Gr 12 AP Maths P1 Sept 2019 Memo

1

$8^n - 7n + 6$

Let $n = 1$ ✓

then $8^n - 7n + 6$

$= 8 - 7 + 6$

$= 7$ *which is div by* 7 ✓

Assume true for $n = k$

Then $8^k - 7k + 6 = 7r$ *for* $r \in N$ ✓

$\therefore\ 8^k = 7r + 7k - 6$ ✓

Now $8^{k+1} - 7(k+1) + 6$ ✓

$= 8 \times 8^k - 7k - 7 + 6$ ✓

$= 8.8^k - 7k - 1$ ✓

$= 8(7r + 7k - 6) - 7k - 1$ ✓

$= 56r + 56k - 48 - 7k - 1$

$= 56r + 49k - 49$ ✓

$= 7(8r + 7k - 7)$ *which is div by* 7 ✓

Thus since the statement is true for $n = 1$ and true for $n = k + 1$ if assumed true for $n = k$, then it is true *for* $n = 2,\ n = 3$ etc. by Mathematical Induction. ✓

(11)

2.1

$z = x + iy$

$|x + iy - 1| = |x + iy|$

3 cases:

$-x - iy + 1 = -x - iy$ ✓

$\therefore 1 = 0$

Invalid ✓

Or

$-x - iy + 1 = x + iy$ ✓

$\therefore -2x - 2iy + 1 = 0$

$\therefore -2(x + iy) = -1$ ✓

$\therefore z = -\frac{1}{-2} = \frac{1}{2}$ ✓

$\therefore \boldsymbol{Re(z)} = \frac{\mathbf{1}}{\mathbf{2}}$

Or

$x + iy - 1 = x + iy$ ✓

$\therefore -1 = 0$

Invalid

(6)

Could also do:

If $|z - 1| = |z|$

then $(z - 1)^2 = z^2$

$z^2 - 2z + 1 = z^2$

$\therefore -2z + 1 = 0$

$\therefore z = \frac{-1}{-2} = \frac{1}{2}$

2.2

$(2 + i)^4$

$(2 + i)^2(2 + i)^2$

$(4 + 4i + i^2)(4 + 4i + i^2)$ ✓

$(4 + 4i - 1)(4 + 4i - 1)$

$(3 + 4i)(3 + 4i)$ ✓

$(9 + 24i + 16i^2)$ ✓

$(9 + 24i - 16)$ ✓

$-7 + 24i$ marked a and b as implied if stop here.

$a = -7$ *and* $b = 24$ ✓

OR

Use Pascal's triangle

1

1 2 1

1 3 3 1

1 4 6 4 1

$= 2^4 + 4(2)^3(i) + 6(2)^2(i)^2 + 4(2)(i)^3 + i^4$ ✓

$= 16 + 32i + 24i^2 + 8 \times i^2 \times i + i^2.i^2$ ✓

$= 16 + 32i - 24 - 8i + (-1 \times -1)$ ✓

$= 16 - 24 + 24i + 1$

$= -7 + 24i$ ✓ ✓

(5)

2.3

$x^2 - 4ix + 5 = 0$

$x^2 - 4ix - 5i^2 = 0$

$(x - 5i)(x + i) = 0$ ✓ ✓

$\therefore\ x = 5i$ *or* $x = -i$ ✓ ✓

OR

Formula

(4)

3.1.1

$f(x) = e^{x-1},\ x \in R \quad g(x) = \sqrt{x - 1}$

$f:\ y = e^{x-1}$

For inverse

$x = e^{y-1}$ ✓

$\ln x = y - 1$ ✓

$f^{-1}(x) = \ln x + 1$ ✓

(3)

3.1.2

$f(g(3)) = e^{\sqrt{3-1}-1}$ ✓

$= e^{\sqrt{2}-1}$ ✓

$= 1{,}51$ ✓

(3)

3.1.3

$f(g(x)) = e^{\sqrt{x-1}-1}$

From g : $x - 1 \geq 0$ ✓

Domain

$\therefore x \geq 1$ ✓

Also $g(x) \geq 0$ ✓

$\therefore\ f(g(x)) \geq e^{0-1}$

$\geq \frac{1}{e}$ ✓

$\geq 0{,}3678 \ldots$

(4)

3.2.1

$\log_3 x + 2\log_x 3 = 3$

$\frac{\log x}{\log 3} + \frac{2\log 3}{\log x} = 3$ ✓ change of base

$\times\ by \log 3 \log x$

$(\log x)^2 + 2(\log 3)^2 = 3\log 3 \log x$ ✓

$(\log x)^2 - 3\log 3 \log x + 2(\log 3)^2 = 0$ ✓

$(\log x - 2\log 3)(\log x - \log 3) = 0$ ✓

$\therefore \log x = 2\log 3 \quad or \quad \log x = \log 3$ ✓

$\therefore \log x = \log 3^2 \quad or \quad \log x = \log 3$

$\therefore x = 9 \quad or \quad x = 3$ ✓ ✓

(7)

[OR $\therefore x = 10^{2\log 3}$ *or* $x = 10^{\log 3}$]

Or change $\log_3 x$ to $\frac{1}{\log_x 3}$

and let $\log_x 3 = k$ etc.

3.2.2

$\ln(\cos x) = -2 \qquad x \in (-2\pi; 0)$

$e^{-2} = \cos x$ ✓

$x = \pm\cos^{-1}(e^{-2}) + 2n\pi \;\; n \in Z$

$x = \pm 1{,}435 + 2n\pi$

✓ for $\pm$ ✓value ✓ $2n\pi$

$x = -1{,}435$ *or* $x = -4{,}848$ ✓ ✓

(6)

4.1.1

Sketch

$f(x) = \ln x$

x-cut ✓

one additional point

$e.g.\ (2; \ln 2) = (2; 0{,}693)\ OR\ (e; 1)$ ✓

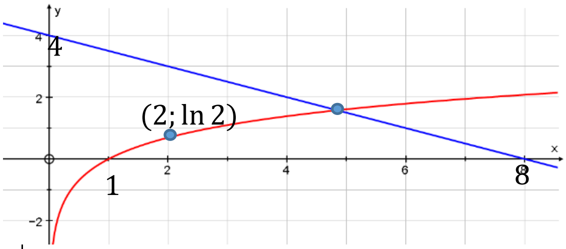
shape and asymptote ✓

$g(x) = -\frac{1}{2}x + 4$

y-cut ✓

x-cut ✓

∴ One root since only one point of intersection ✓



(6)

4.1.2

$\ln x = 4 - \frac{1}{2}x$

Solve for $h(x) = \ln x - \left(4 - \frac{1}{2}x\right) = 0$ ✓

$h(x) = \ln x - 4 + \frac{1}{2}x = 0$ ✓

[OR $h(x) = 4 - \frac{1}{2}x - \ln x = 0$]

$f'(x) = \frac{1}{x} + \frac{1}{2}$ ✓

Choose starting value $x_1 = 4$ ✓ [*or* 5]

$$x_{r+1} = x_r - \frac{\ln x_r - 4 + \frac{1}{2}(x_r)}{\frac{1}{x_r} + \frac{1}{2}}$$ ✓ ✓

$x_2 = 4{,}818274$ ✓

$x_3 = 4{,}844346 \ldots$ ✓

$x_4 = 4{,}844366 \ldots$

$x = 4{,}84437$ ✓

If chose $x_1 = 5$

$x_2 = 4{,}84366012..$

$x_3 = 4{,}84436686 \ldots$

$x = 4{,}84437$

(9)

4.2.

$h(x) = |e^x - 1|$

Critical value where $e^x - 1 = 0$

$\therefore e^x = 1$ ✓

i.e. $x = 0$ ✓

asymptote $y = 1$ ✓

shape on left ✓

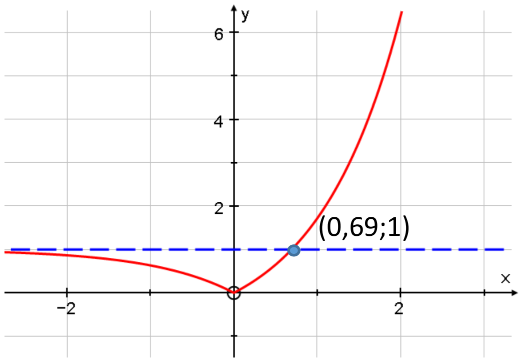
shape on right ✓

point of intercept with asymptote:

$e^x - 1 = 1$ ✓

$\therefore e^x = 2$

$\therefore x = \ln 2 \approx 0{,}69$ ✓



(7)

5.

$$f(x) = \begin{cases} \frac{a}{x} & x \geq 1 \\ b - 2x & x < 1 \end{cases}$$

If Diff then continuous at $x = 1$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$ ✓

$\therefore b - 2(1) = \frac{a}{1}$ ✓

$a = b(1) - 2(1)^2$

$\therefore \quad a = b - 2 \ldots (1)$ ✓

And

$\frac{d}{dx}(ax^{-1}) = -ax^{-2}$ ✓

$\lim_{x\to 1^-} f'(x) = \lim_{x\to 1^+} f'(x)$

$\lim_{x\to 1^-}(-2) = \lim_{x\to 1^+}\left(-\frac{a}{x^2}\right)$ ✓

$\therefore -2 = -a$ ✓

$\therefore a = 2$ *and* $b = 4$ ✓ ✓

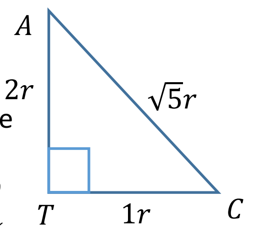
(8)

6.1

$f(x) = \frac{1}{\sqrt{3x-2}}$

$f'(x) = \lim_{h\to 0} \frac{\frac{1}{\sqrt{3(x+h)-2}} - \frac{1}{\sqrt{3x-2}}}{h}$ ✓ subs

	$= \lim_{h\to 0}\left(\frac{1}{\sqrt{3(x+h)-2}}-\frac{1}{\sqrt{3x-2}}\right)\times\frac{1}{h}$ add over LCD $\frac{\sqrt{3x-2}-\sqrt{3(x+h)-2}}{\sqrt{3(x+h)-2}\times.\sqrt{3x-2}}$ ✓ Multiply by conjugate $\frac{\sqrt{3x-2}-\sqrt{3(x+h)-2}}{\sqrt{3(x+h)-2}.\sqrt{3x-2}}\times\frac{\sqrt{3x-2}+\sqrt{3(x+h)-2}}{\sqrt{3x-2}+\sqrt{3(x+h)-2}}$ ✓ $\frac{(3x-2)-(3x+3h-2)}{\sqrt{3(x+h)-2}.\sqrt{3x-2}\times(\sqrt{3x-2}+\sqrt{3(x+h)-2})}$ ✓ $\frac{-3h}{(3x-2)\sqrt{3(x+h)-2}+(3x+3h-2)\sqrt{3x-2}}$ ✓ $\lim_{h\to 0}\frac{1}{h}\times\frac{-3h}{(3x-2)\sqrt{3(x+h)-2}+(3x+3h-2)\sqrt{3x-2}}$ $=\frac{-3}{(3x-2)\sqrt{3x-2}+(3x-2)\sqrt{3x-2}}$ ✓ $=\frac{3}{2(3x-2)\sqrt{3x-2}}$ ✓ $=\frac{-3}{2(3x-2)^{\frac{3}{2}}}$ ✓ (8)
6.2.1	$f(x)=\frac{\sin^2 3x}{x^2}$ $f'(x)=\frac{x^2.2\sin 3x.\cos 3x.3-\sin^2 3x.2x}{x^4}$ ✓ ✓ ✓ ✓ ✓ ✓ ✓ denominator $=\frac{6x^2\sin 3x\,.\cos 3x-2x.\sin^2 3x}{x^4}$ ✓ tidy up. OR $f(x)=\sin^2 3x\,.x^{-2}$ $f'(x)$ $=x^{-2}\times 2\sin 3x\,.\cos 3x\,.3+\sin^2 3x\times -2x^{-3}$ $=\frac{6\sin 3x\,.\cos 3x}{x^2}-\frac{2\sin^2 3x}{x^3}$ (8)
6.2.2	$D_x[x\sqrt{x^2-1}]$ $=D_x[x(x^2-1)^{\frac{1}{2}}]$ ✓ $=x.\frac{1}{2}(x^2-1)^{-\frac{1}{2}}.2x+(x^2-1)^{\frac{1}{2}}.1$ ✓ ✓ ✓ ✓ ✓ $=\frac{x^2}{(x^2-1)^{\frac{1}{2}}}+(x^2-1)^{\frac{1}{2}}$ ✓ ✓ (8)
6.2.3	$\frac{d}{dx}(5e^{3x})$ $=5e^{3x}.3$ $=15e^{3x}$ ✓ ✓ (2)

6.2.4	$\frac{d}{dx}(\log_4 x)$ $=\frac{d}{dx}\left(\frac{\ln x}{\ln 4}\right)$ ✓ $=\frac{1}{\ln 4}\times\frac{1}{x}$ $=\frac{1}{(\ln 4)x}$ ✓ (2)
6.2.5	$y=e^{x.\cos x}$ $\frac{dy}{dx}=e^{x.\cos x}(x(-\sin x)+\cos x.(1))$ ✓ ✓ ✓ ✓ ✓ $\frac{dy}{dx}=e^{x.\cos x}(-x\sin x+\cos x)$ ✓ (6)
6.3	$xy^2=2y$ $x.2y.\frac{dy}{dx}+y^2.1=2.\frac{dy}{dx}$ ✓ ✓ ✓ ✓ ✓ $(2xy-2).\frac{dy}{dx}=-y^2$ ✓ $\frac{dy}{dx}=-\frac{y^2}{(2xy-2)}$ ✓ OR $\frac{dy}{dx}=\frac{y^2}{(2-2xy)}$ (7)
7.1	In ΔATC  $\therefore$ *using Pythagoras* $AC=\sqrt{5}\,r$ ✓ ✓ triangle $\sin 2\theta=2\sin\theta\,.\cos\theta$ $=2\times\frac{1r}{\sqrt{5}r}\times\frac{2r}{\sqrt{5}\,r}$ ✓ ✓ $=\frac{4}{5}$ ✓ (5)
7.2	*Area c = area* ΔATC − *area a − area b* $=\frac{1}{2}b\times h-\frac{1}{2}absin\,C-\frac{1}{2}r^2 2\theta$ $T\hat{O}B=2\theta$ ✓ $\angle$ *at centre* $=2\times\angle$ *at circumference* *Area c* $=\frac{1}{2}r\times 2r-\frac{1}{2}r^2\sin(\pi-2\theta)-\frac{1}{2}r^2 2\theta$ ✓ ✓ ✓ $=r^2-\frac{1}{2}r^2\sin 2\theta-r^2\theta$ ✓ ✓ ✓ $=r^2-\frac{1}{2}r^2\frac{4}{5}-r^2\theta$ ✓ $=r^2(1-\frac{4}{10}-\theta)$ ✓ or $=\frac{1}{5}r^2(5-2-5\theta)$ $=r^2(\frac{3}{5}-\theta)$ $=\frac{1}{5}r^2(3-5\theta)$ (9)
7.3	*Area c* $=4^2(\frac{3}{5}-\theta)$ $\theta=\tan^{-1}\frac{1}{2}=0{,}463647\ldots$ ✓ in radians *Area c* $=4^2\left(\frac{3}{5}-\theta\right)=2{,}18\ units^2$ ✓ (2)

8.1	x –intercept $(1;0)$ ✓ y –intercept $(0;-\frac{1}{4})$ ✓ (2)
8.2.1	$f(x)=\frac{x-1}{(x+2)^2}=\frac{A}{x+2}+\frac{B}{(x+2)^2}$ ✓ $\frac{x-1}{(x+2)^2}=\frac{A(x+2)+B}{(x+2)^2}$ ✓ $x-1=A(x+2)+B$ $\therefore x-1=Ax+2A+B$ ✓ $\therefore A=1$ ✓ $-1=2A+B$ $\therefore B=-1-1=-3$ ✓ $f(x)=\frac{1}{x+2}-\frac{3}{(x+2)^2}$ ✓ (6)
8.2.2	$f(x)=\frac{1}{x+2}-\frac{3}{(x+2)^2}$ $f(x)=(x+2)^{-1}-3(x+2)^{-2}$ ✓ $f'(x)=-(x+2)^{-2}+6(x+2)^{-3}$ ✓ ✓ For $f'(x)=0$ $\frac{-1}{(x+2)^2}+\frac{6}{(x+2)^3}=0$ $\frac{-1(x+2)+6}{(x+2)^3}=0$ ✓ $-x-2+6=0$ ✓ $\therefore 4=x$ ✓ OR $f(x)=\frac{x-1}{(x+2)^2}$ $f'(x)=\frac{(x+2)^2(1)-(x-1)(2x+4)}{(x+2)^4}$ $=\frac{x^2+4x+4-(2x^2+2x-4)}{(x+2)^4}$ $=\frac{-x^2+2x+8}{(x+2)^4}$ For $f'(x)=0$ $x^2-2x-8=0$ etc. OR product rule (6)
8.2.3	$f(4)=\frac{4-1}{(4+2)^2}$ ✓ $T.P.=\left(4;\frac{1}{12}\right)$ ✓ (2)
8.3.1	$\lim_{x\to\infty}\frac{x-1}{(x+2)^2}=\lim_{x\to\infty}\frac{x-1}{x^2+4x+4}$ ✓ $=\lim_{x\to\infty}\frac{\frac{x}{x^2}-\frac{1}{x^2}}{\frac{x^2}{x^2}+\frac{4x}{x^2}+\frac{4}{x^2}}$ $=0$ ✓ (2)
8.3.2	Asymptotes $x=-2$ ✓ $\therefore$ $y=0$ *is a horizontal asymptote* ✓ (2)

8.4	See graph below
9.1.1	$\int\left(12x^3-\frac{2}{x^2}\right)dx$ $=\int(12x^3-2x^{-2})dx$ ✓ $=\frac{12x^4}{4}-\frac{2x^{-1}}{-1}+c$ ✓ ✓ $=3x^4+\frac{2}{x}+c$ ✓ ✓for c (5)
9.1.2	$\int\sin 3x\cos 2x\ dx$ $=\int\frac{1}{2}(\sin(3x+2x)+\sin(3x-2x))\,dx$ ✓ $=\int\frac{1}{2}(\sin(5x)+\sin(x))\,dx$ ✓ $\frac{1}{2}\left(-\frac{\cos 5x}{5}-\cos x\right)+c$ ✓✓ ✓ $=-\frac{\cos 5x}{10}-\frac{\cos x}{2}+c$ ✓ (6)
9.1.3	By parts $\int x\sqrt{2x+9}\,dx$ $\int f.g'\,dx=fg-\int g.f'\,dx+c$ Let $f=x$ $g'=(2x+9)^{\frac{1}{2}}$ $f'=1$ $g=(2x+9)^{\frac{3}{2}}\times\frac{2}{3}\times\frac{1}{2}$ $g=\frac{1}{3}(2x+9)^{\frac{3}{2}}$ $I=\frac{1}{3}x(2x+9)^{\frac{3}{2}}-\int\frac{1}{3}(2x+9)^{\frac{3}{2}}\times 1\,dx+c$ ✓ ✓ ✓ ✓ $I=\frac{1}{3}x(2x+9)^{\frac{3}{2}}-\frac{1}{3}(2x+9)^{\frac{5}{2}}\times\frac{2}{5}\times\frac{1}{2}+c$ ✓ ✓ ✓ $I=\frac{1}{3}x(2x+9)^{\frac{3}{2}}-\frac{1}{15}(2x+9)^{\frac{5}{2}}+c$ ✓ (8) OR But question does not say "or otherwise" $\int x\sqrt{2x+9}\,dx$ Let $2x+9=u$ ✓ Then $x=\frac{u-9}{2}$ ✓ Then $\frac{du}{dx}=2$ $\therefore dx=\frac{du}{2}$ ✓ $I=\int\frac{u-9}{2}\times u^{\frac{1}{2}}.\frac{du}{2}$ ✓ $I=\frac{1}{4}\int u^{\frac{1}{2}}(u-9).du$ ✓ $I=\frac{1}{4}\int u^{\frac{3}{2}}-9u^{\frac{1}{2}}.du$ ✓ $=\frac{1}{4}\left(u^{\frac{5}{2}}\times\frac{2}{5}-9u^{\frac{3}{2}}\times\frac{2}{3}\right)+c$ ✓ $=\frac{1}{10}(2x+9)^{\frac{5}{2}}-\frac{3}{2}(2x+9)^{\frac{3}{2}}+c$ ✓
9.2	$g(x)=4x+20$ and $h(x)=(x-3)(x+5)$. $4x+20-(x^2+2x-15)$ ✓ $=-x^2+2x+35$ ✓

$-x^2 + 2x + 35 = 0$ ✓
$x^2 - 2x - 35 = 0$
$x = 7 \ or \ x = -5$ ✓
$\int_{-5}^{7}(-x^2 + 2x + 35)dx$ ✓ ✓
$= 288 \, units^2$ ✓
(7)

9.3.

$$\int_1^k \frac{12x}{\sqrt{3x^2+1}} dx = 24$$

Let $u = 3x^2 + 1$ ✓
$\frac{du}{dx} = 6x$ ✓
$dx = \frac{du}{6x}$

$\int \frac{12x}{\sqrt{3x^2+1}} dx$

$= \int \frac{12x}{u^{\frac{1}{2}}} \frac{du}{6x}$

$\int 2u^{-\frac{1}{2}} du$ ✓
$= 2u^{\frac{1}{2}} \times 2 + c$ ✓
$= 4(3x^2+1)^{\frac{1}{2}} + c$ ✓(OK if c not present)

$\therefore [4(3x^2+1)^{\frac{1}{2}}]_1^k = 24$
$4(3k^2+1)^{\frac{1}{2}} - 4(3+1)^{\frac{1}{2}} = 24$
$4(3k^2+1)^{\frac{1}{2}} - 4(4)^{\frac{1}{2}} = 24$ ✓
$4(3k^2+1)^{\frac{1}{2}} - 8 = 24$ ✓
$4(3k^2+1)^{\frac{1}{2}} = 32$
$(3k^2+1)^{\frac{1}{2}} = 8$ ✓
$3k^2 + 1 = 64$ ✓
$3k^2 = 63$ ✓
$k^2 = 21$
$\therefore k = \sqrt{21}$ ✓

OR
Let $u = 3x^2 + 1$ ✓
$\frac{du}{dx} = 6x$ ✓
$dx = \frac{du}{6x}$
Boundaries
$u = 3(1)^2 + 1 = 4$
$u = 3k^2 + 1$ ✓
$\int_4^{3k^2+1} \frac{12x}{u^{\frac{1}{2}}} \times \left(\frac{du}{6x}\right) = 24$
$\int_4^{3k^2+1} \left(\frac{2du}{u^{\frac{1}{2}}}\right) = 24$
$\int_4^{3k^2+1} \left(2u^{-\frac{1}{2}} du\right) = 24$ ✓

$\therefore [2u^{\frac{1}{2}} \times 2]_4^{3k^2+1} = 24$ ✓
$\therefore [4u^{\frac{1}{2}}]_4^{3k^2+1} = 24$
$4(3k^2+1)^{\frac{1}{2}} - 4(4)^{\frac{1}{2}} = 24$ ✓
$4(3k^2+1)^{\frac{1}{2}} - 8 = 24$ ✓
$4(3k^2+1)^{\frac{1}{2}} = 32$
$(3k^2+1)^{\frac{1}{2}} = 8$ ✓
$3k^2 + 1 = 64$ ✓
$3k^2 = 63$ ✓
$k^2 = 21$
$\therefore k = \sqrt{21}$ ✓
(11)

8.4
Shape of **left** curve tending towards asymptotes ✓
Shape of **right** curve tending towards $x = -2$ ✓
Asymptote $x = -2$ ✓
y-cut: $y = -\frac{1}{4} = -0,25$ ✓
x-cut: $x = 1$ ✓
Exaggerate the TP $= \left(4; \frac{1}{12}\right)$ ✓
Curve on right tends towards the x-axis asymptote ✓
(7)