MEMO

SAHETI SCHOOL

PRELIMINARY EXAMINATION 2021

GRADE 12 - ADVANCED PROGRAMME MATHEMATICS

PAPER 1- ALEGBRA AND CALCULUS

Time: 2 hours **Total: 200**

Examiner: P R Mhuka

Moderators: S McConkey
N Eleftheriades

PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY

1. This question paper consists of 24 pages. Please check that your paper is complete.
2. Read the questions carefully.
3. Answer all the questions.
4. You may use an approved non-programmable and non-graphical calculator, unless otherwise stated.
5. **Answers must be rounded off to two decimal places**
6. All the necessary working details must be clearly shown.
7. It is in your own interest to write legibly and to present your work neatly.

QUESTION 1:

Electricity on a spacecraft can be produced by a type of nuclear generator. The electrical power produced by this generator can be modelled by

$$P_t = 120e^{-0.0079t}$$

where P_t is the electrical power produced by a type of nuclear generator and t the number of hours.

a) Determine the electrical power initially produced by the generator. (1)

$P_0 = 120$ ✓

b) Calculate how long it takes for the electrical power produced by the generator to reduce by 15%. (4)

✓

$0{,}85 = e^{-0{,}0079t}$

✓

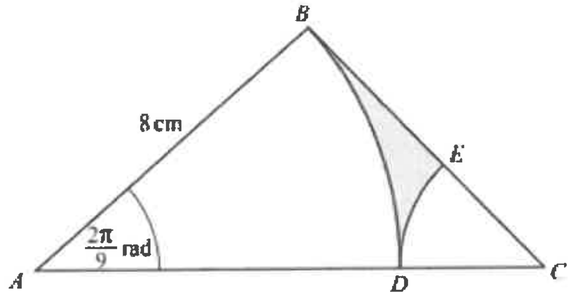
$\ln(0{,}85) = -0{,}0079t$

$t = 20{,}58$ hours ✓✓

[5]

QUESTION 2:

The diagram shows a right-angled triangle ΔABC with $AB = 8\ cm$ and angle $ABC = \frac{\pi}{2}$ radians. The points D and E lie on AC and BC respectively. BAD and ECD are sectors of the circles with centres A and C respectively.
$B\hat{A}D = \frac{2\pi}{9}$ radians.



a) Calculate the area of the shaded region. (7)

$\cos 2\pi/9 = \frac{8}{AC}$

$AC = 10,44$ $\therefore$ $CD = 2,44$

$BC = 6,71$

$\text{Area} = \frac{1}{2}(8)(6,71) - \frac{1}{2}\cdot\left(\frac{2\pi}{9}\right)(8)^2 - \frac{1}{2}(2,44)^2\left(5\pi/18\right)$

$= 1,90$

b) The perimeter of the shaded region. (4)

$$P = \frac{2\pi}{9} \times 8 + \frac{5\pi}{18} \times 2{,}44 + 4{,}27$$

$$= \underline{11{,}98}$$

[11]

QUESTION 3:

Prove by induction that:

$\sum_{r=1}^{n} r!\,r = (n+1)! - 1$ for all positive integers n.

Prove true for $n=1$

$LHS = 1! \times 1 = 1$ $RHS = (1+1)! - 1 = 1$

$\therefore$ true for $n=1$

Assume true for $n=k$

$$\sum_{r=1}^{k} r!\,r = (k+1)! - 1$$

or $1 + 4 + \cdots + k!\,k = (k+1)! - 1$

Prove true for $n = k+1$

$LHS = 1 + 4 + \cdots + k!\,k + (k+1)!(k+1)$

$= (k+1)! - 1 + (k+1)!(k+1)$

$= (k+1)!(k+2) - 1$

$= (k+2)! - 1$

By PMI, it is true for all positive integers on n.

[12]

QUESTION 4:

a) The complex number u is defined by $u = \frac{7+i}{1-i}$.

1) Express u in Cartesian form. (2)

$u = 3 + 4i$

2) Calculate $|u|$ and $arg(u)$ (4)

$|u| = \sqrt{3^2+4^2} = 5$

$arg(u) = \tan^{-1}(4/3) = 0,93$

3) Express u in the Polar form. (3)

$5(\cos(0,93) + i\sin(0,93))$

b) Create a polynomial $f(x)$ with real number coefficients which has all of the desired characteristics.

- The leading term of $f(x)$ is $-2x^3$
- $x = 2i$ is a zero
- $f(0) = -16$

(6)

$x = 2i$, $x = -2i$

$x^2 + 4 = 0$

$f(x) = -2x^3 + ax^2 + bx - 16$

$= (x^2+4)(-2x-4)$

$= -2x^3 - 4x^2 - 8x - 16$

c) Solve for x:

1) $(ln\, x)^2 = ln\, e^3 + ln\, x^2$

(6)

$(\ln x)^2 = 3 + 2\ln x$

$(\ln x)^2 - 2\ln x - 3 = 0$

$\ln x = 3$ $\ln x = -1$

$x = e^3$ $x = e^{-1}$

2) $\frac{2-x}{|3x+1|} \leq 2$ (7)

$x \neq -\frac{1}{3}$

$2-x \leq 2|3x+1|$ ✓

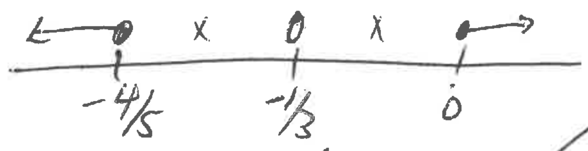
$2-x = 6x+2$

$\underline{x=0}$ ✓

$2-x = -6x-2$

$4 = -5x$

$\underline{-\frac{4}{5} = x}$ ✓



$\underline{x \leq -\frac{4}{5}}$ ✓✓ $\underline{x \geq 0}$ ✓✓

d) Show that $\binom{n+2}{3} - \binom{n}{3} = n^2$, for all integers where $n \geq 3$. (8)

$LHS = \frac{(n+2)!}{(n-1)!\,3!} - \frac{n!}{(n-3)!\,3!}$ ✓✓ ✓✓

$= \frac{(n+2)(n+1)n}{3!} - \frac{n(n-1)(n-2)}{3!}$ ✓ ✓

$= \frac{n}{3!}\left(n^2+3n+2-(n^2-3n+2)\right)$ ✓

$= \frac{n}{3!} \cdot 6n$ ✓

$= \underline{\underline{n^2}}$

[36]

QUESTION 5:

a) Consider the function: $f(x) = \begin{cases} x^2 + 2, & x \leq 2 \\ 3ax - 4, & x > 2 \end{cases}$

1) Determine the value of a if the function is continuous at $x = 2$. (4)

$\lim_{x \to 2^-} (x^2+2) = \lim_{x \to 2^+} (3ax-4) = f(2)$ ✓✓

$6 = 6a - 4$ ✓

$\underline{5/3 = a}$ ✓

2) Using your value of a from (1) determine whether the function is differentiable at $x = 2$. (5)

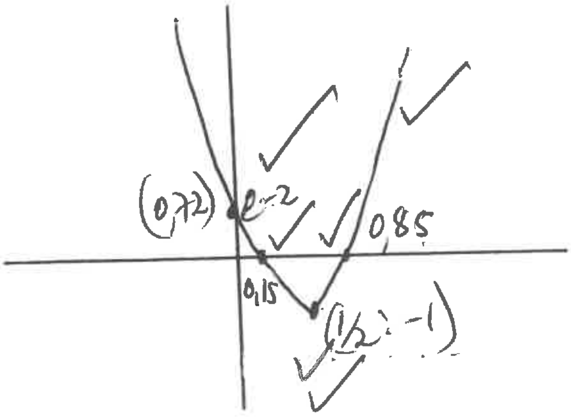
$\lim_{x \to 2^-} (2x) = 4$ ✓✓

$\lim_{x \to 2^+} 5 = 5$ ✓✓

$\therefore$ $4 \neq 5$ not differentiable. ✓

b) A function has equation $f(x) = e^{|2x-1|} - 2$.

1) Sketch $f(x)$, labelling the cusp point and all the points where the graph crosses the axes. (6)



$x = \frac{1}{2};\ y = -1$

y int; $x = 0$

$f(0) = e - 2 = (0{,}72)$

x int: $y = 0$

$|2x-1| = \ln 2$

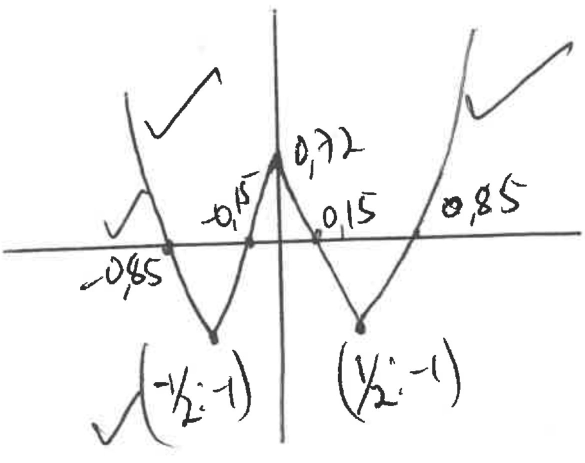
$2x = \ln 2 + 1$

$x = \frac{\ln 2 + 1}{2} = 0{,}85$

$2x = -\ln 2 + 1$

$x = \frac{-\ln 2 + 1}{2} = 0{,}15$

2) Sketch $f(|x|)$ (4)



3) Calculate the equation of $f^{-1}(x)$ for $x < \frac{1}{2}$ (4)

$y = e^{-(2x-1)} - 2$

$-\ln(x+2) = 2y - 1$

$\frac{-\ln(x+2)+1}{2} = y = f^{-1}(x)$

[23]

QUESTION 6:

a) Find the equation of the normal to the curve $y = \frac{2x-1}{\sqrt{x^2+5}}$ at the point where $x = 2$. Give your answer in the form $ax + by = c$, where a, b and c are integers. (8)

$x = 2: \quad y = 1$

$f'(x) = \frac{\sqrt{x^2+5} \cdot 2 - (2x-1)\frac{1}{2}(x^2+5)^{-\frac{1}{2}} \cdot 2x}{(x^2+5)}$

$f'(2) = 4/9$

$m_\perp = -9/4$

$y = -9/4x + c$

$1 = -\frac{9}{4}(2) + c$

$y = -9/4x + 11/2$

$4y + 9x = 22$

b) The parametric equations of a curve are:

$x = 6sin^2t$ and $y = 2sin2t + 3cos2t$, for $0 \leq t < \pi$.

Show that $\frac{dy}{dx} = \frac{2}{3}cos2t - 1$. (8)

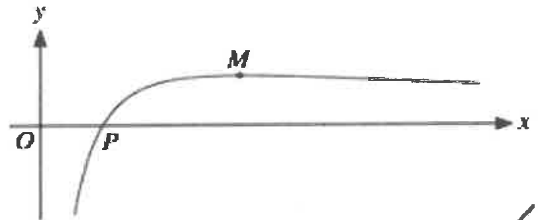
$\frac{dx}{dt} = 12Sint \cdot Cost$

$\frac{dy}{dt} = 4Cos2t - 6Sin2t$.

$\therefore \frac{dy}{dx} = \frac{4Cos2t}{6Sin2t} - \frac{6Sin2t}{6Sin2t}$

$= \frac{2}{3} Cot\, 2t - 1$

c) The diagram shows the curve with equation $y = 5\ln x.(2x+1)^{-1}$. The curve crosses the x-axis at the point P and has a maximum point M.



1) Find the gradient of the curve at the point P. (5)

$$\frac{dy}{dx} = 5\ln x \cdot \left(-(2x+1)^{-2} \cdot 2\right) + (2x+1)^{-1} \cdot \frac{5}{x}$$

$$\therefore f'(1) = 5/3$$

2) Use the Newton Raphson method to find the x-coordinate of M correct to 6 decimal places. (7)

Given $f''(x) = \dfrac{40\ln x}{(2x+1)^3} - \dfrac{20}{x(2x+1)^2} - \dfrac{5}{x^2(2x+1)}$

$x_0 = 2{,}5 = \text{ans}$

$$\therefore x_{n+1} = \text{ans} - \frac{\left(\dfrac{5}{2\text{ans}^2 + \text{ans}} - \dfrac{10\ln \text{ans}}{(2\text{ans}+1)^2}\right)}{\dfrac{40\ln \text{ans}}{(2\text{ans}+1)^3} - \dfrac{20}{\text{ans}(2\text{ans}+1)^2} - \dfrac{5}{\text{ans}^2(2\text{ans}+1)}}$$

$= 3{,}180966$

d) Calculate $\frac{dy}{dx}$ if $2e^{2x}y - y^3.\ln x + 6 = 0$ (7)

$$2e^{2x}\cdot\frac{dy}{dx} + y.4e^{2x} - 3y^2\frac{dy}{dx}.\ln x - y^3.\frac{1}{x} = 0$$

$$\frac{dy}{dx} = \frac{y^3x^{-1} - 4ye^{2x}}{2e^{2x} - 3y^2\ln x}$$

[35]

QUESTION 7:

Given the graph of $t(x) = \frac{8x^2-3x+1}{x-2}$

a) Determine the intercepts with both axes. (2)

$y = -\frac{1}{2}$

no x intercept

b) Find equations of any asymptotes (5)

$x = 2$

$y = 8x + 13$

c) Find the coordinates of the stationary points and state the nature of the stationary points. (8)

$$f'(x) = \frac{(x-2)(16x-3) - (8x^2-3x+1)}{(x-2)^2} = 0$$

$16x^2 - 35x + 6 - 8x^2 + 3x - 1 = 0$

$8x^2 - 32x + 5 = 0$

$x = 3,84$ $\quad$ $x = 0,16$

$y = 58,4$ $\quad$ $y = -0,39$

min TP $\quad$ max TP.

[15]

QUESTION 8:

a) Evaluate the following integrals without the use of a calculator:

1) $\int ln(2e^{sinx})\,dx$ (4)

$= \int \ln 2 + \sin x \, dx$

$= (\ln 2)x - \cos x + c$

2) $\int \left(\frac{8}{4x+1} + \frac{8}{cos^2(4x+1)}\right) dx$ (6)

$\int \frac{8}{4x+1} + 8\sec^2(4x+1)\,dx$

$= \frac{8\ln|4x+1|}{4} + \frac{8\tan(4x+1)}{4} + c$

3) $\int \frac{x+4}{2x^2-5x-12} . dx$ (10)

$$\frac{x+4}{(2x+3)(x-4)} = \frac{A}{2x+3} + \frac{B}{x-4}$$

$$x+4 = A(x-4) + B(2x+3)$$

let $x=4$: $\frac{8}{11} = B$

$x = -3/2$: $\frac{-5}{11} = A$

$$= \frac{-5}{11}\int \frac{1}{2x+3} + \frac{8}{11}\int \frac{1}{x-4} dx$$

$$= \frac{-5}{11} \frac{\ln|2x+3|}{2} + \frac{8}{11} \ln|x-4| + c$$

b) Use the substitution $x = tan\theta$ to determine the exact value of:

$$\int_0^1 \frac{dx}{(1+x^2)^{\frac{3}{2}}}$$ (9)

$x = 0$

$\theta = 0$ ✓

$x = 1$ ✓

$\theta = \pi/4$

$x = \tan\theta$ ✓

$\frac{dx}{d\theta} = \sec^2\theta$ ✓

$dx = \sec^2\theta \, d\theta$

$\int \frac{\sec^2\theta \, d\theta}{(1+\tan^2\theta)^{3/2}}$ ✓ ✓

$= \int \frac{\sec^2\theta \, d\theta}{(\sec^2\theta)^{3/2}}$ ✓

$= \int \cos\theta \, d\theta$ ✓

$= \sin\theta \Big|_0^{\pi/4}$ ✓

$= \frac{1}{\sqrt{2}}$ ✓ $\left(\frac{\sqrt{2}}{2}\right)$

c) Use integration by parts to find the $\int(x^2 - 2x + 1)e^{2x}\,dx$ (10)

$f(x) = x^2 - 2x + 1 \qquad g'(x) = e^{2x}$

$f'(x) = 2x - 2 \qquad g(x) = \frac{e^{2x}}{2}$

$= (x^2 - 2x + 1)\frac{e^{2x}}{2} - \int (2x-2)\frac{e^{2x}}{2}\,dx$

$f(x) = 2x - 2 \qquad g'(x) = \frac{e^{2x}}{2}$

$f'(x) = 2 \qquad g(x) = \frac{e^{2x}}{4}$

$= (x^2 - 2x + 1)\frac{e^{2x}}{2} - \left[\frac{e^{2x}}{4}(2x-2) - \int 2 . \frac{e^{2x}}{4}\,dx\right]$

$= (x^2 - 2x + 1)\frac{e^{2x}}{2} - (2x-2)\frac{e^{2x}}{4} + \frac{e^{2x}}{4} + c$

[39]

QUESTION 9:

a) Calculate the following Reimann Sum by turning it into an integral
$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left(8\left(1+\frac{i}{n}\right)^3 + 3\left(1+\frac{i}{n}\right)^2\right)$ (6)

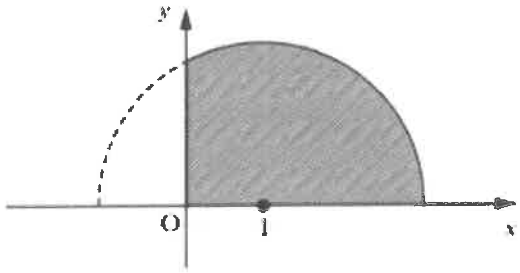
$a = 1 \quad b = 2$

$\int_1^2 (8x^3 + 3x^2)\,dx$

$= 2x^4 + x^3 \Big|_1^2$

$= \underline{37}$

b) A semi-circle with centre $(1;0)$ and radius 2, lies on the x-axis as shown.



Find the volume of the solid of revolution formed when the shaded region is rotated completely about the x-axis. (8)

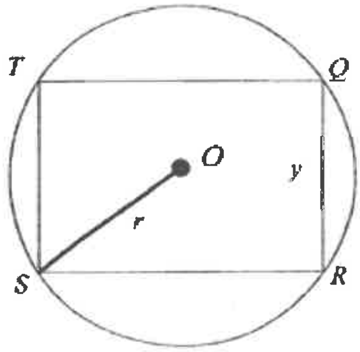
$$(x-1)^2+y^2=4$$

$$V=\pi\int_0^3\left(4-(x-1)^2\right)dx$$

$$=\pi\left[4x-\frac{(x-1)^3}{3}\right]_0^3$$

$$=9\pi \text{ units}^2$$

c) The diagram shows a rectangle $QRST$ inscribed in a circle, centre at O, with fixed radius $r\ cm$. The four corners of the rectangle lie on the circumference of the circle. $QR = y\ cm$.



1) Show that the perimeter, $P\ cm$, of the rectangle is given by: $P = 2y + 2\sqrt{4r^2 - y^2}$. (4)

$QS = 2r$ ✓

$SR = \sqrt{4r^2 - y^2}$ ✓✓

$\therefore P = y + y + 2\sqrt{4r^2 - y^2}$ ✓

$= \underline{2y + 2\sqrt{4r^2 - y^2}}$

2) Find the exact maximum value of P, in terms of r, as y varies.

(7)

$$P'(y) = 2 + \frac{1}{2} \cdot 2(4r^2 - y^2)^{-\frac{1}{2}} \cdot -2y = 0$$

$$2 - \frac{2y}{(4r^2 - y^2)^{\frac{1}{2}}} = 0$$

$$(4r^2 - y^2)^{\frac{1}{2}} = y$$

$$4r^2 - y^2 = y^2$$

$$4r^2 = 2y^2$$

$$r\sqrt{2} = y$$

$$\therefore P = 2r\sqrt{2} + 2\sqrt{4r^2 - 2r^2}$$

$$= 2r\sqrt{2} + 2r\sqrt{2}$$

$$= 4r\sqrt{2}$$

[25]