MEMORANDUM: PRELIM 2019

QUESTION 1

1.1 (a) $2\ln(x-4)=2$

$\ln(x-4)=1$✓

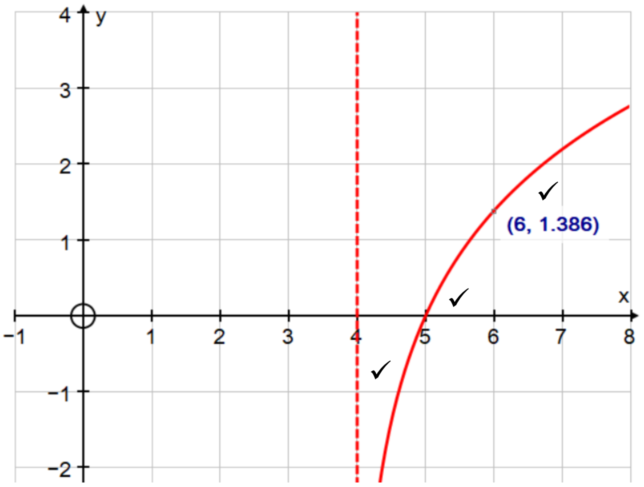
$x-4=e$✓

$x=4+e$✓ $=6{,}72$✓ (4)

(b) Domain: $x-4>0$✓

$x>4$ ✓

Range : $y\in(-\infty;\infty)$ ✓



(6)

1.2 (a) $y(5)=120(1-e^{-0.15\times5})$ ✓

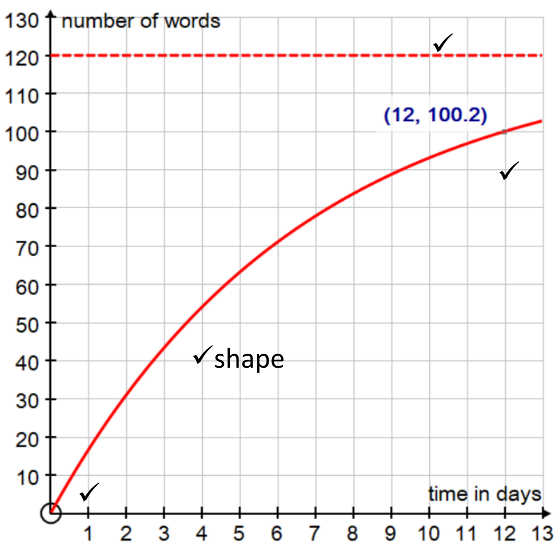
$=63$ words✓ (2)

(b) $100=120(1-e^{-0.15\times t})$ ✓

$\frac{5}{6}=1-e^{-0.15\times t}$✓

$t=\frac{-\ln\frac{1}{6}}{-0{,}15}$✓ $=12$days✓ (4)

(c) (4)



(d) Maximum number is 120 words. ✓

Horizontal asymptote is 120, so it cannot be exceeded. ✓ (2)

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QUESTION 2

$1+\frac{1}{2}+\frac{1}{4}\cdots+2\left(\frac{1}{2}\right)^n=2-2\left(\frac{1}{2}\right)^n$ ✓

When n= 1, LHS= 1✓

RHS= $2-2\left(\frac{1}{2}\right)^1=1$✓

$\therefore\ true\ for\ n=1$

Assume true for $n=k$:✓

$$1+\frac{1}{2}+\frac{1}{4}\cdots+2\left(\frac{1}{2}\right)^k=2-2\left(\frac{1}{2}\right)^k \checkmark$$

Propose a formula for $n=k+1$: $2-2\left(\frac{1}{2}\right)^{k+1}$ ✓

$$1+\frac{1}{2}+\frac{1}{4}\cdots+2\left(\frac{1}{2}\right)^k+2\left(\frac{1}{2}\right)^{k+1} \checkmark =2-2\left(\frac{1}{2}\right)^k+2\left(\frac{1}{2}\right)^{k+1} \checkmark$$

$$=2-2\left(\frac{1}{2}\right)^k+2\left(\frac{1}{2}\right)^k.\left(\frac{1}{2}\right)$$

$$=2+2\left(\frac{1}{2}\right)^k.\left(\frac{1}{2}\right)-2\left(\frac{1}{2}\right)^k$$

✓✓✓ $$=2+2\left(\frac{1}{2}\right)^k.\left(\frac{1}{2}-1\right)$$

manipulation $$=2+2\left(\frac{1}{2}\right)^k.\left(-\frac{1}{2}\right)$$

$$=2-2\left(\frac{1}{2}\right)^k.\left(\frac{1}{2}\right)$$

$$=2-2\left(\frac{1}{2}\right)^{k+1}$$

This is the proposed formula, therefore, it is proven by P.M.I that the formula is true $\forall n\in\mathrm{N}$✓✓ **[13]**

QUESTION 3

3.1 (a) If $2+3i$ is a root then $2-3i$✓ is also a root.

Quadratic factor is $x^2-(2+3i+2-3i)x$✓ $+(2+3i)(2-3i)$✓

$\Longrightarrow x^2-4x+13$ ✓ (4)

(b) $x^4-4x^3+17x^2-16x+52=(x^2-4x+13)(ax^2+bx+4)$ by inspection✓

$=(x^2-4x+13)(x^2+4)$ ✓

$\therefore x=2\pm 3i$✓ or $x=\pm 2i$✓✓ (5)

3.2 $\frac{7-i}{3-4i} = \frac{(7-i)(3+4i)}{(3-4i)(3+4i)}$ ✓

$= \frac{25+25i}{25}$ ✓

$= 1 + i$ ✓

$|1+i| = \sqrt{1^2+1^2}$ ✓ $= \sqrt{2}$ ✓

$\arg(1+i) = tan^{-1}\left(\frac{1}{1}\right)$ ✓ $= \frac{\pi}{4}$ ✓ (7)

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QUESTION 4

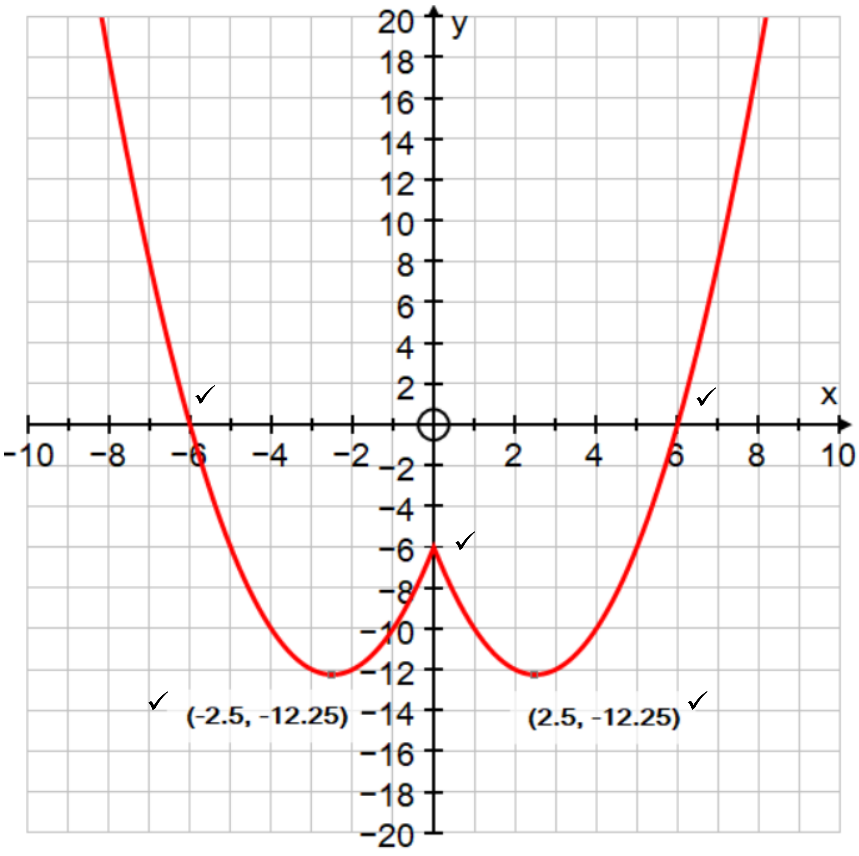
Let $k = |x|$

$k^2 - 5k - 6 = 0$

$k = -1$ and $k = 6$ ✓

$|x| = -1$ or $|x| = 6$

n/a ✓ $x = 6$ ✓ or $x = -6$ ✓



[9]

QUESTION 5

5.1 (a) False✓ $\lim_{x\to -3^-} g(x) \neq \lim_{x\to -3^+} g(x)$✓✓ (3)

(b) True✓ $\lim_{x\to 1^-} g(x) = \lim_{x\to 1^+} g(x)$✓✓ (3)

5.2 Function is continuous since $\lim_{x\to 1^-} g(x) = \lim_{x\to 1^+} g(x) = g(1)$✓

$\lim_{x\to 1^-} g'(x) = e^1 - 1 = 1{,}7$✓✓

$\lim_{x\to 1^+} g'(x) = 4(1) = 4$✓✓

$\lim_{x\to 1^-} g'(x) \neq \lim_{x\to 1^+} g'(x)$ ∴ not differentiable✓ (6) **[12]**

QUESTION 6

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$✓

$9 - 2^2$✓$= 2a + b$✓$\therefore 2a + b = 5$

$\lim_{x\to 2^-} f'(x) = \lim_{x\to 2^+} f'(x)$✓

$-2(2)$✓$= a$✓$\therefore a = -4$✓

$b = 13$✓ **[8]**

QUESTION 7

7.1 (a) $y = \ln\frac{\sin 2x}{2-x}$

$\frac{dy}{dx} = \frac{2\cos 2x \checkmark\checkmark (2-x) - \sin 2x(-1)\checkmark\checkmark}{(2-x)^2 \checkmark} \div \frac{\sin 2x}{2-x}$✓

$= \frac{(4-2x)\cos 2x + \sin 2x}{(2-x)^2} \times \frac{2-x}{\sin 2x}$✓

$= \frac{(4-2x)\cos 2x + \sin 2x}{(2-x)\sin 2x}$✓ (8)

(b) $y = \tan 3x . e^{2x+1}$

$\frac{dy}{dx} = 3$✓$sec^2 3x$✓$. e^{2x+1}$✓$+ \tan 3x$✓$. 2$✓e^{2x+1}✓ (6)

7.2 (a) Let $y = 0$

$0\sin x + \cos x = 3x(0) + \frac{1}{2}x$

$f(x) = cosx - \frac{1}{2}x = 0$✓

$f'(x) = -\sin x - \frac{1}{2}$✓

$a_{n+1} = a_n - \frac{cosa_n - \frac{1}{2}a_n}{-\sin a_n - \frac{1}{2}}$✓✓

Let $a_1 = 1$✓

$a_2 = 1{,}03004 \ldots$

$a_3 = 1{,}02986 \ldots$ ✓✓

$x \approx 1{,}0299$✓

$A(1{,}0299; 0)$✓ (9)

(b) $D_x(y \sin x + \cos x) = D_x(3xy + \frac{1}{2}x)$

$\frac{dy}{dx}$ ✓$\sin x$✓$+ y$ ✓$\cos x$✓$- \sin x$✓$= 3y$✓$+ 3x\frac{dy}{dx}$ ✓$+ \frac{1}{2}$ ✓

$\frac{dy}{dx}\sin x - 3x\frac{dy}{dx} = 3y + \frac{1}{2} - y\cos x + \sin x$ ✓

$\frac{dy}{dx} = \frac{3y+\frac{1}{2}-y\cos x+\sin x}{\sin x - 3x}$ ✓ (10)

(c) $m_T = \frac{3(0)+\frac{1}{2}-(0)\cos 1{,}0299+\sin 1{,}0299}{\sin 1{,}0299-3(1{,}0299)}$ ✓$= -0{,}608$✓

$y - 0 = -0{,}608(x - 1{,}0299)$✓

$y = -0{,}608x + 0{,}6262$✓ (4) **[37]**

QUESTION 8

8.1 $\hat{A} = \frac{\pi-\theta}{2}$ $Int\ <'\ s\ of\ isos\ \Delta$✓

$\frac{\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right)}{2r}$ ✓$= \frac{\sin\theta}{AD}$ ✓

$AD = \frac{2r\sin\theta}{\cos\frac{\theta}{2}}$ ✓

$= \frac{2r\times 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos\frac{\theta}{2}}$ ✓

$= 4r\sin\frac{\theta}{2}$ ✓

or

$AD^2 = (2r)^2 + (2r)^2 - 2(2r)(2r)\cos\theta$✓

$= 8r^2 - 8r^2\cos\theta$✓

$= 8r^2 - 8r^2(1 - 2sin^2\frac{\theta}{2})$✓✓

$= 16r^2 sin^2\frac{\theta}{2}$ ✓

$AD = 4r\sin\frac{\theta}{2}$ ✓

Arc BC $= r\theta$✓

$p = 2r + r\theta + 4r\sin\frac{\theta}{2}$ ✓ (8)

8.2 $A = \frac{1}{2}(2r)^2\sin\theta$✓$- \frac{1}{2}r^2\theta$✓

$= 2r^2\sin\theta - \frac{1}{2}r^2\theta$✓ (3)

8.3 $A = 2(2\theta)^2\sin\theta - \frac{1}{2}(2\theta)^2\theta$✓

$= 8\theta^2\sin\theta - 2\theta^3$ ✓

$A' = 16\theta \checkmark \sin\theta \checkmark + 8\theta^2 \checkmark \cos\theta \checkmark - 6\theta^2 \checkmark = 0 \checkmark$

$\theta = 1{,}857 \checkmark$ (9)

8.4 $A'' = 16sin\theta \checkmark + 16\theta\cos\theta \checkmark + 16\theta\cos\theta \checkmark - 8\theta^2\sin\theta \checkmark - 12\theta \checkmark$

At $\theta = 1{,}857$

$A'' = 16sin1{,}857 + 16(1{,}857)\cos1{,}857 + 16(1{,}857)\cos1{,}857 - 8(1{,}857)^2\sin1{,}857 - 12(1{,}857) \checkmark = - \checkmark$ (7)

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QUESTION 9

9.1 $g(x) = \frac{2x^2+7x-15}{x+2}$

Vertical asymptote: $x = -2 \checkmark\checkmark$

Oblique asymptote: $2x^2 + 7x - 15 \checkmark = (x+2) \checkmark (2x+3) \checkmark + \cdots$

$y = 2x + 3 \checkmark$ (6)

9.2 $g'(x) = \frac{(4x+7) \checkmark (x+2) \checkmark - (2x^2+7x-15) \checkmark (1)}{(x+2)^2 \checkmark}$

$= \frac{2x^2+8x+29}{(x+2)^2} \checkmark > 0 \checkmark$ (6)

9.3 $x - intercepts: 2x^2 + 7x - 15 = 0 \checkmark$

$x = \frac{3}{2} \checkmark \; or \;\; x = -5$

$y - intercepts: g(0) = -\frac{15}{2} \checkmark$ (4)

[16]

QUESTION 10

10.1 (a) $\int \frac{\cos x}{\sin x - 2} dx$

$= \ln|\sin x - 2| \checkmark\checkmark\checkmark + c \checkmark$ (4)

(b) $\int x\sqrt[3]{2+5x}\,dx$

Let $u = 2 + 5x \quad \therefore x = \frac{u-2}{5} \checkmark$

$\frac{du}{dx} = 5 \quad \therefore dx = \frac{du}{5} \checkmark$

$\int \frac{u-2}{5} \times u^{\frac{1}{3}} \times \frac{du}{5} \checkmark$

$= \frac{1}{25}\int\left(u^{\frac{4}{3}} - 2u^{\frac{1}{3}}\right)du$ ✓

$= \frac{1}{25}\left(\frac{3}{7}u^{\frac{7}{3}}$ ✓ $- \frac{3}{2}u^{\frac{4}{3}}$ ✓ $\right) + c$

$= \frac{3}{175}(2+5x)^{\frac{7}{3}} - \frac{3}{50}(2+5x)^{\frac{4}{3}} + c$ ✓ (7)

or

$f(x) = x$ ✓ $\quad f'(x) = 1$ ✓

$g'(x) = (2+5x)^{\frac{1}{3}}$ ✓ $\quad g(x) = \frac{3}{20}(2+5x)^{\frac{4}{3}}$ ✓

$\int x\sqrt[3]{2+5x}\,dx = \frac{3x}{20}(2+5x)^{\frac{4}{3}}$ ✓ $- \int \frac{3}{20}(2+5x)^{\frac{4}{3}}dx$ ✓

$= \frac{3x}{20}(2+5x)^{\frac{4}{3}} - \frac{9}{700}(2+5x)^{\frac{7}{3}} + c$ ✓ (7)

10.2 (a) $\frac{5x^2+20x+6}{x^3+2x^2+x} = \frac{5x^2+20x+6}{x(x+1)^2}$ ✓

$= \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$ ✓

$A = \frac{5x(0)^2+20(0)+6}{(0+1)^2} = 6$ ✓

$C = \frac{5(-1)^2+20(-1)+6}{-1} = 9$ ✓

$\frac{5x^2+20x+6}{x(x+1)^2} = \frac{6}{x} + \frac{B}{x+1} + \frac{9}{(x+1)^2}$ ✓

$5x^2 + 20x + 6 = 6(x+1)^2 + Bx(x+1) + 9x$ ✓

$5x^2 = (6+B)x^2$

$B = -1$ ✓

$\frac{5x^2+20x+6}{x(x+1)^2} = \frac{6}{x} - \frac{1}{x+1} + \frac{9}{(x+1)^2}$ ✓ (8)

(b) $\int\left(\frac{6}{x} - \frac{1}{x+1} + 9(x+1)^{-2}\right)dx$ ✓

$= 6\ln|x|$ ✓ $- \ln|x+1|$ ✓ $- 9(x+1)^{-1}$ ✓✓ $+ c$ (5)

10.3 (a) $A = \lim_{n\to\infty}\left(\frac{48}{5} + \frac{27}{5n} + \frac{9}{5n^2}\right)$ ✓ $= \frac{48}{5}$ ✓ (2)

(b) $\int_0^3 (kx^2 + 2)dx = \frac{48}{5}$ ✓

$\left[\frac{kx^3}{3} ✓ + 2x ✓\right]_0^3 = \frac{48}{5}$

$\frac{k(3)^3}{3} + 2(3) = \frac{48}{5}$ ✓

$9k = \frac{18}{5}$ ✓

$k = \frac{2}{5}$ ✓ (6)

(c) $\pi \int_0^p \left(\frac{2}{5}x^2 + 2\right)^2 dx = \frac{4984}{375}\pi$ ✓✓

$\int_0^p \left(\frac{4}{25}x^4 + \frac{8}{5}x^2 + 4\right) dx = \frac{4984}{375}$ ✓

$\left[\frac{4}{125}x^5 ✓ + \frac{8}{15}x^3 ✓ + 4x ✓\right]_0^p = \frac{4984}{375}$

$\frac{4}{125}p^5 + \frac{8}{15}p^3 + 4p = \frac{4984}{375}$ ✓

$p = 2$✓ (8)

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