Question 1

Prove $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots \frac{1}{n(n+1)} = \frac{n}{n+1}$

① Prove true for $n = 1$

$LHS = \frac{1}{1.2} = \frac{1}{2}$ $\qquad RHS = \frac{1}{1+1} = \frac{1}{2}$

$\therefore$ true for $n=1$

② Assume true for $n = k$

ie $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}$

③ Prove true for $n = k+1$

ie $\underbrace{\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots + \frac{1}{k(k+1)}} + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$

$$\begin{aligned}
LHS &= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k(k+2)+1}{(k+1)(k+2)} \\
&= \frac{k^2+2k+1}{(k+1)(k+2)} \\
&= \frac{\cancel{(k+1)}(k+1)}{\cancel{(k+1)}(k+2)} \\
&= \frac{k+1}{k+2} \\
&= RHS
\end{aligned}$$

$\boxed{14}$

④ Conclusion: By mathematic induction the statement is proved true for $n \in N$

Question 2

a) 1) If $x = 1-2i$ is a zero $\longrightarrow$ $x-1+2i$ is a factor

and $x = 1+2i$ will also be a zero $\longrightarrow$ $x-1-2i$ is a factor

$\therefore (x-1+2i)(x-1-2i)$ is a factor

$\therefore x^2 - \cancel{x} - \cancel{2i}x - \cancel{x} + 1 + \cancel{2i} + \cancel{2i}x - \cancel{2i} - 4i^2$ is a factor

$\therefore x^2 - 2x + 1 - 4(-1)$

$\therefore x^2 - 2x + 5$ is a factor

5

or done with sum + product of roots

2)

$$\begin{array}{r|l} & x^2 - 4x + 5 \\ \hline x^2-2x+5 & x^4 - 6x^3 + 18x^2 - 30x + 25 \\ & x^4 - 2x^3 + 5x^2 \\ \hline & -4x^3 + 13x^2 - 30x \\ & -4x^3 + 8x^2 - 20x \\ \hline & 5x^2 - 10x + 25 \\ & 5x^2 - 10x + 25 \\ \hline \end{array}$$

$\therefore (x^2-4x+5)(x^2-2x+5) = g(x)$

5

other zeros are $2+i$ and $2-i$

b) $(2+i)(x+3i) = 8i+6$

$(2+i)(a+bi+3i) = 8i+6$

$a + (b+3)i = \dfrac{8i+6}{2+i}$

$= 4+2i$

$\therefore a = 4$ $\qquad$ $b+3 = 2$

5 $\qquad$ 15

$b = -1$

Question 3

a) 1) $x = 3$

2) $x = -1$

3) $x = 1$

4) $x = -2$ or $x > 3$ 　 12

5) $x < -1$

b) 1) $x = 3$ 　 $\lim_{x \to 3} x^2 + 1 = 10$ $\therefore$ continuous

2) $\lim_{x \to -1^+} (x^2+1)$ 　 $\lim_{x \to -1^-} \frac{\cancel{(x+3)}(x+1)}{\cancel{(x+3)}}$

$= 2$

$= 0$

$\therefore$ limit doesn't exist

$\therefore$ jump discontinuity

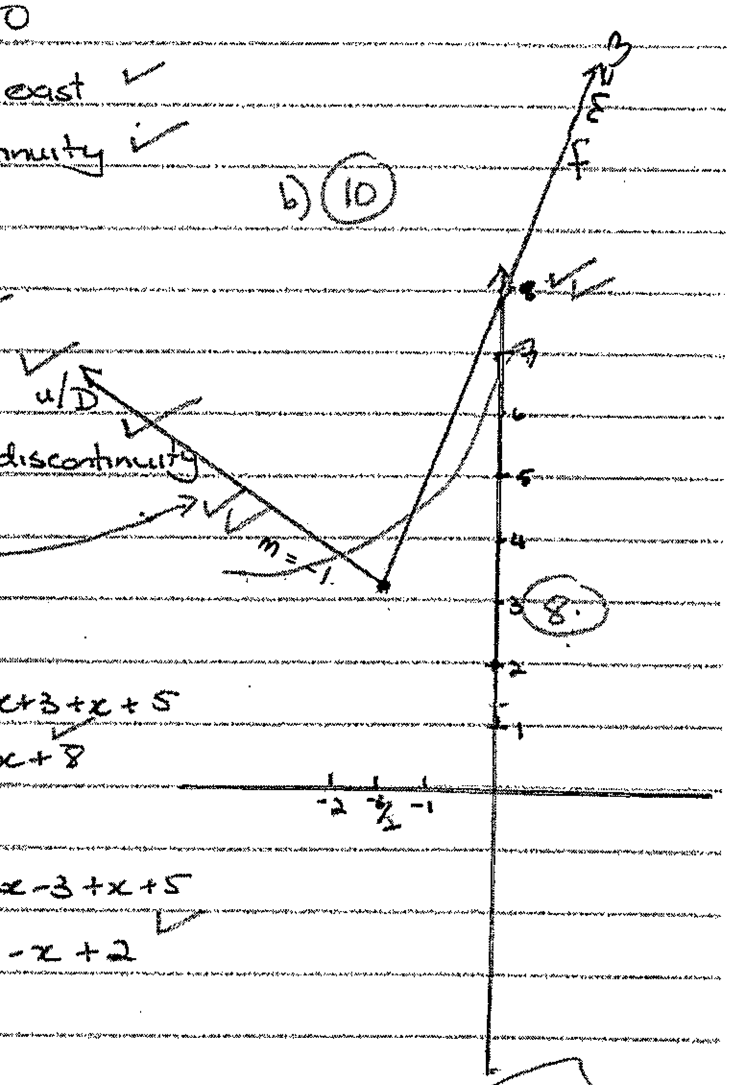
b) 10

3) $x = -3$ 　 $\lim_{x \to -3} \frac{\cancel{(x+3)}(x+1)}{\cancel{(x+3)}}$

$= -2$

but $g(-3)$ is u/D

$\therefore$ removable discontinuity



c) 1) $f(x) = |2x+3| + x + 5$

if $2x + 3 \geq 0$ then $y = 2x + 3 + x + 5$

$x \geq -3/2$ 　 $= 3x + 8$

8

if $2x + 3 < 0$ 　 $y = -2x - 3 + x + 5$

$x < -3/2$ 　 $= -x + 2$

32

2) ~~[illegible]~~ 　 2

$-1 < k < 3$

Question 4

a) $f(x) = e^{x+2} - 1$

i) f^{-1}: $x = e^{y+2} - 1$

$x + 1 = e^{y+2}$

$\ln|x+1| = y + 2$

$y = \ln|x+1| - 2$ ③

$\therefore f^{-1}(x) = \ln|x+1| - 2$

Cut on X-axis: $0 = e^{x+2} - 1$

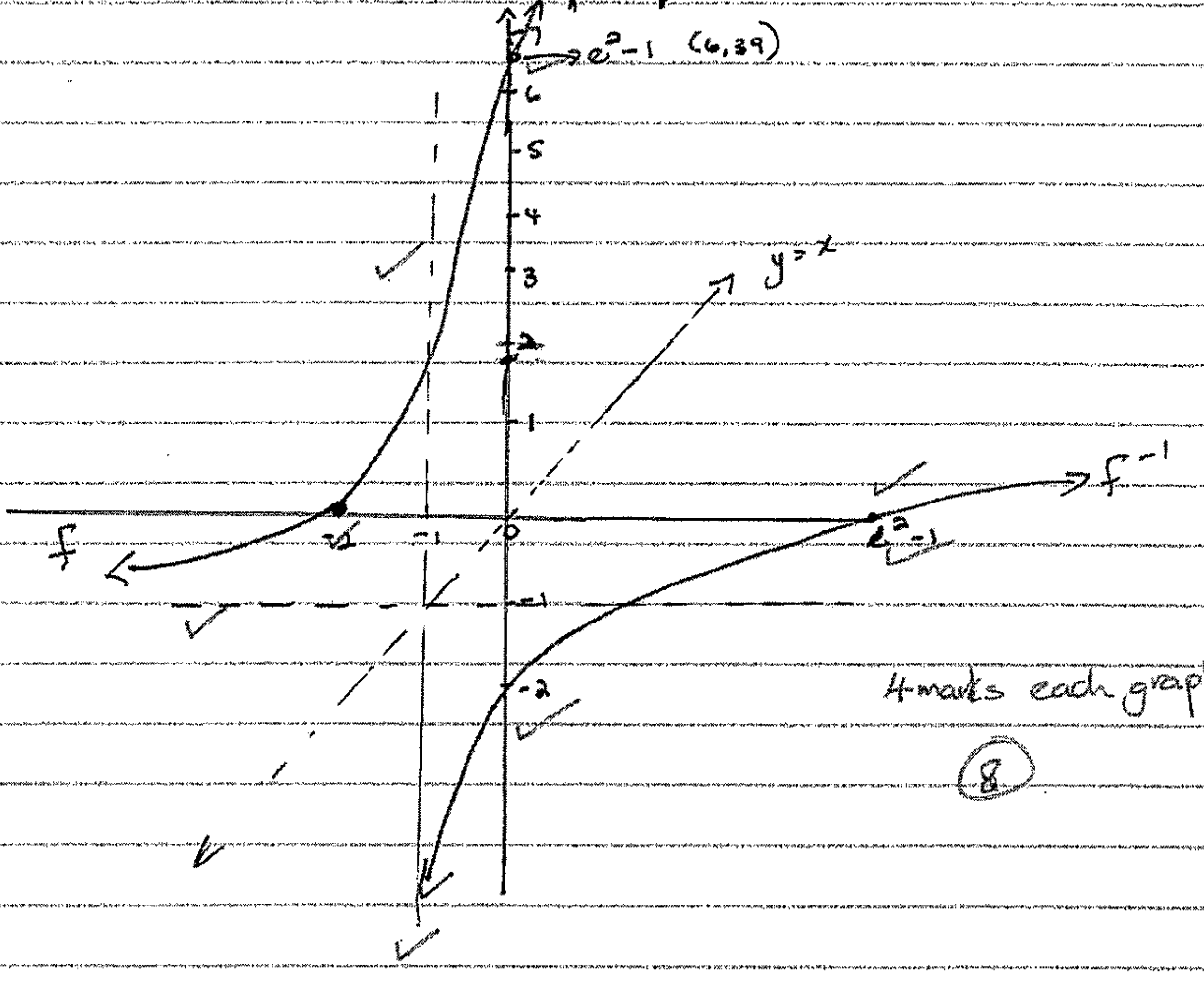
$1 = e^{x+2}$

$\ln 1 = x + 2$

$-2 = x$

Cut on Y-axis: $y = e^2 - 1$

2)



4 marks each graph ⑧

b) $10e^{2x} - 7e^{x} = 26$

Let $k = e^{x}$

$10k^2 - 7k - 26 = 0$

$(10k + 13)(k - 2) = 0$

$k = -13/10$ or $k = 2$ ⑦

$e^{x} = -13/10$ OR $e^{x} = 2$

N/A

$\therefore \ln 2 = x$

$\therefore a = 2$

18

Question 5

a) $\lim\limits_{x\to a} \frac{x^3+1}{x^3-x}$

$= \lim\limits_{x\to a} \frac{(x+1)(x^2-x+1)}{x(x^2-1)}$

$= \lim\limits_{x\to a} \frac{\cancel{(x+1)}(x^2-x+1)}{x(x-1)\cancel{(x+1)}}$

3

lim doesn't exist if $a=0$ or $a=1$

b) i) $f(x) = \sqrt{1-2x}$ $\qquad f(x+h) = \sqrt{1-2(x+h)}$

$= \sqrt{1-2x-2h}$

$f'(x) = \lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$= \lim\limits_{h\to 0} \frac{\sqrt{1-2x-2h}-\sqrt{1-2x}}{h} \times \frac{(\sqrt{1-2x-2h}+\sqrt{1-2x})}{(\sqrt{1-2x-2h}+\sqrt{1-2x})}$

$= \lim\limits_{h\to 0} \frac{1-2x-2h-(1-2x)}{h(\sqrt{1-2x-2h}+\sqrt{1-2x})}$

$= \lim\limits_{h\to 0} \frac{-2\cancel{h}}{\cancel{h}(\sqrt{1-2x-2h}+\sqrt{1-2x})}$

$= \frac{-\cancel{2}}{\cancel{2}\sqrt{1-2x}}$

6

$= -\frac{1}{\sqrt{1-2x}}$

a) $1-2x > 0$

$-2x > -1$

2

$\underline{x < \frac{1}{2}}$

c) 1) $f(x) = e^{3x+\sqrt[4]{x}}$

$f'(x) = e^{3x+\sqrt[4]{x}} \cdot \left(3 + \frac{1}{4}x^{-3/4}\right)$

$= e^{3x+\sqrt[4]{x}} \cdot \left(3 + \frac{1}{4x^{3/4}}\right)$

2) $g(x) = 2x \cdot \sec 3x$

$g'(x) = 2x \cdot \sec 3x \cdot \tan 3x \cdot 3 + 2 \cdot \sec 3x$

$= 6x \tan 3x \sec 3x + 2\sec 3x$

10

3) $h(x) = \sin^2(\tan(2x))$

$= 2\sin(\tan 2x) \cdot \cos(\tan 2x) \cdot \sec^2 2x \cdot 2$

$= 4\sin(\tan 2x) \cdot \cos(\tan 2x) \cdot \sec^2 2x$

d) 1) $x^2 - 4xy + 4y + 8 = 0$

$2x - \left[4x\frac{dy}{dx} + y \cdot 4\right] + 4\frac{dy}{dx} = 0$

$-4x\frac{dy}{dx} + 4\frac{dy}{dx} = 4y - 2x$

$\frac{dy}{dx}(-4x+4) = 4y - 2x$

$\frac{dy}{dx} = \frac{4y-2x}{-4x+4}$ 9

$= \frac{2y-x}{-2x+2}$

36

2) S.P $\longrightarrow$ $\frac{dy}{dx} = 0$

$0 = \frac{2y-x}{-2x+2}$

$\therefore 2y - x = 0$

$y = x/2$

$x^2 - 4x\left(\frac{x}{2}\right) + 4\left(\frac{x}{2}\right) + 8 = 0$

$x^2 - 2x^2 + 2x + 8 = 0$

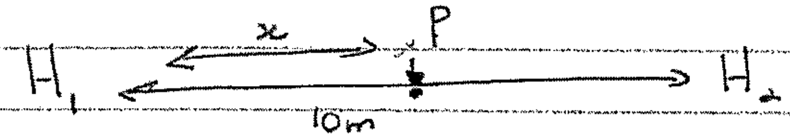
$x^2 - 2x - 8 = 0$ 6

$(x+2)(x-4) = 0$

$x = -2$ or $x = 4$

Question 6

a)



$$T = \frac{8}{x^2} + \frac{1}{(10-x)^2}$$

$$T' = 0 \qquad (\text{at min})$$

$$-16x^{-3} - 2(10-x)^{-3}\cdot(-1) = 0$$

$$\frac{-16}{x^3} + \frac{2}{(10-x)^3} = 0$$

$$-16(10-x)^3 + 2x^3 = 0$$

4

$$2x^3 - 16(10-x)^3 = 0$$

$$\underline{x^3 - 8(10-x)^3 = 0}$$

b)

$$x_{n+1} = x_n - \frac{x_n^3 - 8(10-x_n)^3}{3x_n^2 - 24(10-x_n)^2\cdot -1}$$

$$= x_n - \frac{x_n^3 - 8(10-x_n)^3}{3x_n^2 + 24(10-x_n)^2}$$

$x_0 = 5$

$x_1 = 6,296\ldots$

2

$x_2 = 6,646\ldots$

$x_3 = 6,66\ldots$

$x_4 = 6,\dot{6}$

c) P is $6\frac{2}{3}$m from H_1

1

10

Question 7

1. B removable discontinuity at $x = -2$
Str line $y = x - 2$. 3

2. E vertical asymptotes at $x = -2$ and $x = 3$
Cut on X-axis $\rightarrow$ $x = 2$ $f(x) = \frac{x-2}{(x-3)(x+2)}$
3

3. C vertical asymptote $x = 3$ $f(x) = \frac{(x+3)(x-2)}{(x-3)(x-3)}$
horizontal asymptote $y = 1$ $\rightarrow$ highest power numerator and denominator same
3

4. D. vertical asymptote $x = 3$ 3 $f(x) = \frac{(x-2)(x-2)}{x-3}$

12.

Question 8

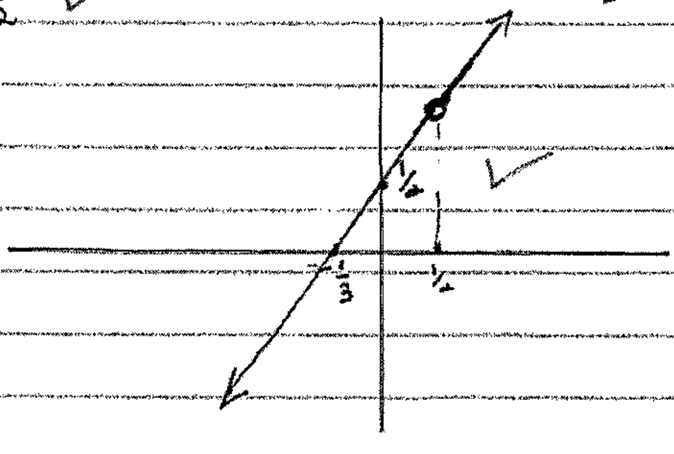
a)
$$\begin{array}{r} 3x - 2 \\ 2x+1 \overline{\smash{)}\, 6x^2 - x + 1} \\ \underline{6x^2 + 3x} \\ -4x - 1 \\ \underline{-4x - 2} \\ 1 \end{array}$$

$\therefore px - 2 = 3x - 2$
$\therefore \frac{p = 3}{1}$ 3

b) $y = \frac{6x^2 - x - 1}{4x - 2}$
$= \frac{(3x + 1)\cancel{(2x - 1)}}{2\cancel{(2x-1)}}$ rem discontinuity at $x = \frac{1}{2}$
$= \frac{3}{2}x + \frac{1}{2}$

4

Cut on Y axis: let $x = 0$
Cut on X axis: let $y = 0$
$0 = \frac{3}{2}x + \frac{1}{2}$
$0 = 3x + 1$
$-1 = 3x$
$x = -\frac{1}{3}$



c)

$$f(x) = \frac{6x^2 - x - 1}{3x - 2}$$

$$f'(x) = \frac{(3x-2)(12x-1) - (6x^2-x-1)(3)}{(3x-2)^2}$$

$$= \frac{36x^2 - 27x + 2 - 18x^2 + 3x + 3}{(3x-2)^2}$$

$$= \frac{18x^2 - 24x + 5}{(3x-2)^2}$$

$f'(x) = 0$ Stationary pts 5 $\boxed{12}$

$\therefore 18x^2 - 24x + 5 = 0$

$x = \frac{4+\sqrt{6}}{6}$ or $\frac{4-\sqrt{6}}{6}$ $\therefore$ two values so 2 S.P's.

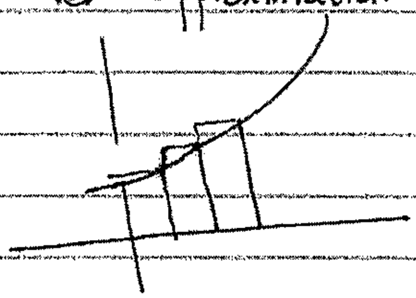
$= 1,0749$ $= 0,2584$

Question 9

a) Area $= \frac{10}{3} + \frac{3}{2(4)} + \frac{1}{6(4)^2} = \frac{119}{32}$. $= 3,72$ 2

b) Over approximation – the rectangles will be above the fn

2



c) $\lim_{n \to \infty} \frac{10}{3} + \frac{3}{2n} + \frac{1}{6n^2} = \frac{10}{3}$

$\therefore$ Exact Area $= \frac{10}{3}$ units. 2 $\boxed{6}$

Question 10

a) 1) $\int \left(2x^3 - \sec^2\left(\frac{x}{2}\right) - \frac{1}{\sqrt[3]{x}}\right) dx$

−1 if c left out

$= \int 2x^3 - \sec^2\left(\frac{x}{2}\right) - x^{-1/3} \, dx$

$= \frac{1}{2}x^4 - 2\tan\frac{x}{2} - \frac{3}{2}x^{2/3} + c$

3

3) $\int (\cos^2 3x)(\sin 3x) \, dx$

Let $u = \cos 3x$

$\frac{du}{dx} = -\sin 3x \cdot 3$

$du = -3\sin 3x \, dx$

$dx = -\frac{1}{3} \, du$

$= \int u^2 \cdot -\frac{1}{3} du$

$= -\frac{1}{3}\int u^2 du$

$= -\frac{1}{3}\left(\frac{1}{3}u^3\right) + c$

4

$= -\frac{1}{9}\cos^2 3x + c$

2) $\int \frac{e^x}{1+e^x} \, dx \longrightarrow \int \frac{du}{u}$

Let $u = 1 + e^x$

$du = e^x \, dx$

$= \ln|u| + c$

$= \ln|1+e^x| + c$

3

4) $\int \cos 4\theta \, \sin 5\theta \, d\theta$

$= \int \sin 5\theta \cos 4\theta \, d\theta$

$= \frac{1}{2}\int \sin 9\theta + \sin\theta \, d\theta$

5

$= \frac{1}{2}\left[-\frac{1}{9}\cos 9\theta - \cos\theta\right] + c$

$= -\frac{1}{18}\cos 9\theta - \frac{1}{2}\cos\theta + c$

5) $\int y\sqrt{y+3} \, dy = \frac{2}{3}y(y+3)^{3/2} - \int \frac{2}{3}(y+3)^{3/2} dy$

$f: y$	$g: \frac{2}{3}(y+3)^{3/2}$
$f': 1$	$g': (y+3)^{1/2}$

$= \frac{2}{3}y(y+3)^{3/2} - \frac{2}{5}\cdot\frac{2}{3}(y+3)^{5/2} + c$

$= \frac{2}{3}y(y+3)^{3/2} - \frac{4}{15}(y+3)^{5/2} + c$

5

b) i) $$\frac{3x+5}{(x+1)(x+2)(x+3)} = \frac{A}{(x+1)} + \frac{B}{(x+2)} + \frac{C}{x+3}$$

$$3x+5 = A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$$

let $x=-2$ $\quad -1 = B(-1)(1)$

$\therefore B = 1$

let $x=-3$ $\quad -4 = C(-2)(-1)$

$-4 = 2C$

$C = -2$

6

let $x = -1$ $\quad 2 = A(1)(2)$

$A = 1$

$$\therefore \frac{3x+5}{(x+1)(x+2)(x+3)} = \frac{1}{x+1} + \frac{1}{x+2} - \frac{2}{x+3}$$

2) $$\int_1^2 \frac{3x+5}{(x+1)(x+2)(x+3)}\,dx = \Big[\ln|x+1| + \ln|x+2| - 2\ln|x+3|\Big]_1^2$$

$= \ln|3| + \ln|4| - 2\ln|5| - [\ln|2| + \ln|3| - 2\ln|4|$

$= -2\ln|5| + 3\ln|4| - \ln 2$

$= 5\ln 2 - 2\ln 5$

$= \ln 32 - \ln 25$

$= \ln\frac{32}{25}$

4

30

Question 11

a) $\int_0^m \frac{x}{\sqrt{2x^2+1}}\,dx$ Consider $\int \frac{x}{\sqrt{2x^2+1}}\,dx$

Let $u = 2x^2+1$

$\frac{du}{dx} = 4x$

$du = 4x\,dx$

$\frac{1}{4}du = x\,dx$

$= \int \frac{\frac{1}{4}du}{\sqrt{u}}$

$= \frac{1}{4}\int u^{-\frac{1}{2}}\,du$

$= \frac{1}{4} \cdot 2u^{\frac{1}{2}} + c$

$= \frac{1}{2}\sqrt{u} + c$

$= \frac{1}{2}\sqrt{2x^2+1} + c$

$\int_0^m \frac{x}{\sqrt{2x^2+1}} = \left[\frac{1}{2}\sqrt{2x^2+1}\right]_0^m$

$= \frac{1}{2}\sqrt{2m^2+1} - \frac{1}{2}\sqrt{1}$

$= \frac{1}{2}\sqrt{2m^2+1} - \frac{1}{2}$

9

b) If $m=2$

Area $= \frac{1}{2}\sqrt{2(2)^2+1} - \frac{1}{2}$

$= \frac{1}{2}\sqrt{9} - \frac{1}{2}$

$= \frac{3}{2} - \frac{1}{2}$

$= 1$

2

c) $V = \pi \int_0^m \left(\frac{x}{\sqrt{2x^2+1}}\right)^2 dx$

4

15