

WESTERFORD HIGH SCHOOL
Advanced Programme Mathematics
Grade 12 P1 Algebra and Calculus

August 2019
Examiner: K Court

Time: 2 hours
Marks: 200

INSTRUCTIONS

1. This paper consists of 9 questions on 4 pages.
 2. Answer all the questions and show necessary working clearly, guided by mark allocation.
 3. Calculators may be used unless stated otherwise.
 4. A **formula sheet** will be provided.
 5. Give answers as simplest fractions, if rational, in surd form if irrational, or to 2 or more decimal places, as specified by the question.
-

QUESTION 1

Prove by Mathematical Induction that $8^n - 7n + 6$ is divisible by 7 for all $n \in \mathbb{N}$.

[11]

QUESTION 2

Do NOT use a calculator in this question.

- 2.1. By letting $z = x + iy$, and if $|z - 1| = |z|$, show that $\operatorname{Re}(z) = \frac{1}{2}$. (6)
- 2.2. Express $(2 + i)^4$ in the form $a + bi$ with a and b real numbers. (5)
- 2.3. Solve for x if $x^2 - 4ix + 5 = 0$ (4)

[15]

QUESTION 3

- 3.1. Given $f(x) = e^{x-1}$, $x \in \mathbb{R}$ and $g(x) = \sqrt{x-1}$
 - 3.1.1. Determine the inverse function of f in the form $f^{-1}(x) = \dots$. (3)
 - 3.1.2. Evaluate $f(g(3))$ correct to 2 decimal places. (3)
 - 3.1.3. Give the domain and range of $f(g(x))$. (4)

3.2. Solve for x to 3 decimal places:

3.2.1. $\log_3 x + 2 \log_x 3 = 3$ (7)

3.2.2. $\ln(\cos x) = -2, \quad x \in (-2\pi; 0)$ (6)

[23]

QUESTION 4

4.1. Consider the equation: $\ln x = 4 - \frac{1}{2}x$

4.1.1. By sketching two suitable graphs, show that this equation has only one root. Be sure to show all intercepts with the axes, any asymptotes, and at least two points per graph. (6)

4.1.2. Using your knowledge of $\frac{d}{dx}(\ln x)$, use the Newton Raphson method with a suitable starting value, to find this root, correct to 5 decimal places. Show all the steps you have taken. (9)

4.2. On a separate set of axes, sketch the graph of $h(x) = |e^x - 1|$. You need to show the shape, any asymptotes, and all intercepts. Also, calculate and indicate on the graph, the point where the graph cuts the asymptote, accurate to 2 decimal places. (7)

[22]

QUESTION 5

Given the function $f(x) = \begin{cases} \frac{a}{x} & x \geq 1 \\ b - 2x & x < 1 \end{cases}$

Determine the value(s) of a and b if $f(x)$ is differentiable at $x = 1$.
Use the correct notation of limits to lay out your answer formally.

[8]

QUESTION 6

6.1. Find the derivative of $f(x) = \frac{1}{\sqrt{3x-2}}$ by first principles. (8)

6.2. Determine the following:

(give your answers with simplified coefficients and positive exponents).

$$6.2.1. \quad f'(x) \text{ if } f(x) = \frac{\sin^2 3x}{x^2} \quad (8)$$

$$6.2.2. \quad D_x[x\sqrt{x^2 - 1}] \quad (8)$$

$$6.2.3. \quad \frac{d}{dx}(5e^{3x}) \quad (2)$$

$$6.2.4. \quad \frac{d}{dx}(\log_4 x) \quad (2)$$

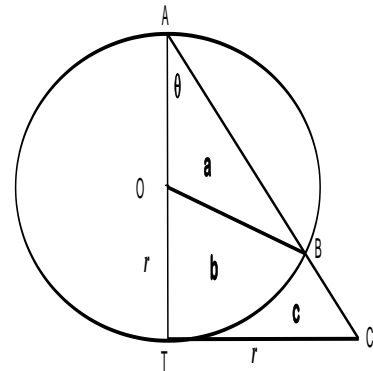
$$6.2.5. \quad \frac{dy}{dx} \text{ if } y = e^{x \cos x} \quad (6)$$

$$6.3. \quad \text{Use implicit differentiation to find } \frac{dy}{dx} \text{ if } xy^2 = 2y. \quad (7)$$

[41]

QUESTION 7

The diagram shows a diameter AT of a circle centre O; ABC is a secant and TC is a tangent to the circle. The length of TC is r , equal to the radius of the circle, and the angle at A is θ . The triangle ATC is divided into three sections with areas a , b , c as shown.



$$7.1. \quad \text{Without using a calculator, determine the value of } \sin 2\theta. \quad (5)$$

$$7.2. \quad \text{Hence, show that the area marked } c \text{ in terms of } r \text{ and } \theta, \text{ equals } \frac{1}{5}r^2(3 - 5\theta). \quad (9)$$

$$7.3. \quad \text{If } r = 4, \text{ use your calculator to find the value of area } c, \text{ correct to 2 d.p.} \quad (2)$$

[16]

QUESTION 8

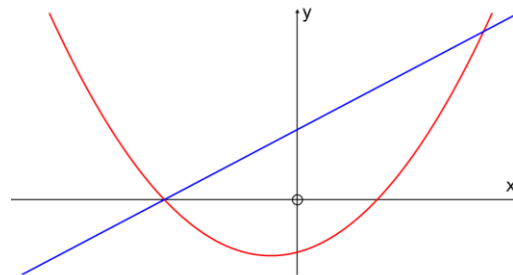
Given $f(x) = \frac{x-1}{(x+2)^2}$

- 8.1. Determine the co-ordinates of the intercepts with the axes. (2)
- 8.2. Using the function $f(x)$,
- 8.2.1. express it in partial fractions. (6)
- 8.2.2. hence, or otherwise, find the value(s) of x for which $f'(x) = 0$. (6)
- 8.2.3. determine any turning point(s). (2)
- 8.3. Determine
- 8.3.1. $\lim_{x \rightarrow \infty} f(x)$. (2)
- 8.3.2. the equation(s) of the function's asymptote(s). (2)
- 8.4. Draw a rough sketch of f , clearly showing all the features determined above. (7)
- (Recall that the sketch does not need to be drawn to scale.)

[27]**QUESTION 9**

- 9.1. Determine the following:
- 9.1.1. $\int \left(12x^3 - \frac{2}{x^2} \right) dx$ (5)
- 9.1.2. $\int \sin 3x \cos 2x \, dx$ (6)
- 9.1.3. $\int x\sqrt{2x+9} \, dx$ using integration by parts. (8)

- 9.2. The sketch alongside shows the functions $g(x) = 4x + 20$ and $h(x) = (x - 3)(x + 5)$. Determine the area bounded by the two functions.



(7)

- 9.3. Find the value(s) of k (with $k > 0$), in simplest surd form (if necessary) which satisfy the equation

$$\int_1^k \frac{12x}{\sqrt{3x^2 + 1}} dx = 24$$

(11)

[37]