Brescia House School

SERVIAM

Gr 12

AUGUST 2021

ADVANCED PROGRAMME MATHEMATICS: PAPER I

MODULE 1: CALCULUS AND ALGEBRA

Time: 2 hours MEMO + Comments. 200 marks

PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY

1. This question paper consists of 20 pages and an Information Booklet of 2 pages (i–ii). Please check that your question paper is complete.
2. Non-programmable and non-graphical calculators may be used, unless otherwise indicated.
3. All necessary calculations must be clearly shown, and writing must be legible.
4. Diagrams have not been drawn to scale.
5. Round off your answers to 2 decimal digits, unless otherwise indicated.

EXAMINATION NUMBER:

Question Number	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**Total**
Mark Achievable	**43**	**24**	**7**	**11**	**28**	**10**	**10**	**28**	**39**	**200**
Mark Attained										

Examiner : Mrs K Raeburn

Moderator : Mr B Dannatt **External Moderator : Mrs C Kennedy**

QUESTION ONE

1.1 Solve for $x \in \mathbb{R}$ in each of the following. Leave answers in terms of ln or e if necessary.

(a) $50 = e^{50x} - 1$ (4)

$51 = e^{50x}$

$\therefore \ln 51 = 50x$.

$\therefore x = \frac{\ln 51}{50}$

generally well answered.

(b) $2e^{x} - 1 = e^{-x}$ (6)

$k = e^{x}$

$2k - 1 = \frac{1}{k}$

$2k^2 - k - 1 = 0$

$(2k + 1)(k - 1) = 0$

$e^{x} = -1/2$ $e^{x} = 1$.

n/s $x = 0$.

k-method.

NB $x \in \mathbb{R}$

$y > 0$

$\therefore e^{x} > 0$.

(c) $x(3|x|-1)=-10$ (8)

$x \geq 0 \quad x(3x-1)=-10$

$3x^2-x+10=0$

$x=\dfrac{1 \pm \sqrt{119}\,i}{6}$

n/s.

$x<0 \quad x(-3x-1)=-10.$

$3x^2+x-10=0.$

$x=5/3$ or -2

$\therefore x=-2.$

must stipulate $x \geq 0$ or $x<0$

& $x \in \mathbb{R}$

$\therefore$ Need conclusion.

1.2 Given: $f(x)=e^{2x}-9$ and $g(x)=\ln(x-1)$ $for\ x>1$

Solve for x if $f(g(x))=0$ (9)

$f(\ln(x-1))=0.$

$e^{2\ln(x-1)}-9=0$

$e^{\ln(x-1)^2}=9.$

$\ln(x-1)^2=\ln 9.$

$(x-1)^2=3^2$

$x-1=\pm 3$

$x=4$ or -2 $\quad x>1$

$\therefore x=4$

Testing e + ln laws with equations.

1.3 The temperature T (in °C) of a cooling cup of tea, after a time t (in minutes), can be modelled by the equation:

$T = 20 + Ae^{-kt}$, where A and k are constants.

(a) Write down the room temperature. (2)

20° C.

asymptote.

(b) Given that the initial temperature is 85 °C and that the temperature is decreasing at the rate of 5 °C per minute, initially, determine the value of k. (9)

$t=0$: $85 = 20 + Ae^{0}$

$\therefore A = 65.$

$\frac{dT}{dt} = -Ak e^{-kt}$

$-5 = -65 \cdot e^{0} \cdot k.$

$k = \frac{1}{13}.$

initial: $t=0$

Rate of change

$\frac{dT}{dt}$

(c) Determine the length of time, to the nearest minute it takes for the tea to cool to 50 °C. (5)

$50 = 20 + 65 \cdot e^{-\frac{1}{13}t}.$

Final $T=50$.

$\frac{30}{65} = e^{-t/13}$

$e^{t/13} = 13/6.$

$\frac{t}{13} = \ln\left(13/6\right)$

$t = 10{,}051.$

$= 10$ minutes.

[43]

QUESTION TWO

2.1 It is given that $f(g(x)) = \frac{1}{x-1} + x^2 - 2x + 1$ and $g(g(x)) = x - 2$.

Determine $g(f(2))$. (6)

$g(x) = x - 1.$

$\therefore f(x) = \frac{1}{x} + x^2.$

$\therefore f(2) = \frac{1}{2} + 2^2 = \frac{9}{2}$

$\therefore g\left(\frac{9}{2}\right) = \frac{9}{2} - 1$

$= \frac{7}{2}$

Needeed to determine g(x) + f(x) first.

2.2 The following function is given:

$$p(x) = \begin{cases} 2x + 1 & if \quad x \leq q \\ x^2 - 4x + 10 & if \quad x > q \end{cases}$$

(a) For what value(s) of q is $p(x)$ continuous at $x = q$? (6)

Use defn of continuity.

For continuity:

$p(q) = \lim_{x \to q^-} (2x+1) = \lim_{x \to q^+} (x^2 - 4x + 10)$

$2q + 1 = q^2 - 4q + 10$

$q^2 - 6q + 9 = 0$

$q = 3.$

(b) Is p differentiable at all points? Motivate your answer. (6)

$$p'(x) = \begin{cases} 2 & x \leq q \\ 2x - 4 & x > q. \end{cases}$$

use defn of differentiability

For differentiability:

$$2 = 2q - 4$$
$$6 = 2q$$
$$q = 3.$$

$\therefore$ As long as $q = 3$, yes.

2.3 Given : $f(x) = \frac{\sqrt{2x-1} - \sqrt{x}}{x-1}$

Rationalise.

Determine : $\lim_{x \to 1} f(x)$ (6)

$$\lim_{x \to 1} \frac{\sqrt{2x-1} - \sqrt{x}}{x-1} \times \frac{\sqrt{2x-1} + \sqrt{x}}{\sqrt{2x-1} + \sqrt{x}}$$

$$= \lim_{x \to 1} \frac{2x - 1 - x}{(x-1)(\sqrt{2x-1} + \sqrt{x})}$$

$$= \lim_{x \to 1} \frac{\cancel{(x-1)}}{\cancel{(x-1)}(\sqrt{2x-1} + \sqrt{x})}$$

$$= \frac{1}{2}.$$

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QUESTION THREE

In using the induction method to prove the accuracy of a statement, you assume the statement is true for $n = k$.

Assuming $2^k + 2^{k+1} + 2^{k+2}$ is divisible by 7, prove that it will be true for the next natural number $n = k + 1$.

i.e. $2^{k+1} + 2^{k+2} + 2^{k+3}$ needs to be div by 7.

$$= 2\underbrace{\left(2^k + 2^{k+1} + 2^{k+2}\right)}_{\text{div by } 7}.$$

$\therefore$ $2 \times$ ~ is div by 7.

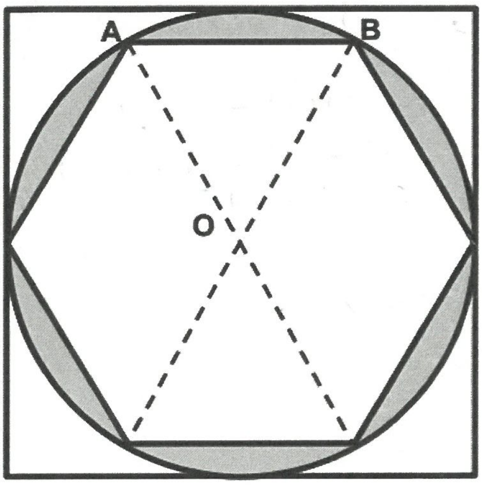
$\therefore$ True for $n = k+1$.

[7]

Only the last part of an induction question.

QUESTION FOUR

In the diagram a square tile is shown, in which a regular hexagon is inscribed in a circle with centre O. The circle fits exactly inside the square. The area of the shaded region is 54 cm^2.



Work in radians. but actually a core question.

4.1 Write down the size of $A\hat{O}B$ in radians, leaving the answer in terms of π if necessary. (2)

$\frac{2\pi}{6} = \frac{\pi}{3}$ radians.

4.2 Determine the area of the square tile, correct to 2 decimal places. (9)

$A_{shaded} = A_{\odot} - A_{hex}$

$54 = \pi r^2 - \cancel{6}^{3}\left(\frac{r^2 \sin \pi/3}{\cancel{2}}\right)$

$54 = \pi r^2 - 3r^2 \cdot \frac{\sqrt{3}}{2}$

$\therefore r^2 = \frac{54}{\left(\pi - \frac{3\sqrt{3}}{2}\right)}$

Area square $= (2r)^2 = 4r^2$

$= 397,412 \ldots$

$= 397,41 \text{ cm}^2$

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QUESTION FIVE

generally ok.

5.1 Determine the derivatives of the following

(a) $y = (3x^4 - 10x)^{15}.\sqrt{4x^4 + 64}$ (No need to simplify) (5)

$f(x) = (3x^4-10x)^{15}$ $\quad g(x) = (4x^4+64)^{1/2}$

$u = 3x^4 - 10x \quad y = u^{15}$ $\quad u = 4x^4 + 64 \quad y = u^{1/2}$

$f'(x) = (12x^3-10).15u^{14}$ $\quad g'(x) = 16x^3.\frac{1}{2}u^{-1/2}$

$= 15(12x^3-10)(3x^4-10x)^{14}$ $\quad g'(x) = 8x^3(4x^4+64)^{-1/2}$.

$\therefore dy/dx = (3x^4-10x)^{15}.8x^3(4x^4+64)^{-1/2} + \sqrt{4x^4+64}.15(12x^3-10)(3x^4-10x)^{14}$

(b) $f(x) = \sin^2(3x - 4)$ (simplify to one trigonometric ratio) (5)

Two composite functions

$u = (3x-4) \quad \therefore \quad y = \sin^2 u$.

$a = \sin u \quad \therefore y = a^2$.

$\therefore f'(x) = du/dx \cdot dy/du$

$= du/dx \cdot dy/da \cdot da/du$.

$= 3.\ 2a.\cos u$

$= 3.\ 2\sin u\cos u = 3\sin 2u$

$= 3\sin(6x-8)$

NB

$\tan A^2 \neq (\tan A)^2$

(c) $y = \tan(3x - 4)^2$ (5)

$u = (3x-4)^2 \quad \therefore \quad y = \tan u$.

$a = 3x-4 \quad \therefore u = a^2$

$dy/dx = du/dx \cdot dy/du$

$= du/da \cdot da/dx \cdot dy/du$

$= 2a.\ 3.\ \sec^2 u$

$= (6x-8)(3)\sec^2(3x-4)^2$

$= (18x-24)\sec^2(3x-4)^2$.

5.2 Determine the gradient of the curve $x^2 - 3x^2y + y^3 = -1$ at (2;3) (8)

$$2x - 3\left(2xy + x^2\, dy/dx\right) + 3y^2\, dy/dx = 0.$$

$$2x - 6xy - 3x^2\, dy/dx + 3y^2\, dy/dx = 0$$

$$dy/dx = \frac{6xy - 2x}{3y^2 - 3x^2}$$

$$\therefore m_T = \frac{6(2)(3) - 2(2)}{3(3)^2 - 3(2)^2}$$

$$= \frac{32}{15}.$$

implicit

5.3 State the equation of the tangent to $f(x) = e^{2x} - 3x$ at the y intercept. (5)

$f(0) = e^0 - 0 = 1$ ie. $(0;1)$.

$f'(x) = 2e^{2x} - 3$.

$\therefore m_T = f'(0) = 2 - 3 = -1$.

$y = -x + 1$.

actually quite easy.

[28]

QUESTION SIX

Newton's — revise this.

Given : $y = x + 1$ and $y = 2tanx$

6.1 Create a function $f(x)$ that can be used to determine the x value of the point of intersection of the two graphs. (2)

$x+1 = 2\tan x$

$\therefore x+1-2\tan x = 0$. or

$f(x) = x+1-2\tan x$

$f(x) = 2\tan x - x - 1$.

6.2 Show that $f(x) = 0$ has a solution in the interval [0 ; 1]. (3)

$f(0) = 1 > 0$

$f(1) = 2 - 2\tan 1 = -1{,}11\ldots < 0$.

$\therefore$ solution exists.

6.3 Calculate this x value using Newton's interpolation method. Round off your answer to 4 decimal places. (5)

$f'(x) = 1 - 2\sec^2 x$ OR $f'(x) = 2\sec^2 x - 1$.

$\therefore x_{r+1} = x_r - \dfrac{x_r + 1 - 2\tan x_r}{1 - 2/\cos^2 x_r}$ or $x_r - \dfrac{2\tan x_r - x_r - 1}{2/\cos^2 x_r - 1}$.

let $x_1 = 1/2$.

$x_2 = 0{,}7551173222$

$x_3 = 0{,}709194777\ldots$

$x_4 = 0{,}70633795\ldots$

$x_5 = 0{,}706328\ldots$

$x_6 = 0{,}706328\ldots$

ie $x = 0{,}7063$.

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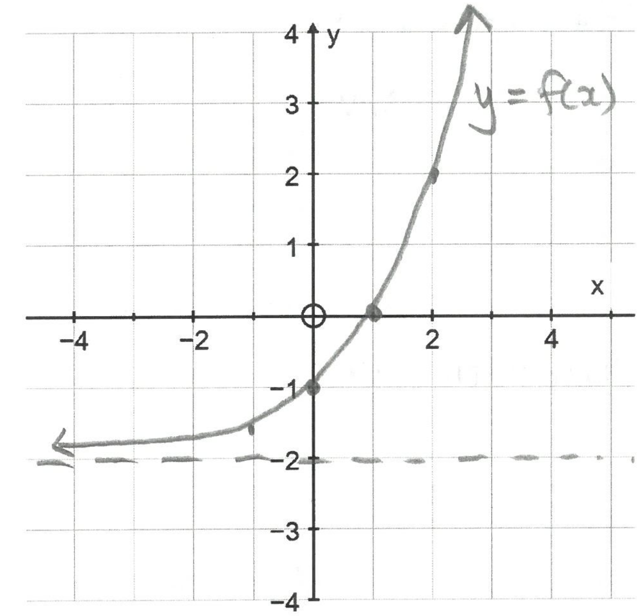
QUESTION SEVEN

Given: $f(x) = 2^x - 2$

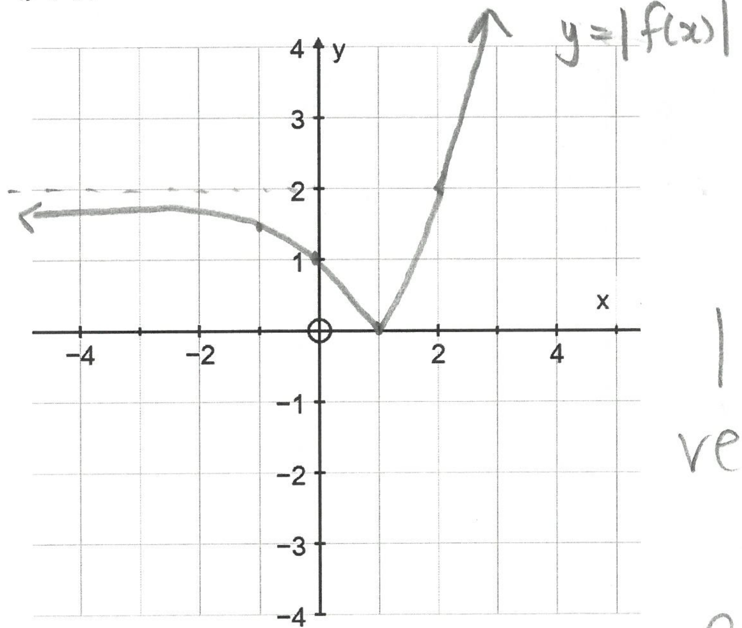
Sketch the following curves on the axes provided. Label your intercepts and asymptotes.

7.1 $y = f(x)$ (2)



This is in fact a Core question which just asks | |.

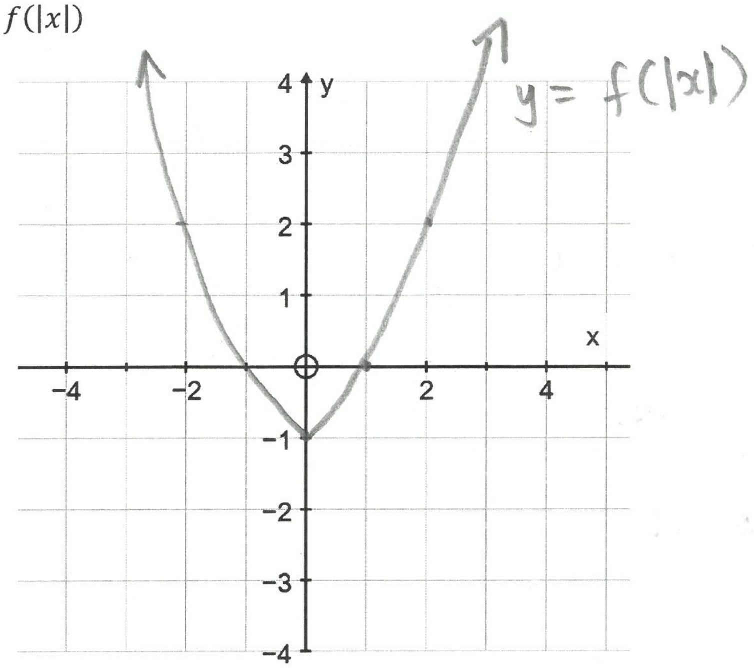
7.2 $y = |f(x)|$ (2)



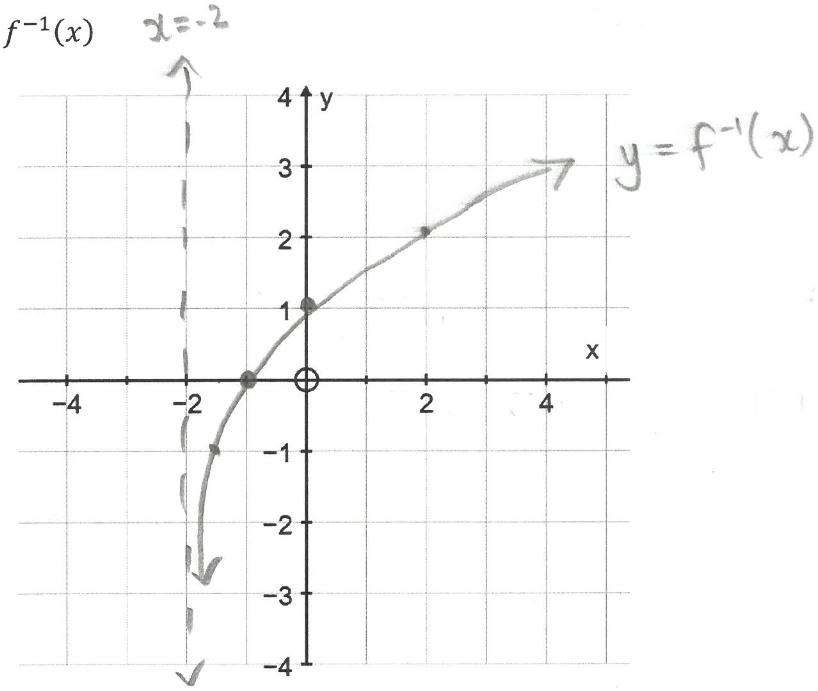
$|f(x)|$ reflects about x-axis

$f(|x|)$ reflects about y-axis.

7.3 $y = f(|x|)$ (3)



7.4 $y = f^{-1}(x)$ (3)



[10]

QUESTION EIGHT

Determine the following integrals:

8.1 $\int \frac{1}{x^2\left(1+\frac{1}{x}\right)^3}\,dx$ (8)

$u = 1 + 1/x = 1 + x^{-1}$

$du/dx = -x^{-2}$

$\therefore -du = \frac{1}{x^2}\,dx$

$= -\int u^{-3}\,du$

$= \frac{u^{-2}}{2} + c$

$= \frac{1}{2(1+1/x)^2} + c$

8.2 $\int \ln x\,dx$ (8)

$f(x) = \ln x$ $\quad g'(x) = 1.$

$f'(x) = 1/x$ $\quad g(x) = x$

We discussed a lot of these types.

$\therefore = x\ln x - \int x \cdot 1/x\,dx$

$= x\ln x - \int 1\,dx$

$= x\ln x - x + c.$

8.3 $\int \frac{1}{a^2-x^2}\,dx$, with a a constant ,using partial fractions. Your final answer must be as simplified as much as possible. (12)

$$\frac{1}{(a-x)(a+x)} = \frac{A}{a-x} + \frac{B}{a+x}.$$

$$1 = A(a+x) + B(a-x)$$

$x=a$: $\quad 1 = 2a.A \qquad \therefore A = \frac{1}{2a}.$

$x=-a$ $\quad 1 = 2a\ B. \qquad \therefore B = \frac{1}{2a}.$

$$\therefore \int \frac{1}{2a(a-x)}\,dx + \int \frac{1}{2a(a+x)}\,dx.$$

$u_1 = a-x \qquad u_2 = a+x$

$du/dx = -1 \qquad du_2/dx = 1.$

$$-\frac{1}{2a}\int \frac{1}{u_1}\,du_1 + \frac{1}{2a}\int \frac{1}{u_2}\,du_2$$

$$-\frac{1}{2a}\ln u_1 + \frac{1}{2a}\ln u_2 + C$$

$$= \frac{1}{2a}\left[\ln(a+x) - \ln(a-x)\right] + C$$

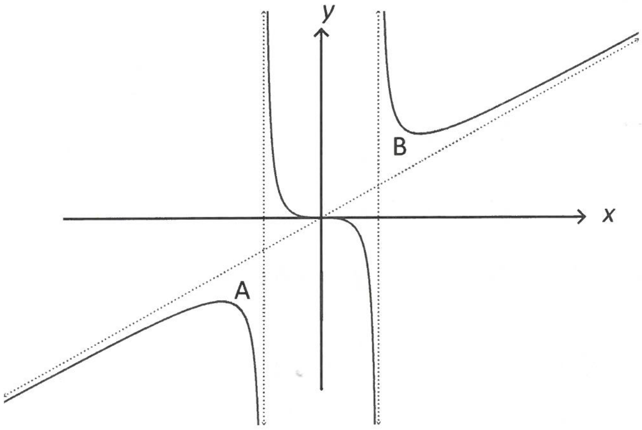
$$= \frac{1}{2a}\ln\frac{(a+x)}{(a-x)} + C$$

Had to do partial fractions first.

[28]

QUESTION NINE

9.1 The sketch given below represents $h(x) = \dfrac{2x^3}{x^2 - 4}$



(a) Calculate the coordinates of A and B, the local maximum and minimum of $h(x)$. (8)

$h'(x) = 0 \qquad \dfrac{(x^2-4).6x^2 - 2x^3.(2x)}{(x^2-4)^2} = 0.$

$6x^4 - 24x^2 - 4x^4 = 0.$

$2x^2(x^2 - 12) = 0$

$x = 0 \qquad x = \sqrt{12} \text{ or } -\sqrt{12}$

pt infl.

$\therefore A(-2\sqrt{3}; -6\sqrt{3})$

$B(2\sqrt{3}; 6\sqrt{3})$

synthetic division poorly done.

(b) Determine the equation of the three asymptotes. (7)

$x = 2$

$x = -2.$

$y = 2x.$

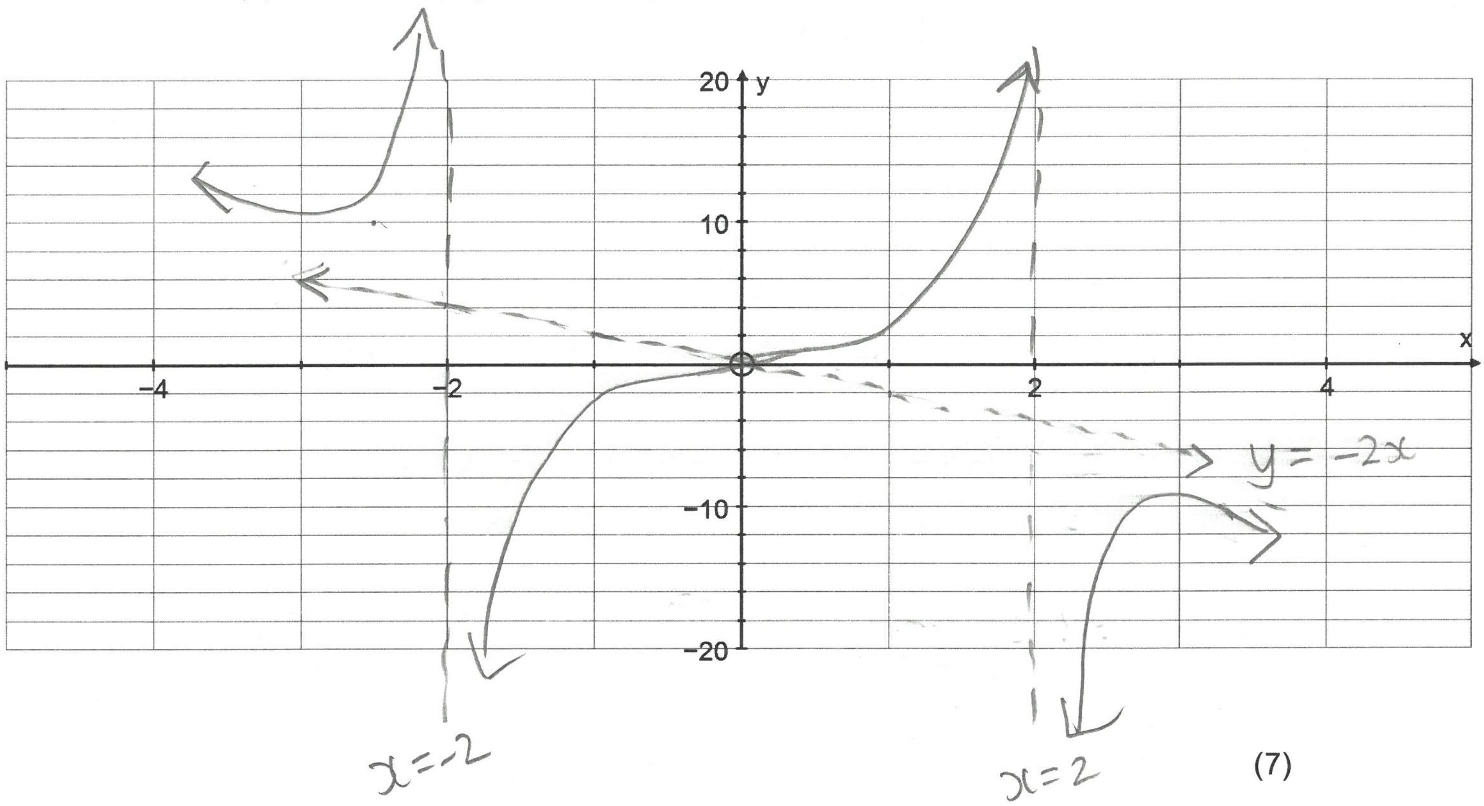
0	4	2	0	0	0
				8	0
			0	0	
		2	0	8	0

(c) From the graph, calculate the value(s) of x if $\sqrt{\dfrac{2x^3}{x^2-4}}$ is real. (3)

ie. $\dfrac{2x^3}{x^2-4} \geqslant 0$.

$x \in (-2;0] \cup (2;\infty)$

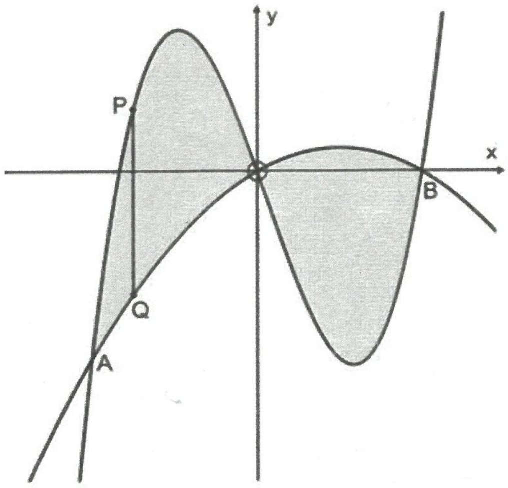
(d) Sketch $h(-x)$, on the diagram provided. Clearly show the asymptotes.



(7)

9.2 Refer to the figure below showing the graphs of:

$f(x) = 3x^3 - x^2 - 10x$ and $g(x) = -x^2 + 2x$



(a) Calculate the area of the shaded region. (8)

Need to determine co-ords A & B. (x value)

$3x^3 - x^2 - 10x = -x^2 + 2x$

$3x^3 - 12x = 0$

$3x(x^2 - 4) = 0$

$x = 0 \quad x = \pm 2.$

Area $= \int_{-2}^{0} f(x) - g(x)\,dx + \int_{0}^{2} g(x) - f(x)\,dx$

$= \int_{-2}^{0} 3x^3 - 12x\,dx + \int_{0}^{2} 12x - 3x^3\,dx.$

$= 12 + 12$

$= 24$ units^2

(b) PQ is a vertical line with P on f, Q on g between A and O. Determine the maximum length of PQ. (6)

$PQ = 3x^3 - 12x.$

$dPQ/dx = 0 \Rightarrow 9x^2 - 12 = 0$

$x^2 = \frac{4}{3}$

$x = \pm \frac{2}{\sqrt{3}}. \quad x < 0$

$x = -\frac{2}{\sqrt{3}}.$

$\therefore \max PQ = 3\left(-\frac{2}{\sqrt{3}}\right)^3 - 12\left(-\frac{2}{\sqrt{3}}\right)$

$= 3\left(\frac{-8}{3\sqrt{3}}\right) + \frac{24}{\sqrt{3}}$

$= \frac{16}{\sqrt{3}}$ or $\frac{16\sqrt{3}}{3}$ units.

This is in fact a Core Maths question.

[39]

Total : 200 marks