REDHILL HIGH SCHOOL

GRADE 12

AP Maths Prelim

September 2019

Examiner: G Evans

Moderators: V van Rooy

Time: 3 hours

300 marks

PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY

1. This paper consists of 25 pages. Please check that your question paper is complete. Answer on the question paper.
2. Read the questions carefully.
3. All questions are compulsory.
4. It is in your own interests to write legibly and present your work neatly.
5. Calculators should be used where appropriate. Round to 2 decimal places.

Student Name: MEMO **Class:** ________

SECTION A - CALCULUS & ALGEBRA (200 marks)

Question 1 (33 marks)

(a) Solve for x, without the use of a calculator:

$\log_3 x - 4\log_x 3 = -3$ (7)

$\therefore \dfrac{\log_3 x}{\log_3 3} - \dfrac{\log_3 3^4}{\log_3 x} = -3$

$\therefore (\log_3 x)^2 - \log_3 3^4 . \log_3 3 = -3 \log 3 . \log x$

$\therefore (\log_3 x)^2 - 4 = -3 \log x$

$\therefore (\log_3 x)^2 + 3 \log x - 4 = 0$

$\therefore (\log_3 x + 4)(\log_3 x - 1) = 0$

$\therefore x = 3^{-4} = \frac{1}{81}$ or $x = 3$

(b) Solve for x, proving that the equation has only one solution:

$x(3|x|-1) = -10$ (7)

$x < 0$	$x \geq 0$
$x(-3x-1) = -10$	$x(3x-1) = -10$
$\therefore -3x^2 - x + 10 = 0$	$\therefore 3x^2 - x + 10 = 0$
$\therefore 3x^2 + x - 10 = 0$	No real solution
$\therefore (3x-5)(x+2) = 0$	
$\therefore x \neq \frac{5}{3}$ or $x = -2$	

(c) If $a-\frac{2}{(1-i)^2}=\frac{1+bi}{1-i}$, where i is the imaginary number, find the value(s) of a and b respectively. (9)

$$LHS = \frac{a(1-2i+i^2)-2}{(1-i)^2}$$

$$= \frac{-2ai-2}{(1-i)^2}$$

$$RHS = \frac{1+bi}{1-i} \times \frac{1+i}{1+i}$$

$$= \frac{1+bi-i-bi^2}{(1-i)^2}$$

$$= \frac{1+b+(b-1)i}{(1-i)^2}$$

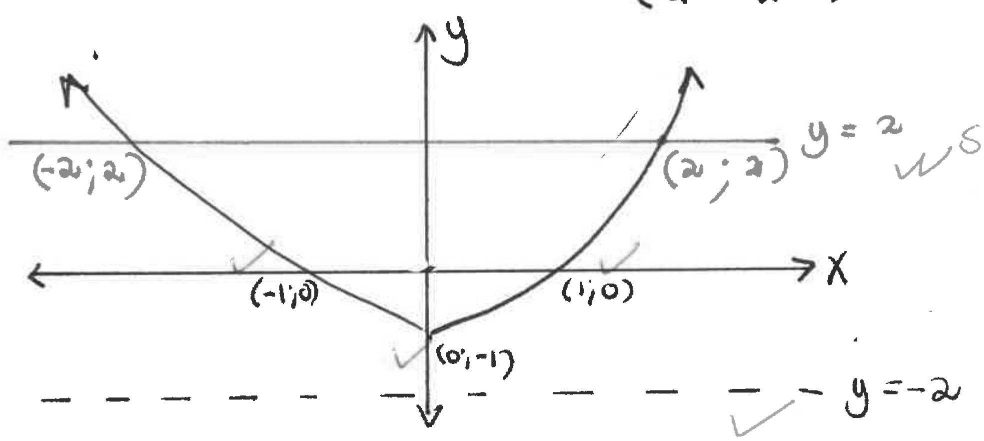
$1+b=-2$

$\therefore b=-3$

$-4=-2a$

$\therefore 2=a$

(d) (i) Sketch the graph: $f(x)=2^{|x|}-2 = \begin{cases} 2^{x}-2, & x \geq 0 \\ 2^{-x}-2, & x<0 \end{cases}$ (6)



(ii) Hence, solve the inequality: $2^{|x|}<4$ (4)

$\therefore 2^{|x|}-2<4-2$

$\therefore 2^{|x|}-2<2$

$\therefore x \in (-2, 2)$

Question 2 (15 marks)

The temperature, T (°C) of a cooling cup of tea, after a time t (minutes), can be modelled by the equation:

$T = 20 + Ae^{-kt}$, where A, ~~B~~ and k are constants.

(a) Write down the room temperature. (2)

$20^\circ C$ ✓✓

(b) Given that the initial temperature is 85 °C and that the temperature is decreasing at the rate of 5 °C per minute, initially, determine the value of k. (9)

$\frac{dT}{dK} = -k \cdot Ae^{-kt}$ ✓✓ and $\frac{dT}{dk} = -5$ ✓✓

$\therefore 65 \cdot e^{-kt} \cdot (-k) = -5$ ✓✓

$A = 65$ ✓

$\therefore \quad k = \frac{1}{13}$ ✓✓

(c) Determine the length of time, to the nearest minute it takes for the tea to cool to 50 °C. (4)

$50 = 20 + 65 \cdot e^{-\frac{1}{13}t}$ ✓ ✓

$\therefore 30 = 65 \cdot e^{-t/13}$ ✓

$\therefore -t/13 = \ln\left(\frac{30}{65}\right)$

$\therefore \quad t = 10$ minutes ✓

Question 3 (12 marks)

(a) It is given that $f(g(x)) = \dfrac{1}{x-1} + x^2 - 2x + 1$ and $g(g(x)) = x - 2$

Determine $g(f(2))$ (6)

$g(x) = x - 1$

$f(x) = \dfrac{1}{x} + x^2$

$\therefore\ f(2) = \dfrac{1}{2} + (2)^2$

$= 4{,}5$

$g(f(2)) = 4{,}5 - 1$

$= 3{,}5$

(b) For which value(s) of x will y be defined:

$$y = \ln\left(\frac{2x-1}{x^2+4x+4}\right)$$ (6)

$$\frac{2x-1}{(x+2)^2} > 0$$

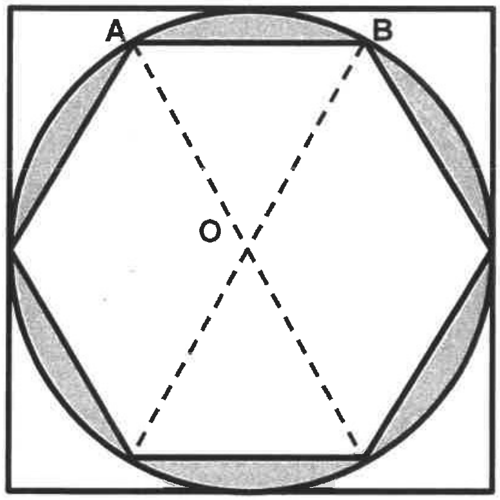
$-\quad 0 \quad -\quad 0 \quad +$

$-2 \qquad \tfrac{1}{2}$

$\therefore\ x > \dfrac{1}{2}$

Question 4 (11 marks)

In the diagram a square tile is shown, in which a regular hexagon is inscribed in a circle with centre O. The circle fits exactly inside the square. The area of the shaded region is 54 cm^2.



(a) Write down the size of $\angle AOB$, giving your answer in radians. (2)

$$\frac{2\pi}{6} = \frac{\pi}{3}$$

(b) Determine the area of the square tile. (9)

$$\frac{1}{2} r^2 \times \frac{\pi}{3} - \frac{1}{2} r^2 . \sin \frac{\pi}{3} = 9$$

$$\therefore r^2 \left(\frac{\pi}{3} - \sin \frac{\pi}{3} \right) = 18$$

$$\therefore r^2 = 99{,}35 \ldots$$

$$\therefore r = 9{,}97 \text{ cm}$$

Side of tile = 19,94 cm

Area of tile = 397,4121 cm^2

Question 5 (12 marks)

Prove by induction that:

$$\sum_{r=1}^{r=n} \frac{2}{(r+1)(r+2)} = \frac{n}{n+2} \quad \text{for all natural values of } n.$$

① Prove true for $n = 1$:

$$LHS = \sum_{r=1}^{1} \frac{2}{(r+1)(r+2)} \qquad RHS = \frac{1}{3}$$

$$= \frac{2}{(2)(3)}$$

$$= \frac{1}{3}$$

$\therefore$ True for $n = 1$

② Assume true for $n = k$:

$$\therefore \sum_{r=1}^{k} \frac{2}{(r+1)(r+2)} = \frac{k}{k+2}$$

③ Prove true for $n = k+1$:

$$LHS = \sum_{r=1}^{k+1} \frac{2}{(r+1)(r+2)}$$

$$= \frac{k}{k+2} + \frac{2}{(k+1+1)(k+1+2)}$$

$$= \frac{k}{k+2} + \frac{2}{(k+2)(k+3)}$$

$$= \frac{k(k+3) + 2}{(k+2)(k+3)}$$

$$= \frac{k^2+3k+2}{(k+2)(k+3)} = \frac{(k+1)(k+2)}{(k+2)(k+3)}$$

$$= \frac{k+1}{(k+1)+2}$$

④ By the principle of induction we have proved that if the formula is true for one natural number, then it is also true for all natural values of n.

Question 6 (18 marks)

(a) The following function is given:

$$g(x) = \begin{cases} 2x+1 & \text{if } x \le p \\ x^2 - 4x + 10 & \text{if } x > p \end{cases}$$

(i) For which value(s) of p is $g(x)$ continuous at $x = p$? (6)

$$\lim_{x \to p^-} g(x) = \lim_{x \to p^+} g(x)$$

$$\therefore 2p + 1 = p^2 - 4p + 10$$

$$\therefore 0 = p^2 - 6p + 9$$

$$\therefore 0 = (p-3)^2$$

$$\therefore \underline{p = 3}$$

(ii) Is f differentiable at all points? Motivate your answer. (6)

g

$$\lim_{x \to p^-} g'(x) = 2$$

$$\lim_{x \to p^+} g'(x) = 2p - 4$$

$$= 2(3) - 4$$

$$= 2$$

$\therefore$ $g(x)$ is differentiable at $p = 3$

(b) Determine the limit:

$$\lim_{x\to1}\frac{\sqrt{2x-1}-\sqrt{x}}{x-1} \times \frac{\sqrt{2x-1}+\sqrt{x}}{\sqrt{2x-1}+\sqrt{x}}$$ (6)

$$= \lim_{x\to1} \frac{2x-1-x}{(x-1)\left[\sqrt{2x-1}+\sqrt{x}\right]}$$

$$= \lim_{x\to1} \frac{1}{\sqrt{2x-1}+\sqrt{x}}$$

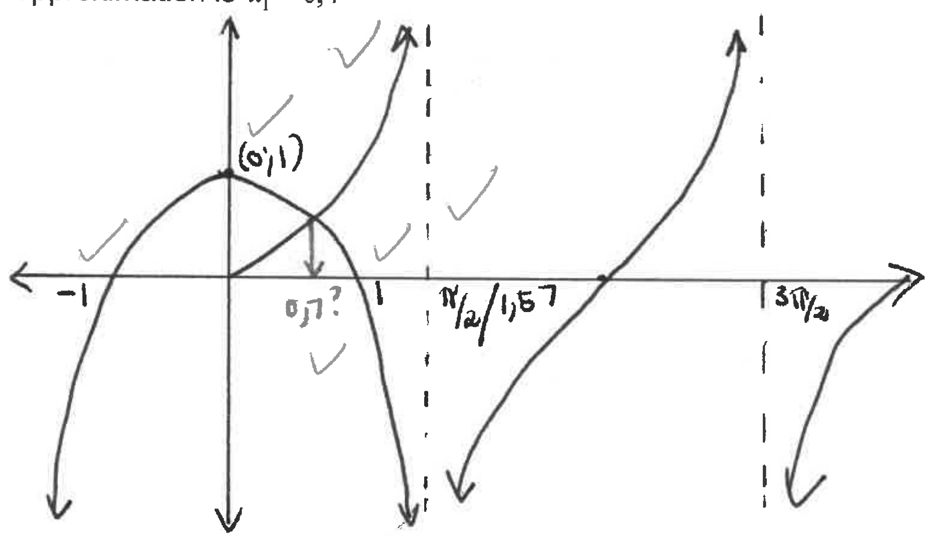
$$= \frac{1}{1+1}$$

$$= \frac{1}{2}$$

Question 7 (18 marks)

It is required to solve: $\tan x = 1-x^2$ using Newton's method.

(a) By drawing a suitable rough sketch, show that a reasonable first approximation is $x_1 = 0,7$ (6)



(b) Hence solve for the nearest solution, correct to 5 decimal places (8)

$f(x) = \tan x - 1 + x^2$ ✓

$f'(x) = \sec^2 x + 2x$ ✓✓

$x_{r+1} = x_r - \frac{f(x_r)}{f'(x_r)}$ ✓

$x_0 = 0,7$

$x_1 = 0,59313$

$x_2 = 0,58332$

$x_3 = 0,58324$ ✓✓✓✓

$x_4 = 0,58324$

(c) Does the initial guess in Newton's method need to be close to the actual answer? Explain why and/or why not? (4)

No - depends on the slope of the tangent drawn on the graph at the initial guess ✓✓✓

Question 8 (20 marks)

A curve has equation $y = \dfrac{x^2}{x-2}$.

(a) Find the equations of he asymptotes (5)

VA: $x = 2$

Oblique asymptote:

$$\begin{array}{r|l} & x + 2 \\ \hline x-2 & x^2 \\ & x^2 - 2x \\ \hline & 2x \\ & 2x - 4 \\ \hline & 4 \end{array}$$

$\therefore\ y = x + 2$

(b) Determine the coordinates of the stationary points (8)

$\dfrac{dy}{dx} = 0$

$\therefore \dfrac{(x-2)(2x) - x^2(1)}{(x-2)^2} = 0$

$\therefore\ 2x^2 - 4x - x^2 = 0$

$\therefore\ x^2 - 4x = 0$

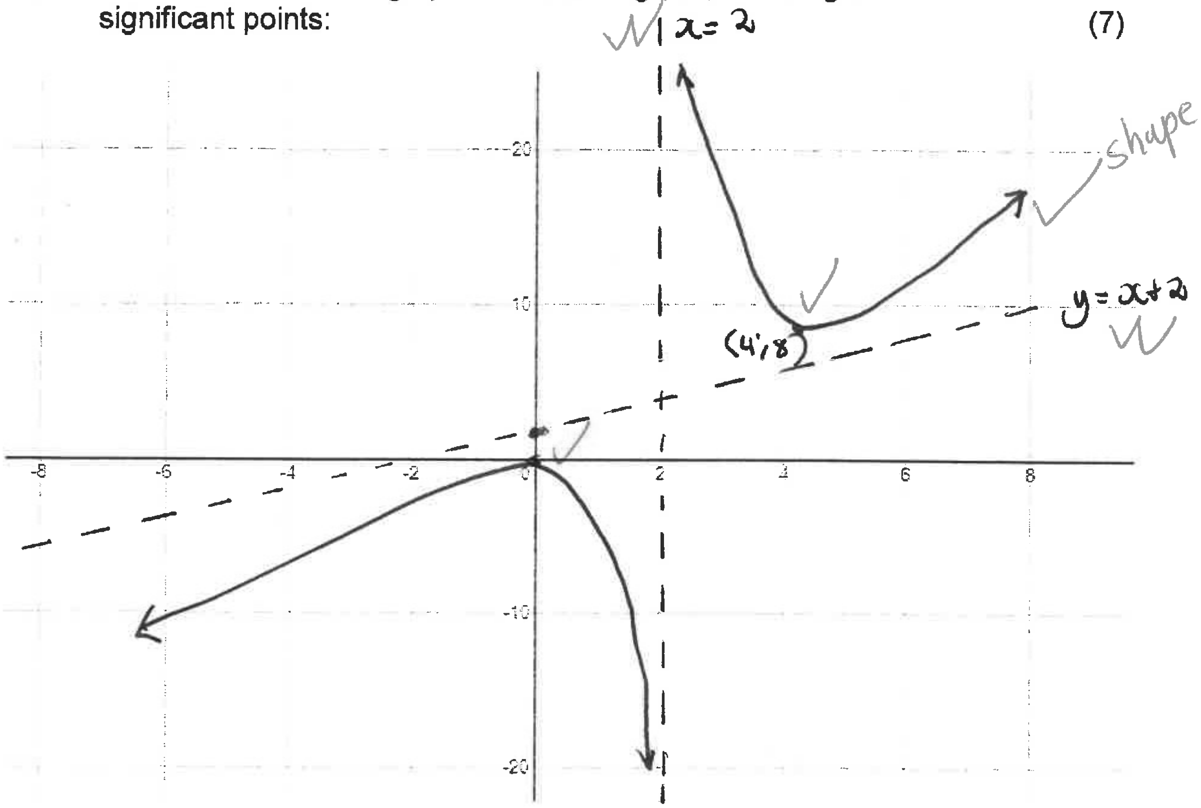
$\therefore\ x(x-4) = 0$

$\therefore\ x = 0$ or $x = 4$

$y = 0$ $\quad$ $y = 8$

$(0;0)$ $\quad$ $(4;8)$

(c) Make a sketch of the graph on the axes given, including all significant points: (7)



Question 9 (14 marks)

Cars are passing a particular point on a bridge. For the purposes of this problem we will assume that each car is 3,5 metres long and that there is a distance of d metres between each car. The cars are all travelling at v km/h.

(a) Show that the number of cars passing through the point on the bridge each hour (i.e. the *flow rate*) is given by:

$$F = \frac{1000v}{d+3,5}$$ (4)

v km/h

$= \frac{v}{1} \times \frac{1000\text{ m}}{\text{h}}$

$= 1000v$ m/h

$t = \frac{D}{S} = \frac{t+3,5}{1000v}$

$F = \frac{1}{t}$

$= 1 \times \frac{1000v}{t+3,5}$

$= \frac{1000v}{t+3,5}$

(b) To ensure safe driving, we insist that cars keep a safe following distance, represented by the value d. Let us assume that a car travelling at v km/h requires a following distance of $d = 0{,}006v^2$.

Now find the velocity that maximises the flow rate of traffic over the bridge (9)

$$F = \frac{1000V}{0{,}006v^2 + 3{,}5}$$

$$\frac{dF}{dv} = 0$$

$$\therefore \frac{(0{,}006v^2+3{,}5)(1000) - (1000v)(0{,}012v)}{(0{,}006v^2+3{,}5)^2} = 0$$

$$\therefore 6v^2 + 3500 - 12v^2 = 0$$

$$\therefore 3500 = 6v^2$$

$$\therefore 24{,}15\,m/h = v$$

Question 10 (6 marks)

The equation of a circle centre (a; b) and radius r is given by: $(x-a)^2 + (y-b)^2 = r^2$

Show, by differentiation, that $\frac{dy}{dx} = \frac{a-x}{y-b}$

$$2(x-a) + 2(y-b)\frac{dy}{dx} = 0$$

$$\therefore \frac{2(y-b)\frac{dy}{dx}}{2(y-b)} = \frac{-2(x-a)}{2(y-b)}$$

$$\therefore \frac{dy}{dx} = \frac{a-x}{y-b}$$

Question 11 (26 marks)

Integrate the following functions:

(a) $\int x.\cos\left(5x^2\right)dx$ (6)

$$= \frac{\sin 5x^2}{10} + C$$

(b) $\int \frac{1}{x^2\left(1+\frac{1}{x}\right)^3} dx$ (8)

$$= \int x^{-2} \cdot \left(1 + x^{-1}\right)^{-3} dx$$

$$= -\frac{\left(1+\frac{1}{x}\right)^{-2}}{-2} + C$$

$$= \frac{1}{2\left(1+\frac{1}{x}\right)^2} + C$$

(c) (i) Use partial fractions to show that: $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\ln\frac{a+x}{a-x}+C$ (9)

$$\frac{1}{(a-x)(a+x)}$$

$$= \frac{A}{a-x} + \frac{B}{a+x}$$

$$\therefore \ A = \frac{1}{2a} \text{ and } B = \frac{1}{2a}$$

$$= \frac{1}{2a(a-x)} + \frac{1}{2a(a+x)}$$

$$= \int \frac{1}{2a(a-x)} + \frac{1}{2a(a+x)} \, dx$$

$$= -\frac{1}{2a}\ln|a-x| + \frac{1}{2a}\ln|a+x| + C$$

$$= \frac{1}{2a}\left[\ln|a+x| - \ln|a-x|\right] + C$$

$$= \frac{1}{2a}\cdot\ln\frac{a+x}{a-x} + C$$

(ii) Use this result to find the value of

$$\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{1}{9-4x^2}\,dx$$ (7)

$$= \left[\frac{1}{6}\ln\frac{3+2x}{3-2x}\right]_{1/4}^{1/2}$$

$$= \frac{1}{6}\ln\frac{3+1}{3-1} - \frac{1}{6}\ln\frac{3+\frac{1}{2}}{3-\frac{1}{2}}$$

$$= \frac{1}{6}\ln 2 - \frac{1}{6}\ln\frac{7}{2}\times\frac{2}{5}$$

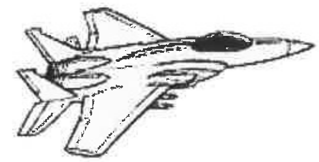
$$= \ln\sqrt[6]{2} - \ln\sqrt[6]{7/5}$$

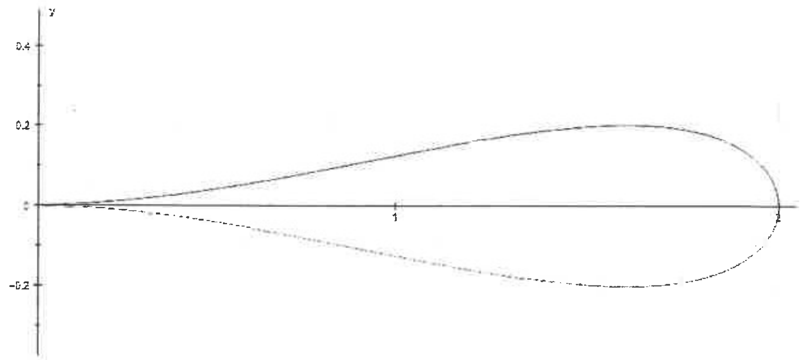
$$= \ln\sqrt[6]{\frac{2}{1}\times\frac{5}{7}}$$

$$= 0{,}0544$$

Question 12 (12 marks)

The fuel tank on the wing of a jet is formed by rotating the region bounded by the curve $y = \frac{1}{8}x^2.\sqrt{2-x}$ about the x-axis between $x = 0$ and $x = 2$, where the units are measured in metres.





Determine the volume of the fuel tank.

$$V = \pi \int_0^2 \left(\frac{1}{8}x^2\sqrt{2-x}\right)^2 dx$$

$$= \pi \int_0^2 \frac{x^4(2-x)}{64} dx$$

$$= \frac{\pi}{64}\int_0^2 2x^4 - x^5 \, dx$$

$$= \frac{\pi}{64}\left[\frac{2x^5}{5} - \frac{x^6}{6}\right]_0^2$$

$$= \frac{\pi}{64}\left[\frac{2(2)^5}{5} - \frac{(2^6)}{6}\right]$$

$$= \frac{\pi}{30} \text{ units}^3$$