h\to 0} \frac{\cancel{1} - \cancel{x} - \cancel{h} - \cancel{1} + \cancel{x}}{\cancel{h}\left(\sqrt{(1-x-h)} + \sqrt{1-x}\right)}$$

$$= \frac{-1}{2\sqrt{1-x}}$$

(8)

g. $x^3 - 3x^2y + y^3 = -1$ $(2;3)$

$$3x^2 - 3(2x)y - 3x^2(1)\left(\frac{dy}{dx}\right) + 3y^2\left(\frac{dy}{dx}\right) = 0$$

$$\therefore \frac{dy}{dx} = \frac{-3x^2 + 6xy}{-3x^2 + 3y^2}$$

@ $(2;3)$

$$\frac{dy}{dx} = \frac{-3(2)^2 + 6(2)(3)}{-3(2)^2 + 3(3)^2}$$

$$\frac{dy}{dx} = \frac{8}{5}$$

(12)

$\therefore$ Equation of tangent

$$y - 3 = \frac{8}{5}(x-2)$$

$$y = \frac{8}{5}x - \frac{1}{5}$$

unnecessary

1h. $f(x) = e^{2x} - 3x$

$f'(x) = 2e^{2x} - 3$

$f'(0) = 2 - 3$

$= -1$

$f(0) = e^{2(0)} - 3(0)$

$= 1$

$\therefore y$ int $= (0;1)$

Equation of tangent :

$\therefore \quad y - 1 = -1(x - 0)$

$y = -x + 1$

(7)

2a) $\int x^{\frac{1}{2}}\sqrt{1+x^{\frac{3}{2}}}\,dx$

$u = 1 + x^{\frac{3}{2}}$

$\frac{du}{dx} = \frac{3}{2}x^{\frac{1}{2}}$

$\frac{2}{3}du = x^{\frac{1}{2}}dx$

$= \frac{2}{3}\int u^{\frac{1}{2}}\,du$

$\frac{2}{3}u^{\frac{3}{2}} + c$

$= \frac{4}{9}\left(1+x^{\frac{3}{2}}\right)^{\frac{3}{2}} + c$

(12)

b) $\int \ln x\,dx$

$= \int 1 . \ln x\,dx$

$= x . \ln x - \int x . \frac{1}{x}\,dx$

$= x\ln x - x + c.$

(6)

c) $\int \operatorname{cosec}^2(2-3x)\,dx.$

$u = 2 - 3x$

$\frac{du}{dx} = -3$

$\therefore -\frac{1}{3}du = dx$

$= -\frac{1}{3}\int \operatorname{cosec}^2 u\,du$

$-\frac{1}{3}(-\cot u) \quad +c$

$= +\frac{1}{3}\cot(2-3x) + c.$

(6)

d) $\int \sin\theta\cos^2\theta\,d\theta$

$u = \cos\theta$

$\frac{du}{d\theta} = -\sin\theta$

$-du = \sin\theta\,d\theta$

$-\frac{\cos^3\theta}{3} + c.$

(11)

e. $F(x) = \dfrac{2x^2 - 5x + 2}{(x-2)^3}$

(1) $2x^2 - 5x + 2 = (2x - 1)(x - 2)$ ✓ ✓ (2)

$F(x)$: $\dfrac{2x^2 - 5x + 2}{(x-2)^3} = \dfrac{2x-1}{(x-2)^2}$

(2) $\therefore \dfrac{2x-1}{(x-2)^2} = \dfrac{A}{(x-2)^2} ✓ + \dfrac{B}{(x-2)}$ ✓

$2x - 1 = 3 ✓ + B(x-2)$ ✓

$2x - 1 = 3 + Bx - 2B$

$\therefore B = 2$ ✓

$\therefore F(x) = \dfrac{3}{(x-2)^2} + \dfrac{2}{(x-2)}$ ✓ (6)

(3) $\int F(x)\,dx$

$= \int \dfrac{3}{(x-2)^2}\,dx ✓ + \int \dfrac{2}{(x-2)}\,dx$ ✓

$= -3(x-2)^{-1} + 2\ln|x-2| + C$ ✓✓ ✓ ✓ ✓ ✓ ✓ ✓ (10)

$= \dfrac{-3}{(x-2)} + 2\ln|x-2| + c$

Q3. $f(x) = x^3 + 2x^2 - 2x - 4$

If $x = -2$

$f(x) = (x+2)(x^2 + kx - 2)$

$\therefore f(x) = (x+2)(x^2 - 2)$

$g(x) = x^4 - 4$

$= (x^2+2)(x^2-2)$

$2k - 2 = -2$

$2k = 0$

$k = 0$

8

a) $h(x) = \dfrac{(x+2)\cancel{(x^2-2)}}{(x^2+2)\cancel{(x^2-2)}}$

$= \dfrac{(x+2)}{(x^2+2)}$

b) $\lim\limits_{x \to \infty} h(x)$

$= \lim\limits_{x \to \infty} \dfrac{\frac{x}{x^2} + \frac{2}{x^2}}{\frac{x^2}{x^2} + \frac{2}{x^2}}$

$= \dfrac{0}{1}$

$= 0.$

(10)

c) $h(x) = \dfrac{(x+2)}{(x^2+2)}$

$h'(x) = \dfrac{(1)(x^2+2) - (2x)(x+2)}{(x^2+2)^2}$

$= \dfrac{x^2 + 2 - 2x^2 - 4x}{(x^2+2)^2}$

$= \dfrac{-x^2 - 4x + 2}{(x^2+2)^2}$

St. pts: $x^2 + 4x - 2 = 0$

$x = -2 \pm \sqrt{6}$

$\therefore x \approx 0,45$ oR $-4,45$

(10)

d) $h(-2) = 0$

$h(0) = 1.$

(4)

QUESTION 3 [42]

Given that $f(x) = x^3 + 2x^2 - 2x - 4$ and $g(x) = x^4 - 4$. If $h(x) = \frac{f(x)}{g(x)}$.

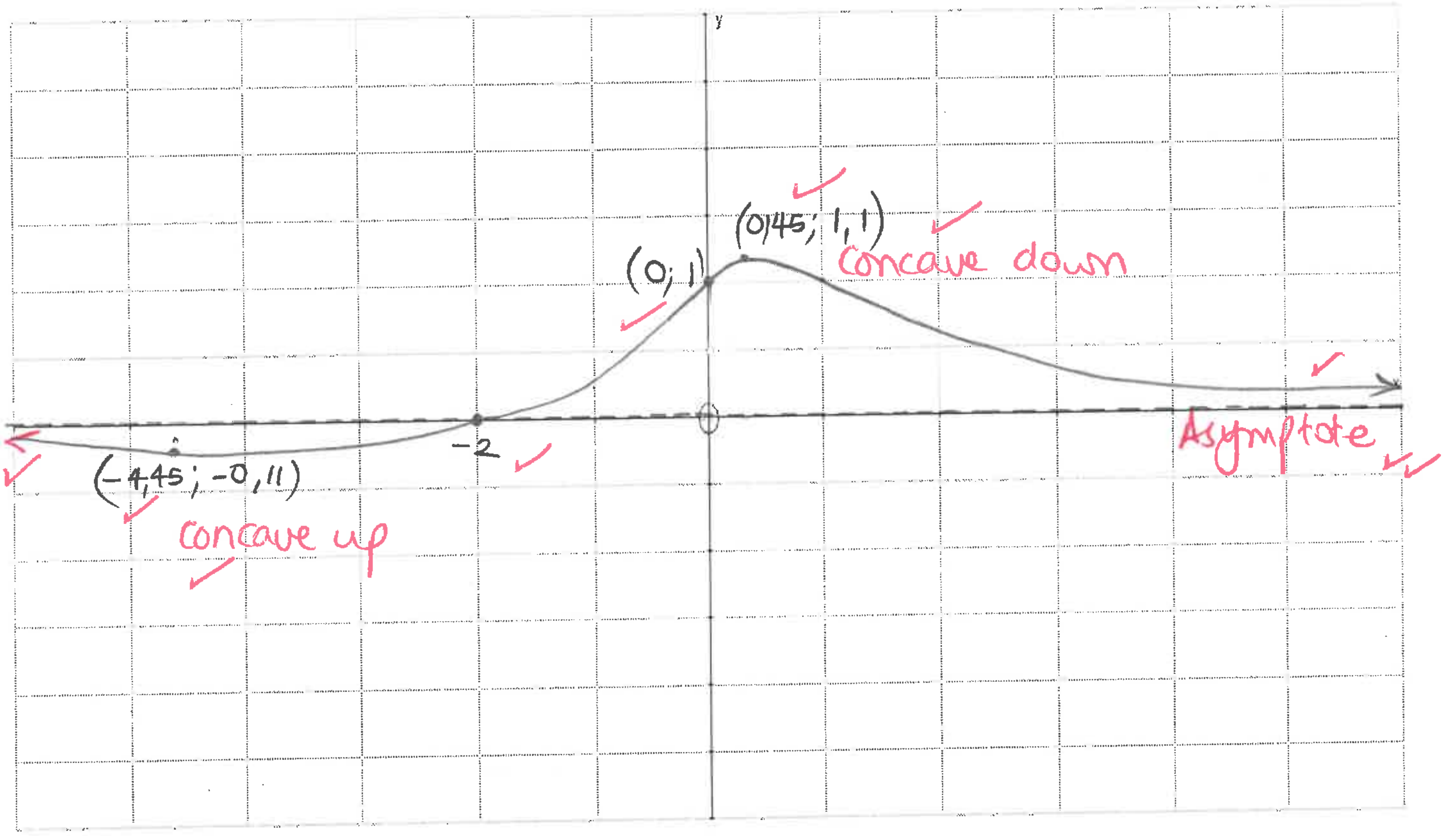
a) Show that $h(x) = \frac{x+2}{x^2+2}$ (8)

b) Define the horizontal asymptotes of $h(x)$ (10)

c) Calculate $h'(x) = 0$ (10)

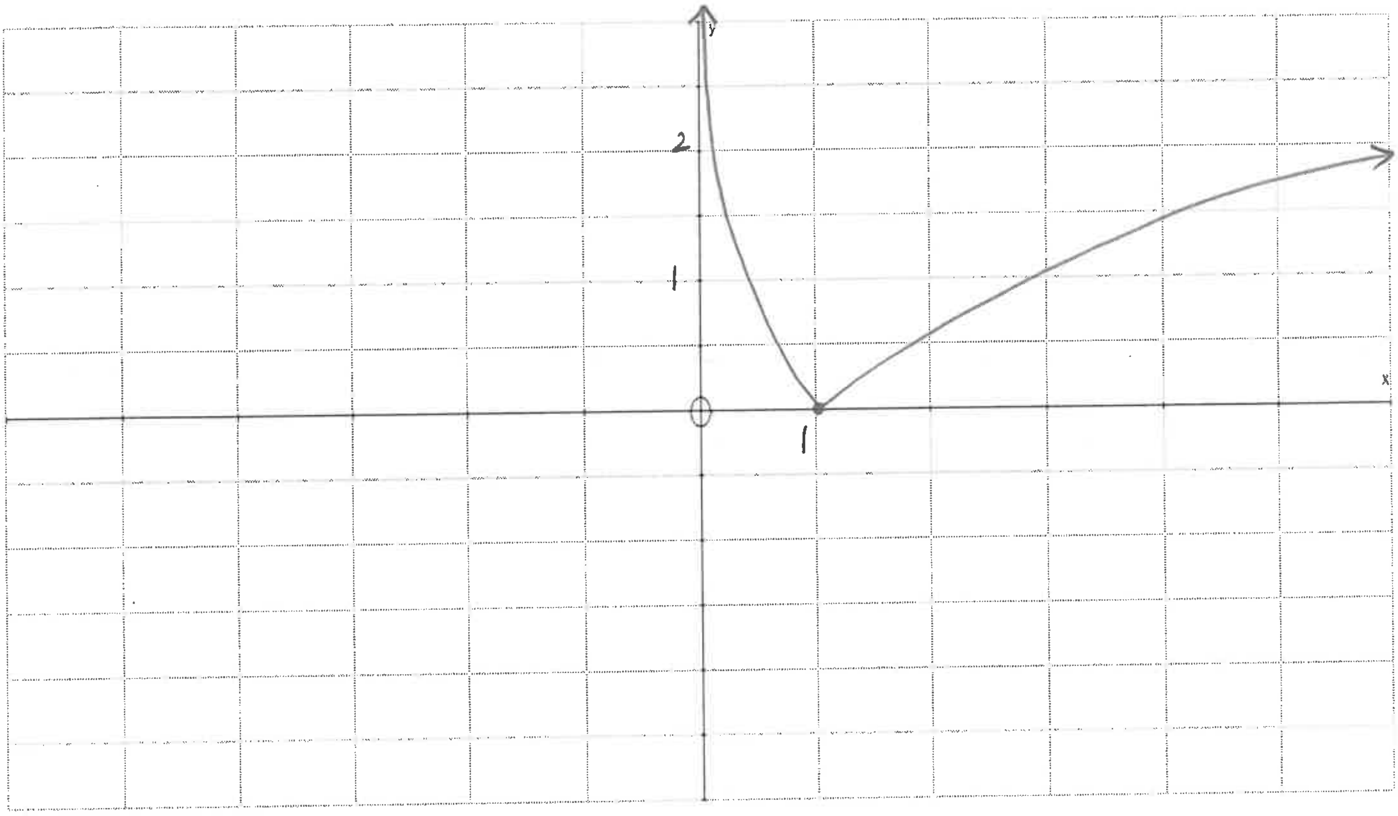
d) Determine the intercepts of $h(x)$ (4)

e) Sketch $h(x)$ (10)

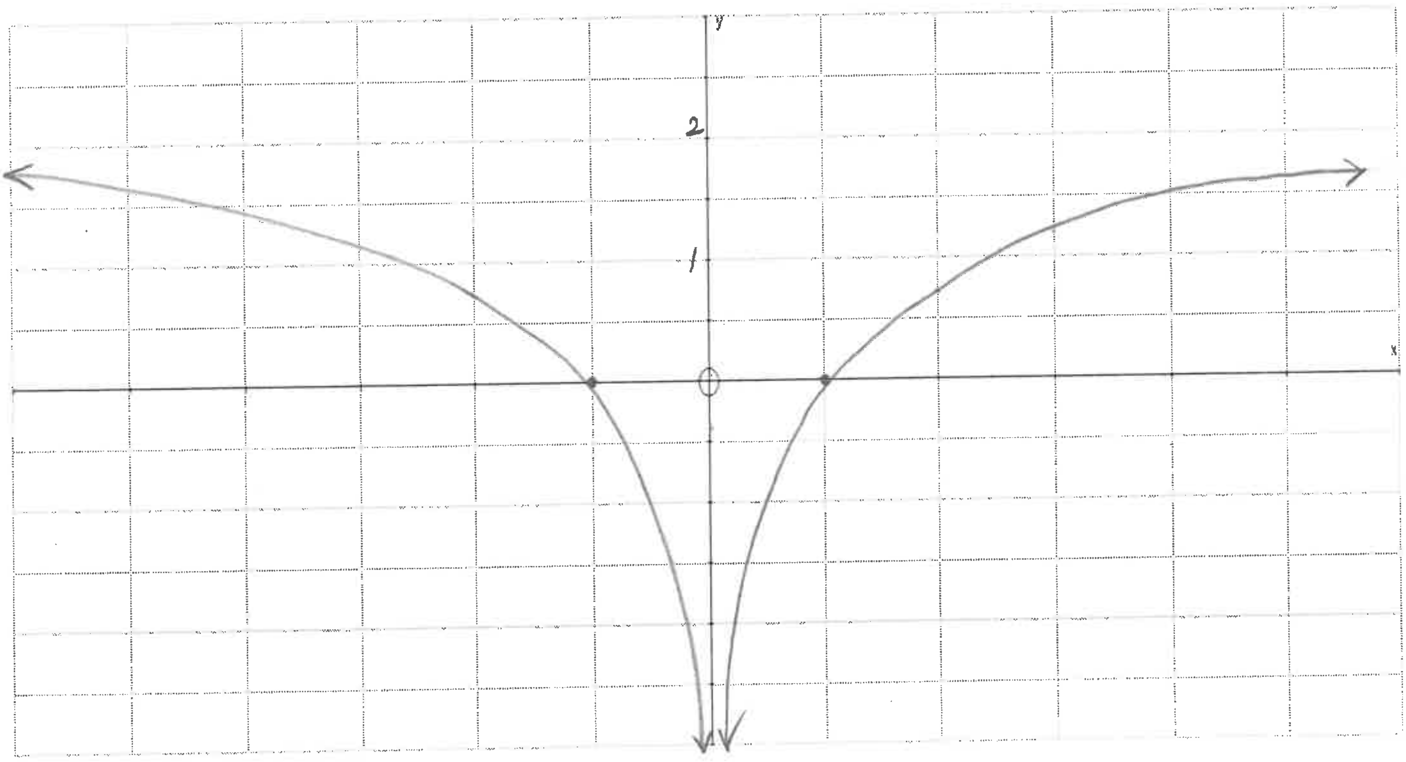


QUESTION 4 [4]

a) Sketch $y = |\ln x|$ and (2)



b) $y = \ln|x|$ (2)



Q5. 1) Greatest Value for $f(x)$ is at A; the function is decreasing on the given domain. (2)

2. B; D; E are points of inflection (3)
Stationery point on the derivative graph indicates a change in concavity

3. f has no turning point, $f'(x) \neq 0$ (2)

Q6. Given to $6\sum_{r=1}^{n} r(r+2) = n(n+1)(2n+7)$
prove: expand.
$6[3 + 8 + 15 + \ldots + n(n+2)] = n(n+1)(2n+7)$
If $n = 1$
LHS $= 6[1(1+2)] = 18$
RHS $= 1(1+1)(2(1)+7)$
$= 18$
$\therefore$ LHS = RHS
$\therefore$ The statement is true for $n = 1$

Assume that the statement is true for $n = k$
$\therefore 6[3 + 8 + 15 + \cdots + k(k+2)] = k(k+1)(2k+7)$

Prove that the statement is true for $n = k+1$
ie Prove that
$18 + 48 + 90 + \ldots + 6k(k+2) + 6(k+1)(k+3) = (k+1)(k+2)(2k+9)$

LHS $:= k(k+1)(2k+7) + 6(k+1)(k+3)$
$= (k+1)(2k^2 + 7k + 6k + 18)$
$= (k+1)(2k^2 + 13k + 18)$
$= (k+1)(2k+9)(k+2)$
$=$ RHS (10)
$\therefore$ By process of Mathematical Induction the statement is true for all $n \in \mathbb{N}$

Q7 $D\hat{C}B = 90°$ ($\angle$ in semi-circle)

1) $\cos\theta = \frac{BC}{2\alpha}$

$BC = 2\alpha\cos\theta$

Area $\triangle$ BCD $= \frac{1}{\cancel{2}}(\cancel{2}\alpha)(2\alpha\cos\theta)\ \sin\theta$

$\frac{1}{2}\pi r^2$

Area of sectN BCD $= \frac{1}{2}\pi(\alpha)^2$ (4)

$\therefore$ Area of shaded region $= \frac{1}{2}\pi\alpha^2 - \alpha^2\sin 2\theta$

$= \alpha^2\left(\frac{1}{2}\pi - \sin 2\theta\right)$

2) $\frac{\cancel{\alpha^2}\left(\frac{\pi}{2} - \sin 2\theta\right)}{\cancel{\alpha^2}} = \frac{\frac{1}{2}\pi\cancel{\alpha^2}}{\cancel{\alpha^2}}$

$\sin 2\theta = 0$

$\therefore\ 2\theta = 0°$ (3)

$A = \frac{\pi}{2}\alpha^2 - \alpha^2\sin 2\theta$

3) $\frac{dA}{d\theta} = -2\alpha^2\cos 2\theta$

If $\frac{dA}{d\theta} = 0$

$-2\alpha^2\cos 2\theta = 0$

$\cos 2\theta = 0$

$\therefore\ 2\theta = \frac{\pi}{2}$ (3)

$\theta = \frac{\pi}{4}$.