MEMORANDUM: PRELIM 2020 CORE

QUESTION 1

1.1 (a) $|x^2 - 7| = 2x + 1$

$x^2 - 7 = 2x + 1$ ✓ or $x^2 - 7 = -2x - 1$ ✓

$x^2 - 2x - 8 = 0$ ✓ $x^2 + 2x - 6 = 0$ ✓

$(x - 4)(x + 2) = 0$ $x = \frac{-2 \pm \sqrt{4 - 4(-6)}}{2}$ ✓

$x = 4$ ✓or $x = -2$ n/a ✓ $x = -1 + \sqrt{7}$ ✓ (8)

(b) $e^x - 6e^{-x} - 5 = 0$

Let $e^x = k$ ✓

$k - \frac{6}{k} - 5 = 0$ ✓

$k^2 - 5k - 6 = 0$ ✓

$(k - 6)(k + 1) = 0$ ✓

$e^x = 6$ ✓

$x = \ln 6$ ✓ (6)

1.2 (a) $y = e^{x+3} - 2$

$x = e^{y+3} - 2$ ✓

$x + 2 = e^{y+3}$ ✓

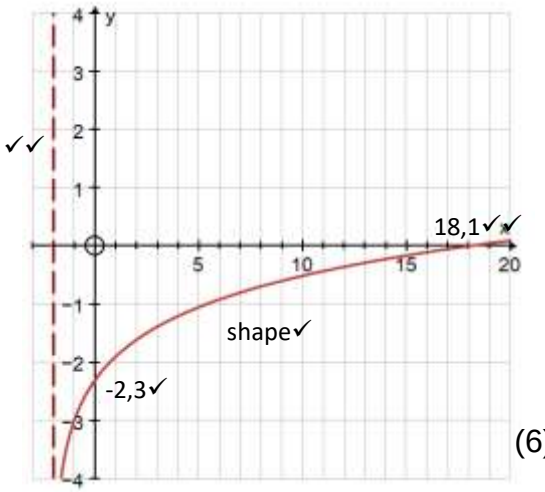
$y + 3 = \ln(x + 2)$ ✓

$f^{-1}(x) = \ln(x + 2) - 3$ ✓✓

Domain: $x \in (-2; \infty)$ ✓

Range: $y \in R$ ✓ (7)

(b)



(6)

1.3 $x = 2 - i$ ✓

$x^2 - (2 + i + 2 - i)x + (2 + i)(2 - i)$ is factor ✓

$x^3 + px + q = (x^2 - 4x + 5)(x + 4$ ✓$)$

$= x^3 - 11x + 20$ ✓ $\therefore p = -11$ ✓$q = 20$ ✓ (6) **[33]**

QUESTION 2

$\ln x + 2\ln x + 3\ln x + \cdots + n\ln x = \frac{n}{2}\ln x^{n+1}$ ✓

Let $n = 1$

LHS = $\ln x$ RHS = $\frac{1}{2}\ln x^{1+1} = \ln x$✓

Formula true for $n = 1$✓

Assume formula true for $n = k$✓

$\ln x + 2\ln x + 3\ln x + \cdots + k\ln x = \frac{k}{2}\ln x^{k+1}$✓

Prove true for $n = k + 1$. Proposed formula: $\frac{k+1}{2}\ln x^{k+2}$ ✓ $= \frac{(k+1)(k+2)}{2}\ln x$ ✓

$\ln x + 2\ln x + 3\ln x + \cdots + k\ln x + (k+1)\ln x$ ✓ $= \frac{k}{2}\ln x^{k+1} + (k+1)\ln x$ ✓

$= \frac{k(k+1)}{2}\ln x + (k+1)\ln x$ ✓

$= \left(\frac{k(k+1)}{2} + k + 1\right)\ln x$

$= \frac{k^2+k+2k+2}{2}\ln x$ ✓

$= \frac{k^2+3k+2}{2}\ln x$

$= \frac{(k+1)(k+2)}{2}\ln x$ ✓

✓

This is the proposed formula so by M.I we have proven the formula true for all $n \in N$

[13]

QUESTION 3

3.1 $g(-5) = -|-5+2| + 1 = -2 \quad (-5;-2)$ ✓

$g(0) = -|0+2| + 1 = -1 \quad (0;-1)$ ✓

Subst. $(-5;-2)$

$-2 = |-5+p| + q$ ✓ $\therefore -2 = 5 - p + q \quad LH\ arm$

$-7 = -p + q$ ✓

Subst. $(0;-1)$

$-1 = |0+p| + q$ ✓$\therefore -1 = p + q$ ✓ $RH\ arm$

$p = 3$✓$; q = -4$✓ (8)

3.2 $|x+2| + |x+3| \leq 5$

$|x+3| - 4 \leq -|x+2| + 1$✓✓

Points of intersection:

$x = -5; 0$✓✓

$x \in [-5; 0]$✓✓ (6)

QUESTION 4

$Area\ sector\ OAB = \frac{1}{2} \times 12^2 \times \frac{2\pi}{3} = 48\pi$✓✓

$Area\ silt = \frac{1}{2} \times 12^2 \times \frac{\pi}{3} - \frac{1}{2} \times 12^2 \times \sin\frac{\pi}{3} = 13{,}04439\ \ldots$✓✓

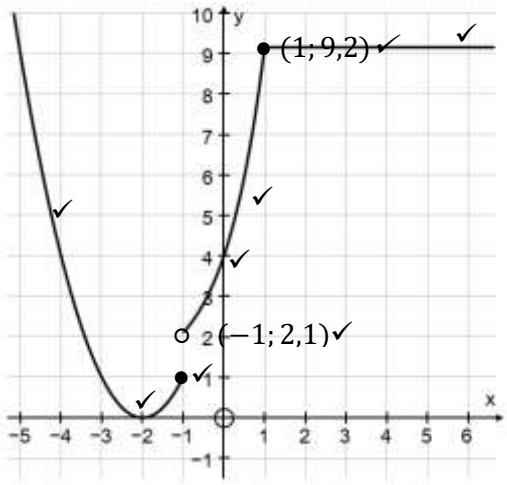
$Area\ \Delta AOB = \frac{1}{2} \times 12^2 \times \sin\frac{2\pi}{3} = 36\sqrt{3}$✓✓

$Area\ water = 48\pi - 36\sqrt{3} - 13{,}04439 = 24\pi$✓✓

$Volume = 24\pi \times 200$✓$= 4800\pi\ ml \approx 15\ litres$✓ **[10]**

QUESTION 5

5.1



(8)

5.2 Discontinuous at $x = -1$ ✓

$\lim_{x\to -1^-} f(x) = 1$ ✓✓

$\lim_{x\to -1^+} f(x) = 2{,}1$ ✓✓

Jump discontinuity.✓ (6)

5.3 $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1) = 9{,}1$ ✓✓ Therefore f is continuous at $x = 1$

$\lim_{x\to 1^-} f'(x) = 3e$ ✓✓

$\lim_{x\to 1^+} f'(x) = 0$ ✓

f is not differentiable at $x = 1$ ✓ (6) **[20]**

QUESTION 6

6.1 $f(x) = \frac{1}{\sqrt{2-x}}$

$f'(x) = \lim_{h\to 0} \frac{\frac{1}{\sqrt{2-(x+h)}} - \frac{1}{\sqrt{2-x}}}{h}$ ✓

$= \lim_{h\to 0} \frac{\sqrt{2-x}-\sqrt{2-x-h}}{h\sqrt{2-x-h}.\sqrt{2-x}}$ ✓✓ $\times \frac{\sqrt{2-x}+\sqrt{2-x-h}}{\sqrt{2-x}+\sqrt{2-x-h}}$ ✓

$= \lim_{h\to 0} \frac{2-x-2+x+h}{h\sqrt{2-x-h}.\sqrt{2-x}\left(\sqrt{2-x}+\sqrt{2-x-h}\right)}$ ✓✓

$= \lim_{h\to 0} \frac{1}{\sqrt{2-x-h}.\sqrt{2-x}\left(\sqrt{2-x}+\sqrt{2-x-h}\right)}$ ✓✓

$= \frac{1}{2\sqrt{2-x}(2-x)}$ ✓✓ (10)

6.2 (a) $f(x) = \sin(\tan(2x))$

$f'(x) = \cos(\tan(2x))$ ✓✓ $\times\ sec^2(2x)$ ✓✓ $\times\ 2$ ✓

$g(x) = x^{\frac{2}{3}}\left(x + \sqrt[4]{x}\right) = x^{\frac{5}{3}} + x^{\frac{11}{12}}$ ✓

$g'(x) = \frac{5}{3}x^{\frac{2}{3}}$ ✓ $+ \frac{11}{12}x^{-\frac{1}{12}}$ ✓

$h(x) = \frac{2x}{\cos x}$

$h'(x) = \frac{2\cos x + 2x\sin x}{(\cos x)^2}$ ✓✓ ✓

(b) $f'(\pi) = \cos(\tan(2\pi)) \times sec^2(2\pi) \times 2 = 2$ ✓

$g'(1) = \frac{5}{3}(1)^{\frac{2}{3}} + \frac{11}{12}(1)^{-\frac{1}{12}} = 2\frac{7}{12}$ ✓

$h'(\pi) = \frac{2\cos\pi + 2x\sin\pi}{(\cos\pi)^2} = -2$✓

$g'(1) > f'(\pi) > h'(\pi)$ ✓ (4)

6.3 (a) $f(x) = 6\ln x - (-x^2 + 8x - 3) = 0$

$f(5) = 6\ln 5 + 5^2 - 8(5) + 3 = -2{,}3$ ...✓

$f(6) = 6\ln 6 + 6^2 - 8(6) + 3 = 1{,}7$ ... ✓

Function is continuous in interval and therefore a point of intersection in interval✓✓ (4)

(b) $f'(x) = \frac{6}{x} + 2x - 8$✓✓

$a_{n+1} = a_n - \frac{6\ln a_n + {a_n}^2 - 8a_n + 3}{\frac{6}{a_n} + 2a_n - 8}$ ✓✓

$a_1 = 5{,}5$

$a_2 = 5{,}9833588$ ✓✓

$a_3 = 5{,}6477908$ ...

$x \approx 5{,}6240846$✓ (7) **[37]**

QUESTION 7

7.1 $f(x) = \frac{x^3 + ax^2 + bx + c}{x^2 - 1}$

$x = 1$✓; $x = -1$✓ (2)

7.2 $x^3 + ax^2 + bx + c = (x^2 - 1)(x + 2)$✓ $+ 2x + 1$✓

$= x^3 + 2x^2 - x - 2 + 2x + 1$✓

$= x^3 + 2x^2 + x - 1$✓

$a = 2$✓; $b = 1$✓; $c = -1$✓ (7)

7.3 $f(x) = \frac{x^3 + 2x^2 + x - 1}{x^2 - 1}$

$f'(x) = \frac{(3x^2 + 4x + 1)$✓$(x^2 - 1)$✓$ - 2x$✓$(x^3 + 2x^2 + x - 1)$✓$}{(x^2 - 1)^2} = 0$✓

$3x^4 + 4x^3 + x^2 - 3x^2 - 4x - 1 - 2x^4 - 4x^3 - 2x^2 + 2x = 0$✓

$x^4 - 4x^2 - 2x - 1 = 0$ ✓

$x \approx -1{,}79$ ✓ (8)

7.4

x	$x < -1{,}79$	$x = -1{,}79$	$x > -1{,}79$
$f'(x)$	+✓	0✓	-✓

Local maximum turning point.✓ (4) **[21]**

QUESTION 8

8.1 (a) $y = \frac{3x+5}{x^3+5x^2+7x+3}$

$y = x^3 + 5x^2 + 7x + 3 = (x+1)(x^2+4x+3)$ ✓

$= (x+1)(x+1)(x+3)$ ✓

$\frac{3x+5}{(x+1)^2(x+3)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+3}$ ✓✓

$B = \frac{3(-1)+5}{-1+3} = 1$ ✓✓

$C = \frac{3(-3)+5}{(-3+1)^2} = -1$ ✓✓

$\frac{3x+5}{(x+1)^2(x+3)} = \frac{A}{x+1} + \frac{1}{(x+1)^2} - \frac{1}{x+3}$ ✓

$3x + 5 = A(x+1)(x+3) + (x+3) - (x+1)^2$ ✓

$3x + 5 = Ax^2 + 4Ax + 3A + x + 3 - x^2 - 2x - 1$

$A - 1 = 0 \ \therefore A = 1$ ✓✓

$\frac{3x+5}{(x+1)^2(x+3)} = \frac{1}{x+1} + \frac{1}{(x+1)^2} - \frac{1}{x+3}$ ✓ (13)

(b) $\int \left(\frac{1}{x+1} + \frac{1}{(x+1)^2} - \frac{1}{x+3}\right) dx$ ✓

$= \int \left(\frac{1}{x+1} + (x+1)^{-2} - \frac{1}{x+3}\right) dx$ ✓

$= \ln|x+1|$ ✓$- (x+1)^{-1}$ ✓$- \ln|x+3|$ ✓$+ c$ ✓ (6)

8.2 (a) $\int 4x(x^2+2)^5 dx$

$= 2\checkmark \int 2x\checkmark (x^2+2)^5 dx$

$= 2 \times \frac{1}{6}(x^2+2)^6 + c\checkmark\checkmark$

$= \frac{1}{3}(x^2+2)^6 + c\checkmark\checkmark$ (6)

(b) $\int \frac{1}{1+\cot^2 x} dx$

$= \int \frac{1}{cosec^2 x} dx\checkmark\checkmark$

$= \int sin^2 x\, dx\checkmark\checkmark$

$= \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)\checkmark\checkmark dx$

$= \frac{x}{2} - \frac{1}{4}\sin 2x + c\ \checkmark\checkmark$ (8) **[33]**

QUESTION 9

9.1 $\int x\, sec^2 x\, dx$

Let $f(x) = x\ \checkmark$ $g'(x) = sec^2 x\checkmark$

$f'(x) = 1\ \checkmark$ $g(x) = \tan x\checkmark$

$\int x\, sec^2 x\, dx = x\tan x\checkmark - \int \tan x\, dx\ \checkmark$

$= x\tan x - \int \frac{\sin x}{\cos x} dx\ \checkmark$

$= x\tan x + \checkmark\int \frac{-\sin x\checkmark}{\cos x} dx$

$= x\tan x + \ln|\cos x| + c\checkmark$ (10)

9.2 $V = \pi \int_0^k \left(5\sqrt{x}\sec x\right)^2 dx\checkmark\checkmark$

$73{,}97\checkmark = 25\pi \int_0^k x\, sec^2 x\, dx$

$73{,}97 = 25\pi[x\tan x + \ln|\cos x|]_0^k\checkmark\checkmark$

$73{,}97 = 25\pi[(k\tan k + \ln|\cos k|\ \checkmark) - 0 - \ln\cos 0\checkmark]$

$k = 1\checkmark\checkmark$ (9) **[19]**