St Andrew's School for Girls

SKILLED FOR LIFE

ADVANCED PROGRAMME MATHEMATICS
PAPER 1

Grade 12

Preliminary Examinations

September 2019

Examiner	Lucea Pepper
Moderator/s	Gary Kelly
Marks	200
Time	2 hours
Number of Pages	(11 pages of questions)
Instructions	**1 Show all working.** **2 All answers to 2 dp unless otherwise stated.** **3 Check your calculator is in radian mode** **4 Start each question on a new page.**

Name:

QUESTION 1 [22 marks]

1.1 Solve for x (in terms of e) without using a calculator and showing all working :

$\ln x - 4\log_x e + 3 = 0$ (6)

1.2 The following formula models the number of years (t), from now in terms of the number of people (P) that stay in a town at that time :

$$t = 100\ln\left(\frac{4}{3} - \frac{P}{60000}\right)$$

(a) Determine how many people initially live in the town, when t = 0. (4)

(b) As a result of migration to the cities, the town's population is decreasing. Calculate after how many years (to the nearest year) there will be no residents left in the town. (3)

(c) Change the subject of the formula to P, hence write the formula as P = ... (4)

(d) Hence, or otherwise, determine the initial **rate** at which the population decreases (this is when t = 0 years). (5)

QUESTION 2 [19 marks]

2.1 Given: $P(x) = 4x^4 - 8x^3 + 33x^2 - 8x + 29 = 0$

(a) If it is further given that $1 - \frac{5}{2}i$ is a root of the above equation, use this root to determine a quadratic factor of $P(x)$ in the form $ax^2 + bx + c$ where a, b and c are integers. (4)

(b) Hence determine all the roots of the equation in the complex number system. (4)

2.2 (a) Represent the complex number a = 2 + 3i in the Argand plane. (2)

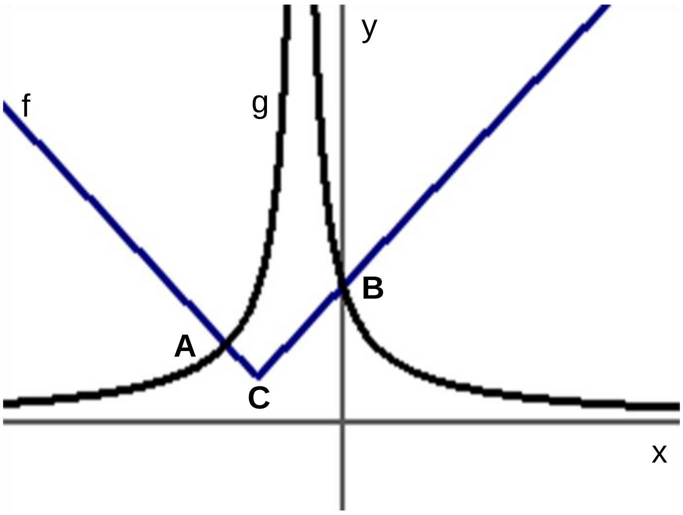
(b) Determine $|a|$ and arg (a) (3)

(c) Write a in polar form. (2)

(d) If a is rotated anti-clockwise through $\frac{\pi}{3}$ radians to become b. Write b in rectangular form. (4)

QUESTION 3 [9 marks]

The following sketch shows the graph of : $f(x) = |x + 2| + 1$ and $g(x) = \left|\frac{3}{x + 1}\right|$



3.1 Write down the coordinates of the salient point C of f. (2)

3.2 The graphs intersect at A and B. B is also the y-intercept of both graphs.
Write down the coordinates of B. (1)

3.3 Determine the x-coordinate of A (leave your answer in surd form). (6)

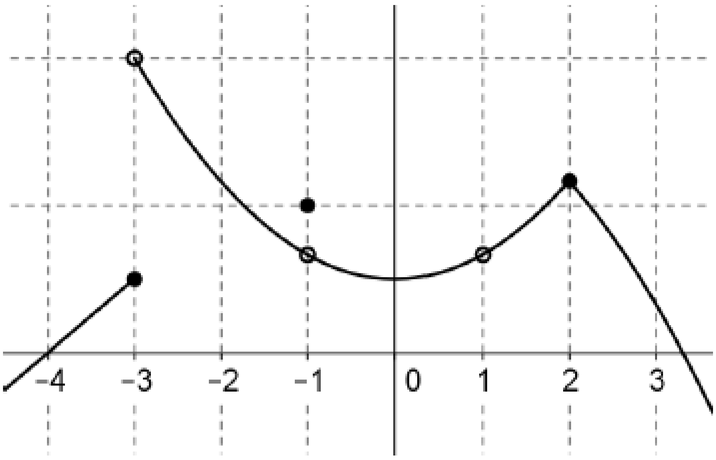
QUESTION 4 [12 marks]

Use mathematical induction to prove that the following statement is true for all $n \in N$.

$$\left(1-\frac{1}{2}\right)\times\left(1-\frac{1}{3}\right)\times\left(1-\frac{1}{4}\right)\times\ldots\ldots\times\left(1-\frac{1}{n+1}\right) = \frac{1}{n+1}$$ (12)

QUESTION 5 [19 marks]

5.1 The sketch shows the graph of $y = f(x)$



(a) The function is not continuous everywhere for $x \in [-4\,;3]$.
Give the points and type of discontinuity. (6)

(b) Motivate why the function is not differentiable at the following points :

(1) $x = -1$ (1)

(2) $x = 2$ (2)

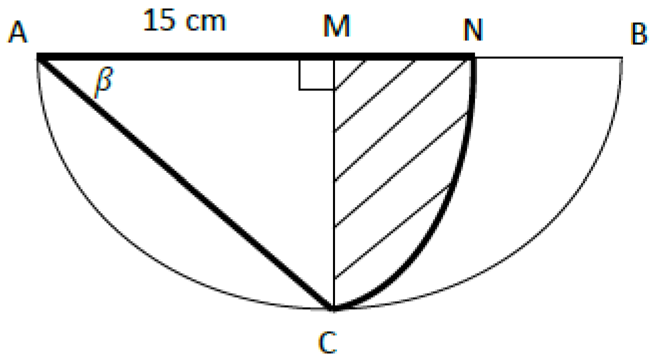
5.2 Consider : $f(x) = \begin{cases} m\,.\ln(x+2) & \text{for} \quad x > -1 \\ e^{x+1} + k & \text{for} \quad x \le -1 \end{cases}$

Given that f is differentiable for all values of x. Determine the values of m and k. (10)

QUESTION 6 [7 marks]

ABC is a semi-circle with centre M. ANC is a sector with centre A and correspnding arc NC.

AM = 15cm, $A\hat{M}C = \frac{\pi}{2}$ radians and $M\hat{A}C = \beta$



6.1 Give a reason why $\beta = \frac{\pi}{4}$ radians. (1)

6.2 Determine the area of the shaded region MNC. (6)

QUESTION 7 [9 marks]

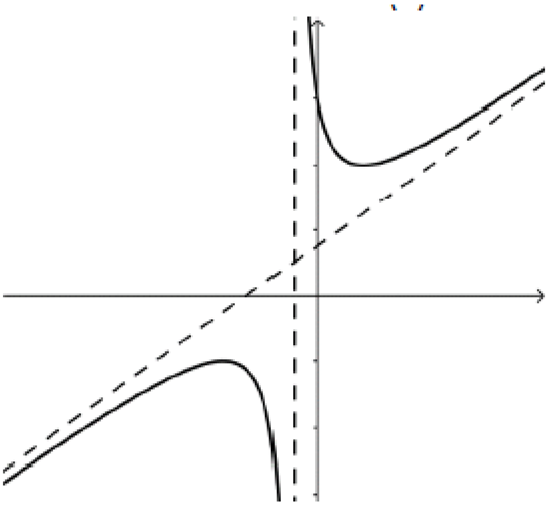
The equation of a graph is $x^2 - 3xy + 25y^2 = 91$.

7.1 Determine $\frac{dy}{dx}$ by using implicit differentiation. (6)

7.2 Determine the equation of the tangent to the graph at the point (3 ; 2). (3)

QUESTION 8 [12 marks]

The sketch shows the graph of $f(x) = \dfrac{2x^2 + 4x + 6}{2x + 1}$ and the asymptotes of f.



8.1 Determine the equations of the asymptotes of f. (4)

8.2 (a) Determine the coordinates of the turning points of f. (6)

(b) Hence write down the range of f. (2)

QUESTION 9 [18 marks]

9.1 Differentiate with respect to x:

(a) $y = \dfrac{4x}{(\ln x)^3}$ (5)

(b) $y = e^{-3x}\cos^5 4x$ (6)

9.2 The functions cosh(x) and sinh(x) are defined as follows:

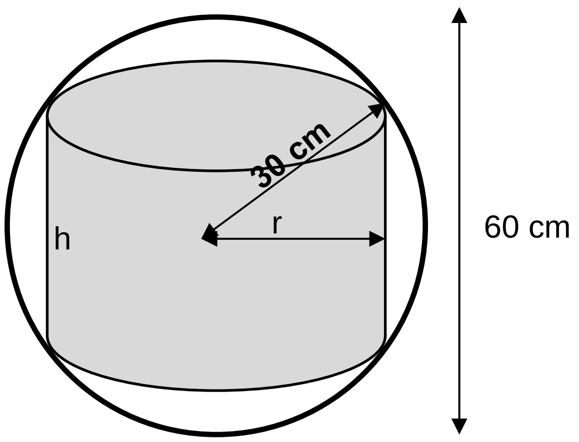
$$\cosh(x) = \frac{1}{2}\left(e^{x} + e^{-x}\right) \quad \text{and} \quad \sinh(x) = \frac{1}{2}\left(e^{x} - e^{-x}\right)$$

(a) If $f(x) = \cosh(x)$, show that $f'(x) = \sinh(x)$. (2)

(b) Show that the gradient of the graph of $f(x) = \cosh(x)$ at the point where $x = \ln a$, is given by: $\dfrac{a^2 - 1}{2a}$ (5)

QUESTION 10 [10 marks]

A cylinder is to be fitted into a sphere of radius 30cm. The cylinder has height h cm and base radius r cm.



10.1 Use Pythagoras' theorem to show that : $r^2 + \dfrac{h^2}{4} = 900$ (1)

10.2 Express the volume of the cylinder, V cm^3, in terms of h only. (3)

10.3 Find the value of h that maximizes the volume V. What is the maximum volume? (6)

QUESTION 11 [6 marks]

Consider : $\int_a^b f(x)dx = \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\left(2 + \frac{3i}{n}\right)^2 - 2\left(2 + \frac{3i}{n}\right) + 2 \right]$

11.1 Determine the values of a and b. (2)

11.2 Write down the function $f(x)$. (2)

11.3 Calculate the area enclosed by the graph of f, the x-axis and the lines x = a and x = b. (You can use your calculator). (2)

QUESTION 12 [38 marks]

12.1 Determine the following integrals :

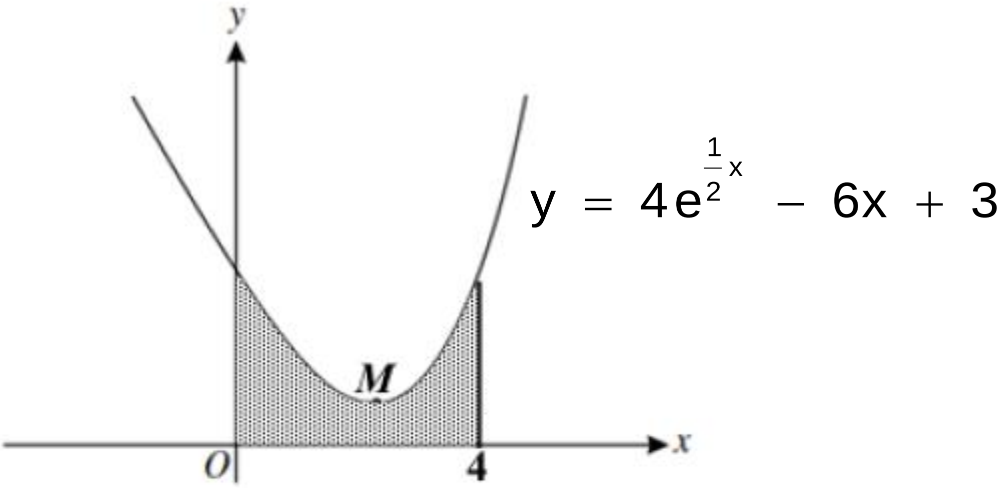
(a) $\int \sqrt{\sin 4x}\,\cos 4x\;dx$ (6)

(b) $\int \sin(x+1)\cos(3x-2)dx$ (8)

(c) $\int x\,.\,e^x\,dx$ (hint: use parts) (8)

(d) $\int \frac{3}{2x^2 - x - 1}dx$ (hint : use partial fractions) (8)

12.2 The diagram below shows the graph of $y = 4e^{\frac{1}{2}x} - 6x + 3$.

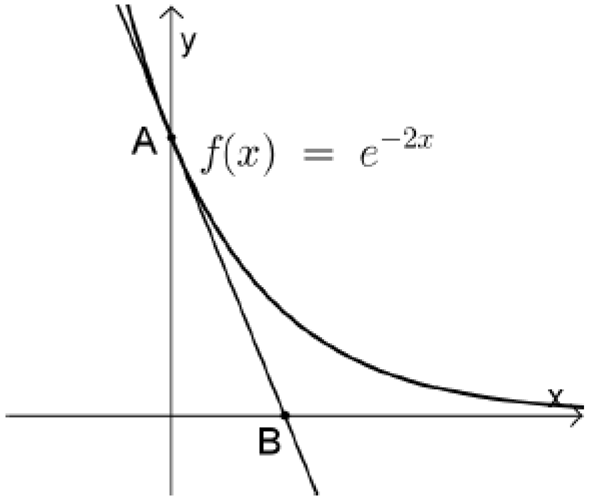


The area bordered by the graph, the x- and y-axis and the line $x = 4$ is shaded and is equal to $8e^2 + k$. Determine the value of k. (8)

QUESTION 13 [11 marks]

The sketch shows the function $f(x) = e^{-2x}$ **.** A is the x-intercept of f with the y-axis.

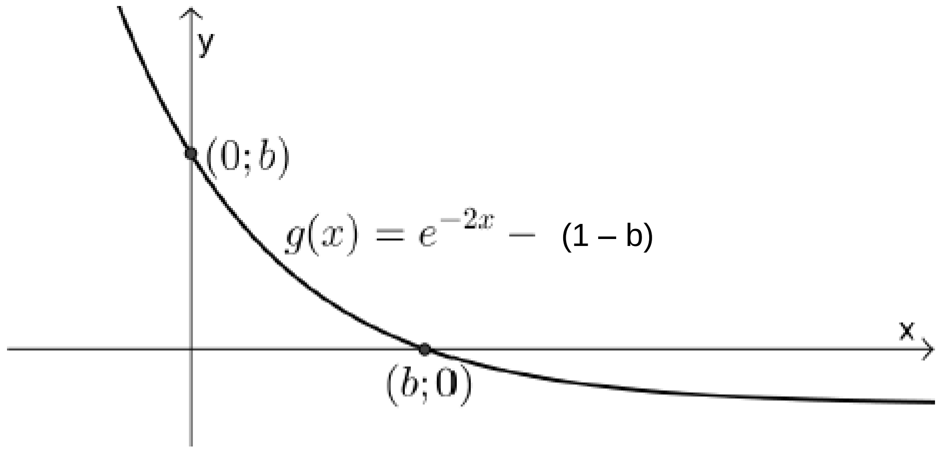
The tangent to f at A intersects the x-axis at B.



13.1 Determine the equation of the tangent at A and hence the coordinates of B. (5)

13.2 The graph of g results from the graph of f above which has been translated down. The graph of g then intercepts the x-axis as well as the y-axis. These intercepts are equidistant from the origin, at (0 ; b) and (b ; 0).

The following sketch shows the graph of this function:



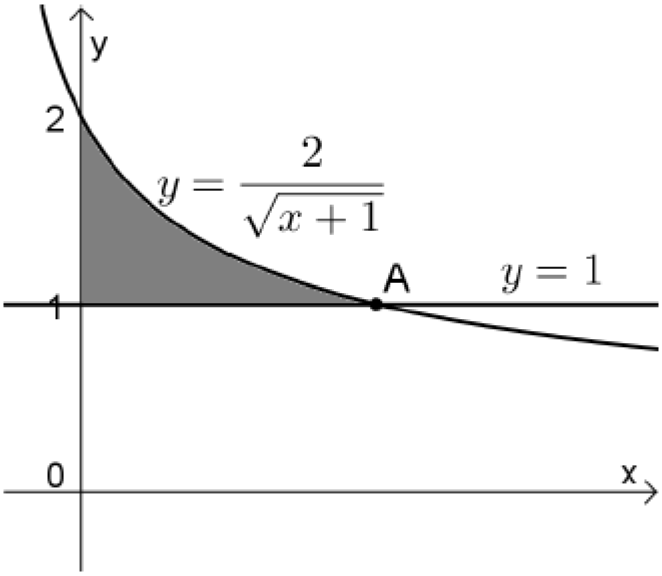
The following equation can be used to calculate b : $e^{-2b} - 1 + b = 0$.

Use Newton's method and calculate this value, correct to four decimal digits.

Use b = 0,5 as a first approximation. (6)

QUESTION 14 [8 marks]

The diagram below shows the graph of $y = \dfrac{2}{\sqrt{x+1}}$ and the line $y = 1$.



14.1 Show that the intersection A is at (3 ; 1). (2)

14.2 Determine the volume of the solid of revolution which develops if the shaded region is rotated about the x-axis. (6)