ST. DAVID'S MARIST INANDA

ADVANCED PROGRAMME MATHEMATICS

PRELIMINARY EXAMINATION
PAPER 1: CALCULUS and ALGEBRA

GRADE 12

3 SEPTEMBER 2019

EXAMINER: MRS S RICHARD **MODERATOR: MRS C KENNEDY**	**MARKS: 200** **TIME: 2 hours**

NAME: Memo

INSTRUCTIONS:

- ✓ This paper consists of 24 pages and a separate 2-page formula sheet. Please check that your paper is complete. The last 2 pages are blank for working out.
- ✓ Please answer all questions on the Question Paper.
- ✓ You may use an approved non-programmable, non-graphics calculator unless otherwise stated. PLEASE ENSURE YOUR CALCULATOR IS IN **RADIAN** MODE.
- ✓ Round answers to 2 decimal places, unless stated otherwise.
- ✓ It is in your interest to show all your working details.
- ✓ Work neatly. Do NOT answer in pencil.
- ✓ Diagrams are not drawn to scale.

QUESTION	Q1 [29]	Q2 [16]	Q3 [11]	Q4 [24]	Q5 [10]	Q6 [17]	Q7 [23]	Q8 [10]	Q9 [9]	Q10 [33]	Q11 [8]	Q12 [10]	TOTAL [200]
LEARNER'S MARKS													

QUESTION 1

a) Solve for x, without using a calculator and showing all working:

i) $|x|^2 - |x| - 20 = 0$ (6)

If $x \geq 0$

$x^2 - x - 20 = 0$ ✓

$(x-5)(x+4) = 0$

$x = 5$ ✓ or $x \neq -4$ ✓

or

$x < 0$

$(-x)^2 - (-x) - 20 = 0$ ✓

$x^2 + x - 20 = 0$

$(x+5)(x-4) = 0$

$x = -5$ ✓ or $x \neq 4$ ✓

OR

$|x| = k$ ✓

$k^2 - k - 20 = 0$ ✓

$(k-5)(k+4) = 0$

$k = 5$ or $k = -4$ ✓

$|x| = 5$ ✓ or $|x| \neq -4$ ✓

$x = \pm 5$ ✓

ii) $\log_3 x + 2\log_x 3 = 3$ (6)

$\log_3 x + \dfrac{2}{\log_3 x} = 3$ ✓

Let $k = \log_3 x$

$k + \dfrac{2}{k} = 3$

$k^2 + 2 = 3k$

$k^2 - 3k + 2 = 0$ ✓

$(k-2)(k-1) = 0$

$k = 2$ ✓ or $k = 1$ ✓

$\therefore \log_3 x = 2$ or $\log_3 x = 1$

$3^2 = x$ or $3^1 = x$. ✓

$9 = x$ ✓

b) Sketch $y = |2^x - 1| - 1$ on the axes provided. (5)

$y = 0$:

$0 = |2^x - 1| - 1$

$1 = |2^x - 1|$ or

$2^x - 1 = 1$ or $2^x - 1 = -1$

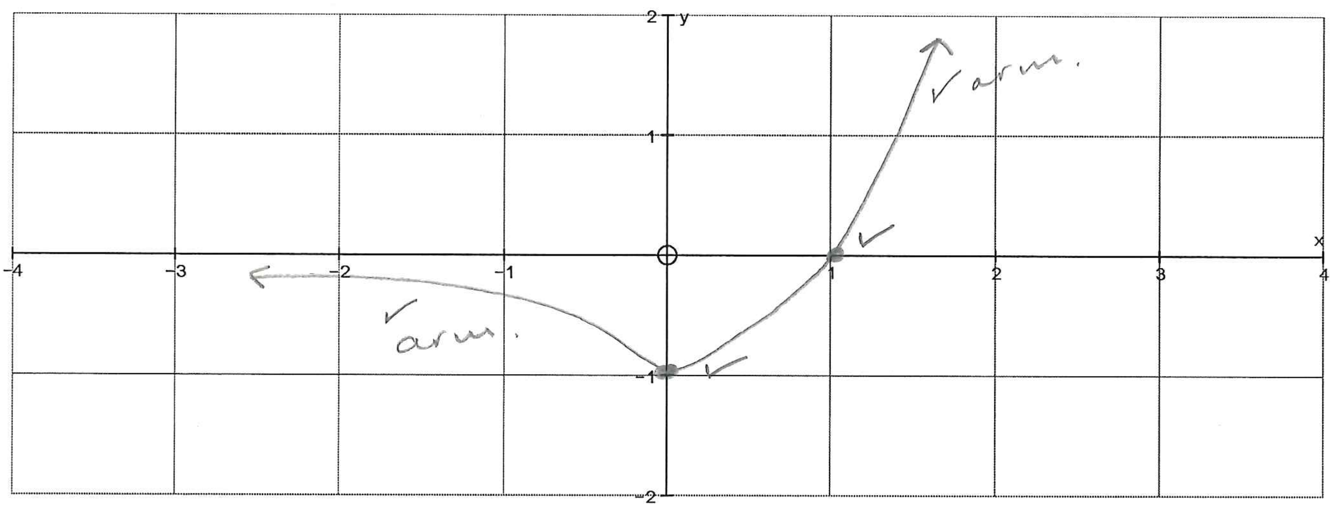
$2^x = 2^1$ $\quad$ $2^x = 0$

$x = 1$ $\quad$ no soln.

$x = 0$: $y = |2^0 - 1| - 1$

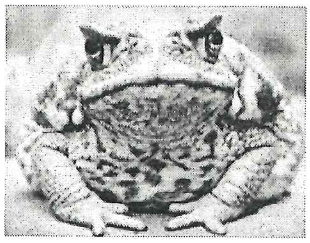
$= |1 - 1| - 1$

$= -1$



✓ concavity.

c) The number of Cane toads K, in a particular area of Western Australia after t years can be determined by using the equation:



$$K = \frac{3000}{3+7e^{-0,05t}}$$

i) Determine the number of toads at the start of the study, that is at t = 0. (2)

$$K = \frac{3000}{3+7e^{0}} \checkmark$$

$$= 300 \checkmark \text{ toads.}$$

ii) How many toads, to the nearest integer, will there be after exactly one year? (2)

$$K = \frac{3000}{3+7e^{-0,05}} \checkmark$$

$$= 310,60$$

$$= 310 \text{ toads.} \checkmark \text{ or } 311$$

iii) It will be considered an epidemic if there are more than 900 toads in a particular area. After how many years will there be 900 toads? (6)

$$900 \checkmark = \frac{3000}{3+7e^{-0,05t}}$$

$$3+7e^{-0,05t} \checkmark = \frac{30}{9} \checkmark$$

$$e^{-0,05t} = \frac{1}{21} \checkmark$$

$$\ln \frac{1}{21} = -0,05t \checkmark$$

$$60,9 \text{ years} = t \checkmark$$

61 years.

iv) Determine the number of toads in the area as $t \to \infty$? (2)

No of toads as $t \to \infty$

$$= \frac{3000}{3} \checkmark$$

$$= 1000 \text{ toads} \checkmark$$

[29]

QUESTION 2 Note $i = \sqrt{-1}$

a) Determine, in standard form $ax^3 + bx^2 + cx + d = 0$ with a, b, c and d real co-efficients, a cubic equation which has roots -2 and $-1-3i$ (8)

$x = -2$; $x = -1-3i$ $\therefore$ $x = -1+3i$ ✓

$$\begin{array}{r} -1-3i \\ -1+3i \\ \hline -2 \end{array} ✓$$

$(-1-3i)(-1+3i)$ ✓
$= 1-9i^2$
$= 1+9$
$= 10$ ✓

$(x+2)(x^2 + 2x + 10) = 0$ ✓

$x^3 + 2x^2 + 10x + 2x^2 + 4x + 20 = 0$ ✓

$x^3 + 4x^2 + 14x + 20 = 0.$

✓ ✓

b) Determine the values of a and b, where a and b are real numbers that satisfy

the equation: $\dfrac{a+3i}{2-5i} \times bi = -11-13i$ (8)

$(a + 3i) \times bi = (-11 - 13i)(2 - 5i)$ ✓

$abi + 3bi^2 = -22 + 55i - 26i + 65i^2$ ✓ ✓ ✓

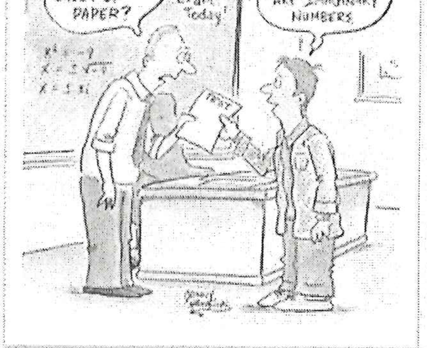
$abi - 3b = -22 + 29i - 65$

$ab = 29$ ✓ $\qquad -3b = -87$ ✓

$b = 29$ ✓

$\therefore a = 1$ ✓

[16]



QUESTION 3

Use Mathematical induction to prove that for all $n \in \mathbb{N}$ that:

$$(1\times 2)+(2\times 3)+(3\times 4)+...+n(n+1)=\frac{1}{3}n(n+1)(n+2) \qquad (11)$$

- Prove true for $n=1$:

LHS : 1×2 $= 2$ ✓

RHS $= \frac{1}{3}(1)(1+1)(1+2)$
$= \frac{1}{3}(1)(2)(3)$
$= \frac{6}{3}$
$= 2$ ✓

$\therefore$ true for $n=1$ ✓

- Assume true for $n=k$: $k \in \mathbb{N}$ ✓

ie. $\underbrace{(1\times 2)+(2\times 3)+...+k(k+1)}_{*} = \underbrace{\frac{1}{3}k(k+1)(k+2)}$ ✓

- Prove true for $n=k+1$ ✓

ie

$\underbrace{(1\times 2)+(2\times 3)+...+k(k+1)}_{*} + (k+1)(k+2)$ ✓ $= \frac{1}{3}(k+1)(k+2)(k+3)$ ✓

LHS : $\frac{1}{3}k(k+1)(k+2) + (k+1)(k+2)$ ✓

$= (k+1)(k+2)\left[\frac{1}{3}k+1\right]$ ✓

$= \frac{1}{3}(k+1)(k+2)(k+3)$

$=$ RHS

$\therefore$ true for $n=1$

true for $n=k+1$ iff true for $n=k$

$n, k \in \mathbb{N}$

$\therefore$ true for all $n \in \mathbb{N}$ by MI ✓

[11]

QUESTION 4

a) Given $f(x)=\sqrt{x+2}$ and $g(x)=\ln(1-x^2)$

Determine $g(f(x))$ in simplest from and state its domain. ✓ (8)

$$g(\sqrt{x+2}) = \ln(1-(\sqrt{x+2})^2)$$
$$= \ln(1-(x+2)) ✓$$
$$= \ln(-x-1) ✓$$

$$-x-1 > 0 ✓$$
$$-x > 1$$
$$x < -1 ✓ \quad \text{and} \quad x+2 \geq 0$$
$$x \geq -2 ✓$$

$$\therefore \; x \in [-2;-1) ✓ ✓$$

b) Determine the value of the limit if it exists:

$$\lim_{x\to1}\frac{\sqrt{x+3}-2}{x-1} \times \frac{\sqrt{x+3}+2}{\sqrt{x+3}+2} ✓$$ (4)

$$= \lim_{x\to1}\frac{x+3-4}{(x-1)(\sqrt{x+3}+2)} ✓$$

$$= \lim_{x\to1}\frac{\cancel{(x-1)}}{\cancel{(x-1)}(\sqrt{x+3}+2)} ✓$$

$$= \frac{1}{\sqrt{1+3}+2}$$

$$= \frac{1}{4} ✓$$

c) Given the function:

$$h(x)=\begin{cases}2x^3+x^2-3, & x<-1\\ -x+k, & x\ge -1\end{cases}$$

i) Determine the real value of k if h(x) is continuous at $x=-1$. (5)

$$\lim_{x\to -1^-} 2x^3+x^2-3 = 2(-1)^3+(-1)^2-3$$
$$= -2+1-3$$
$$= -4$$

$$\lim_{x\to -1^+} -x+k = -(-1)+k$$
$$= 1+k$$

and $h(-1) = -(-1)+k$
$$= 1+k$$

without this it could be a removable discontinuity

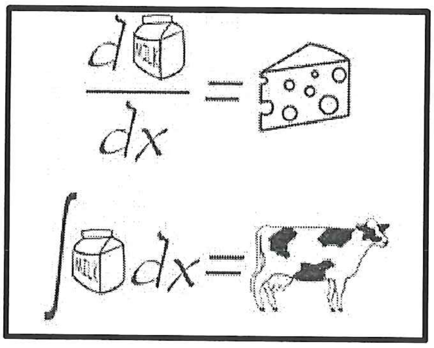
$$\therefore\ 1+k = -4$$
$$k = -5$$

ii) Is h(x) differentiable at $x=-1$? Assume $k=-5$ Justify your answer. (7)

$$\lim_{x\to -1^-} 6x^2+2x = 6(-1)^2+2(-1)$$
$$= 6-2$$
$$= 4$$

$$\lim_{x\to -1^+} -1 = -1$$

For differentiable: $\lim_{x\to -1^-} f'(x) = \lim_{x\to -1^+} f'(x)$



But $4 \neq -1$

$\therefore$ not diff at $x=-1$.

[24]

QUESTION 5

a) Given $f(x) = 4x - 4\sin x - 3\pi$. Use Newton's method to solve $f(x) = 0$

Use $x = 2$ as an initial value. Give your answer correct to 3 decimal places. (6)

$f'(x) = 4 - 4\cos x$ ✓ ✓

$x_1 = 2$

$x_2 = 2 - \frac{f(2)}{f'(2)}$ ✓ $= 2{,}8936\ldots$ ✓

$x_2 = 2{,}893616\ldots$

$x_3 = 2{,}745359\ldots$

$x_4 = 2{,}743685\ldots$

✓ at least 3.

$x_n = 2{,}744$ ✓

b) A chord of a circle which subtends an angle of θ at the centre, cuts off a segment equal in area to $\frac{3}{8}$ of the area of the whole circle.

Show that $4\theta - 4\sin\theta = 3\pi$ (4)

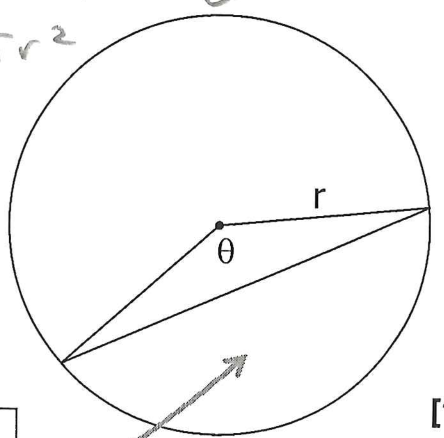
$A = \frac{1}{2}r^2\theta$ ✓ $- \frac{1}{2}r^2\sin\theta$ ✓ $= \frac{3}{8}\pi r^2$ ✓

(sector $-$ triangle $= \frac{3}{8}$ whole circle)

$r^2\theta - r^2\sin\theta = \frac{3}{4}\pi r^2$

$4r^2\theta - 4r^2\sin\theta = 3\pi r^2$ ✓

$4\theta - 4\sin\theta = 3\pi$



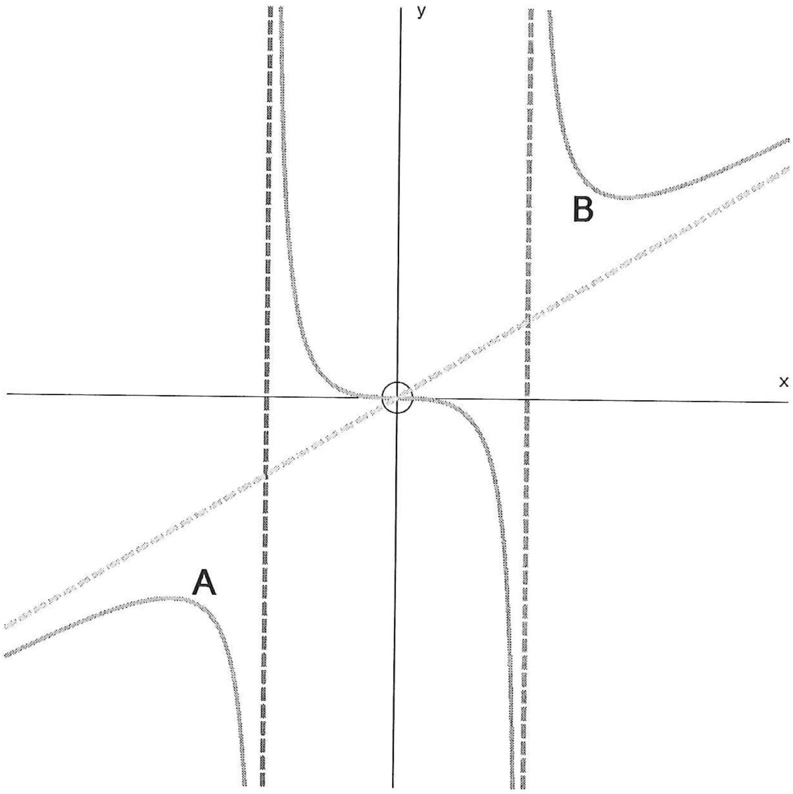
$Area\ of\ segment = \frac{3}{8} area\ of\ whole\ circle$

[10]

QUESTION 6

The sketch, not necessarily drawn to scale, represents:

$h(x)=\dfrac{2x^3}{x^2-4}$



a) Calculate the coordinates of A and B, the relative maximum and relative minimum of h(x). (9)

$$h'(x) = \frac{(x^2-4)(6x^2) - 2x(2x^3)}{(x^2-4)^2} = 0$$

$$6x^4 - 24x^2 - 4x^4 = 0$$

$$2x^4 - 24x^2 = 0$$

$$2x^2(x^2-12) = 0$$

$x = 0$ or $x = \pm\sqrt{12} = \pm 2\sqrt{3}$

$y = \pm 6\sqrt{3}$

$B(2\sqrt{3};\ 6\sqrt{3})$ $\quad A(-2\sqrt{3};\ -6\sqrt{3})$

$(3{,}46;\ 10{,}39)$ $\quad (-3{,}46;\ -10{,}39)$

b) Give the equations of the three asymptotes. (4)

$x = 2$ or $x = -2$ vertical asymptotes

$$x^2 - 4 \,\overline{)\, 2x^3} \quad 2x \ldots$$

M

$y = 2x$ oblique asymptote

c) From the graph, determine the value(s) of x if $\sqrt{\dfrac{2x^3}{x^2-4}}$ is real. (4)

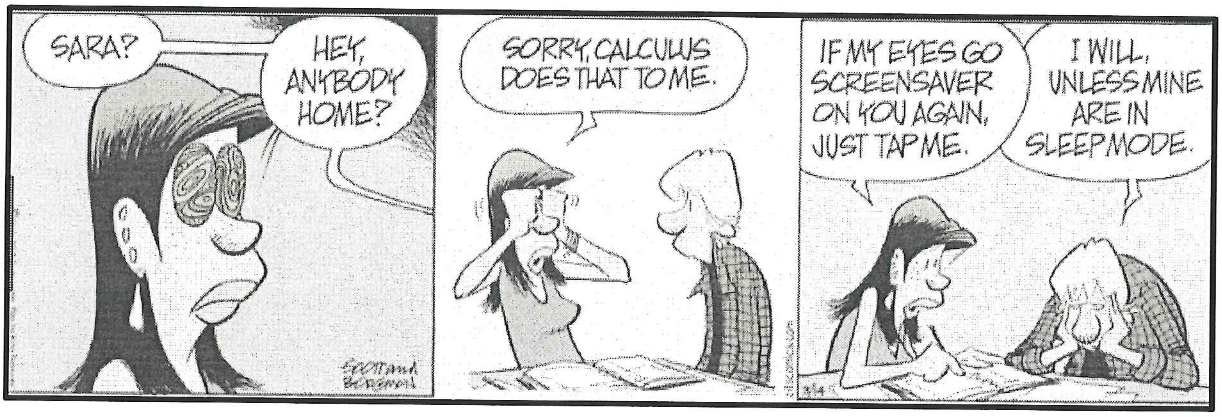
$$\frac{2x^3}{x^2-4} \geq 0$$

$-$ $+$ $-$ $+$

-2 0 2

$x \in (-2; 0] \cup (2; \infty)$

[17]



QUESTION 7

a) Determine the equation of the tangent to the curve defined by the equation

$x^2 - xy^2 + 3y^2 = xy + 12$ at the point $(\cancel{-3;1})$. $(3;-1)$ (10)

$$2x - \left(1.y^2 + x.2y\frac{dy}{dx}\right) + 6y\frac{dy}{dx} = 1y + x.\frac{dy}{dx}$$

$$2x - y^2 - 2xy\frac{dy}{dx} + 6y\frac{dy}{dx} = y + x\frac{dy}{dx}$$

$$2(3) - (-1)^2 - 2(3)(-1)\frac{dy}{dx} + 6(-1)\frac{dy}{dx} = (-1) + 3\frac{dy}{dx}.$$

$$6 - 1 + 6\frac{dy}{dx} - 6\frac{dy}{dx} = -1 + 3\frac{dy}{dy}$$

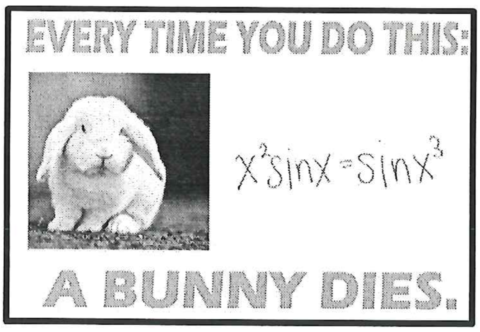
$$5 + 1 = 3\frac{dy}{dx}.$$

$$2 = \frac{dy}{dx}$$

$$y - (-1) = 2(x - 3)$$

$$y + 1 = 2x - 6$$

$$y = 2x - 7$$



b) Determine the n^{th} derivative of $f(x) = (2-x)^{-5}$ (6)

$n=1$: $f'(x) = -5(2-x)^{-6}(-1)$

$n=2$: $f''(x) = -5 \cdot -6(2-x)^{-7}(-1)(-1)$

$n=3$: $f'''(x) = -5 \cdot -6 \cdot -7 (2-x)^{-8} \cdot (-1)(-1)(-1)$

$\vdots$

nth : $f^{(n)}(x) = \dfrac{(n+4)!\,(2-x)^{-5-n}}{4!}$

OR $\quad {}^{n+4}P_n \, (2-x)^{-5-n}$

c) Determine $D_x\left[\ln\left(\dfrac{x-2}{x-1}\right)\right]$. Simplify your answer. (7)

$= \dfrac{1}{\frac{x-2}{x-1}} \cdot D_x\left[\dfrac{x-2}{x-1}\right]$

$= \dfrac{x-1}{x-2} \cdot \dfrac{1}{(x-1)^2}$

$= \dfrac{1}{(x-2)(x-1)}$

$D_x\left[\dfrac{x-2}{x-1}\right] = \dfrac{(x-1)(1) - (x-2)(1)}{(x-1)^2}$

$= \dfrac{x-1-x+2}{(x-1)^2}$

$= \dfrac{1}{(x-1)^2}$

OR

$D_x\left[\ln(x-2) - \ln(x-1)\right]$

$= \dfrac{1}{x-2} - \dfrac{1}{x-1}$

$= \dfrac{x-1-(x-2)}{(x-2)(x-1)}$

$= \dfrac{1}{(x-2)(x-1)}$

[23]

QUESTION 8

The concentration $C(T)$ in milligrams/cm^3 of a particular drug in a patient's bloodstream is determined using the formula:

$C(T)=\dfrac{0{,}16T}{(T+2)^2}$,where T is the time in hours which has elapsed since the drug was taken.

a) If the patient took the drug at 7:00 in the morning, at what time will the concentration be a maximum? (8)

$$c'(T) = 0$$

$$= \frac{(T+2)^2(0{,}16) - 0{,}16T(2(T+2).1)}{[(T+2)^2]^2}$$

$$0 = 0{,}16(T+2)^2 - 0{,}16T(2(T+2))$$

$$= 0{,}16(T+2)[T+2-T(2)]$$

$$= 0{,}16(T+2)[2-T]$$

$T = -2$ or $T = 2$ hours

At 9am.

b) What is this maximum concentration? (2)

$$C(2) = \frac{0{,}16(2)}{(2+2)^2}$$

$$= \frac{0{,}32}{16}$$

$= 0{,}02$ mg/cm^3

[10]

QUESTION 9

The area under the curve $f(x) = x^2 + 3$ above the x-axis and in the interval [0;3] is calculated using Riemann sums. If n rectangles are used, then an expression for the area simplifies to:

$$A = 18 + \frac{27}{2n} + \frac{9}{2n^2}$$

a) Calculate the area if 18 rectangles are used. (3)

$$A = 18 + \frac{27}{2(18)} + \frac{9}{2(18)^2} \checkmark$$

$$= \frac{1351}{72} \checkmark = 18,76 \checkmark$$

b) Calculate the exact area under the curve. (1)

$$\lim_{n \to \infty} 18 + \frac{27}{2n} + \frac{9}{2n^2}$$

$$= 18 \checkmark$$

OR

$$\int_0^3 x^2 + 3\,dx$$

$$= 18$$

c) Why do the answers in a) and b) differ in value? (2)

Fewer ✓ rectangles in a), greater area as there is part of each ✓ rectangle above curve.

d) Calculate the percentage error if 18 rectangles are used. (3)

$$\frac{\frac{1351}{72} - 18}{18} \checkmark = 4,24\% \checkmark \text{ error (over)}$$

OR

$$\frac{\frac{1351}{72}}{18} = 1,042438...$$

$\therefore$ 4,24% error

OR

$$\frac{18,76 - 18}{18} \times 100 = 4,22\% \text{ error (over)}$$

[9]

QUESTION 10

a) Determine the following integrals:

i) $\int 4x(x^2+2)^5 dx$ (6)

Let $u = x^2 + 2$ ✓

$\frac{du}{dx} = 2x$

$du = 2x\, dx$ ✓

$\int 2 u^5 du$

$= 2$✓ $\frac{u^6}{6} + c$

$= \frac{2(x^2+2)^6}{6}$✓ $+ c$

$= \frac{(x^2+2)^6}{3}$✓ A $+ c$ ✓

ii) $\int \frac{1}{1+\cot^2 x} dx$ (Hint: $1+\cot^2 x = \operatorname{cosec}^2 x$) (8)

learn how to do this!

$$\int \frac{1}{\operatorname{cosec}^2 x} dx = \int \sin^2 x \, dx$$

$$\cos 2x = 1 - 2\sin^2 x$$

$$\frac{\cos 2x - 1}{-2} = \sin^2 x$$

$$-\frac{1}{2}\int \cos 2x - 1 \, dx$$

$$= -\frac{1}{2}\left[\frac{\sin 2x}{2} - x\right] + c$$

$$= -\frac{1}{4}\sin 2x + \frac{1}{2}x + c$$

iii) $\int xe^{-4x}dx$ (by parts) (8)

let $f(x) = x$ $\quad f'(x) = 1$

$g(x) = \frac{e^{-4x}}{-4}$ $\quad g'(x) = e^{-4x}$

$$= \frac{x \cdot e^{-4x}}{-4} - \int \frac{e^{-4x}}{-4} dx$$

$$= -\frac{x}{4}e^{-4x} + \frac{1}{4}\frac{e^{-4x}}{-4} + c$$

$$= -\frac{x}{4}e^{-4x} - \frac{1}{16}e^{-4x} + c$$

b) i) Decompose $\frac{6x-23}{(x+2)(2x-3)}$ into partial fractions. (6)

$$\frac{6x-23}{(x+2)(2x-3)} = \frac{A}{x+2} + \frac{B}{2x-3} \checkmark$$

$$6x-23 = A(2x-3) + B(x+2) \checkmark$$
$$6x-23 = 2Ax - 3A + Bx + 2B$$

$$6 = 2A + B \checkmark \quad \therefore \quad 12 = 4A + 2B \quad \ldots (1)$$
$$-23 = -3A + 2B \checkmark \quad 23 = 3A - 2B \quad \ldots (2)$$

*

$$35 = 7A$$
$$5 = A \checkmark A$$

$$\therefore \quad 6 = 2(5) + B$$
$$-4 = B \checkmark A$$

*fx-991ES PLUS can solve 2 linear equations simult.

$$\frac{6x-23}{(x+2)(2x-3)} = \frac{5}{x+2} + \frac{-4}{2x-3}$$

MODE
5: EQN
1: $a_nX + b_nY = c_n$.

ii) Hence determine the following integral:

$$\int \frac{6x-23}{(x+2)(2x-3)}dx$$ (5)

$$= \int \frac{5}{x+2}\,dx \checkmark - 4\int \frac{1}{2x-3}\,dx \checkmark$$

$$= 5\ln|x+2| \checkmark - \frac{4\ln|2x-3|}{2} \checkmark \checkmark + c$$

$$= 5\ln|x+2| - 2\ln|2x-3| + c$$

[33]

Can be or $= 5\log(x+2) - 2\log(2x-3) + c$

ln ln

QUESTION 11

The sketch shows the graphs of $y = \sin 2x$ and $y = \cos x$ for the interval $x \in \left[0; \frac{\pi}{2}\right]$

a) Determine the x-coordinate of the point of intersection P. (4)

$\sin 2x = \cos x$

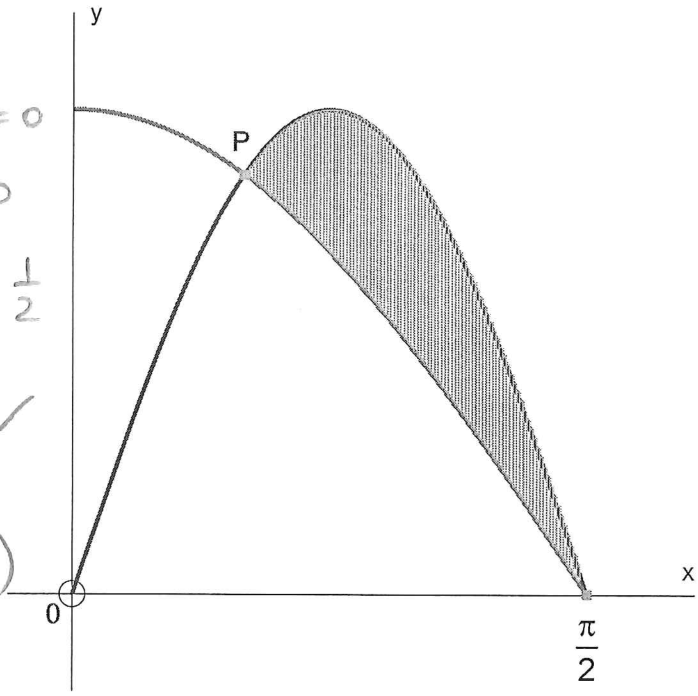
$2 \sin x \cos x - \cos x = 0$ ✓

$\cos x (2 \sin x - 1) = 0$ ✓

$\cos x = 0$ or $\sin x = \frac{1}{2}$ ✓

$x = \frac{\pi}{2}$ or $x = \frac{\pi}{6}$ ✓

$P\left(\frac{\pi}{6} ; y_P\right)$



* Do not divide by $\cos x$, only if changing to $\tan x$.

b) Hence or otherwise determine the area of the shaded region. (4)

✓ $\int_{\frac{\pi}{6}}^{\pi/2} \sin 2x$ ✓ $- \cos x$ ✓ dx

use calculator.

$= \frac{1}{4}$ ✓ units^2

[8]

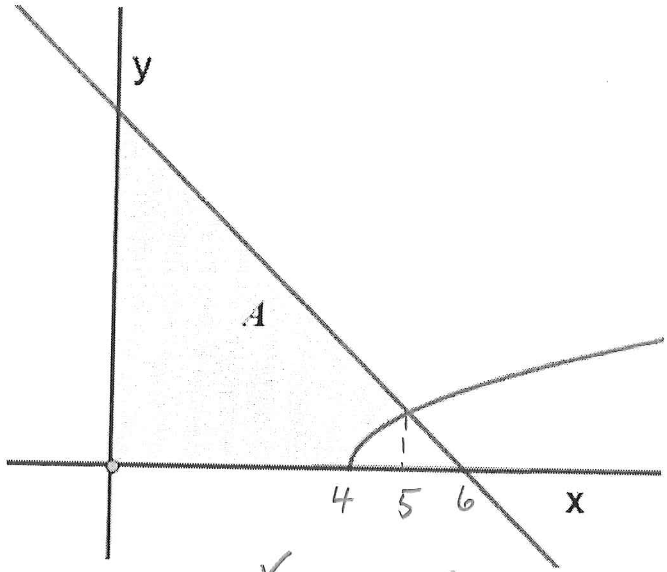
QUESTION 12

The diagram shows part of the curve $y = \sqrt{x-4}$ and the line $x + y = 6$

An area A is formed bounded by the axes and the graphs of the curve $y = \sqrt{x-4}$ and the line $x + y = 6$

Calculate the volume of the solid formed when the shaded area A is rotated about the x-axis.

(10)



$6 - x = \sqrt{x-4}$

$36 - 12x + x^2 = x - 4$

$x^2 - 13x + 40 = 0$

$(x-8)(x-5) = 0$

$x \neq 8$ or $x = 5$

$\text{Vol} = \pi \int_0^5 (6-x)^2 \, dx - \pi \int_4^5 (\sqrt{x-4})^2 \, dx$

$= \frac{215}{3}\pi - \frac{1}{2}\pi = 225,147.. - 1,57..$

$= \frac{427\pi}{6}$

$= 223,58$ units^3

[10]

[Total: 200 marks]

8a) Alternate memo.

$$c(T) = 0{,}16T\,(T+2)^{-2}$$

$$c'(T) = 0{,}16\,(T+2)^{-2} + 0{,}16T\left(-2(T+2)^{-3}\right)$$

$$= \frac{0{,}16}{(T+2)^2} + \frac{0{,}16T(-2)}{(T+2)^3}$$

$$0 = \frac{0{,}16}{(T+2)^2} - \frac{0{,}32T}{(T+2)^3}$$

$$0 = 0{,}16(T+2) - 0{,}32T$$

$$0 = 0{,}16\left[T+2-2T\right]$$

$$0 = 0{,}16\left[2-T\right]$$

$T = 2$ hours.

$\therefore$ 9 am.

Q 2b) Alternate (not recommended)

$$\frac{a+3i}{2-5i} \times \frac{2+5i}{2+5i} \times bi = -11-13i$$

$$\Rightarrow \frac{2abi + 5abi^2 - 6b - 15bi}{29} = -11-13i$$

$$2abi - 5ab - 6b - 15bi = -319 - 377i$$

$$-5ab - 6b = -319$$

$$\times 2$$

$$-10ab - 12b = -638 \quad \ldots (1)$$

$$2ab - 15b = -377$$

$$\times 5$$

$$10ab - 75b = -1885 \quad \ldots (2)$$

$$-10ab - 12b = -638$$

$$10ab - 75b = -1885$$

$$-87b = -2523$$

$$b = 29$$

$$2a(29) - 15(29) = -377$$

$$58a - 435 = -377$$

$$58a = 58$$

$$a = 1$$

$$= 2a + 5ai + 6i + 15i^2$$

$$= 2a + 5ai + 6i - 15$$