ST. DAVID'S MARIST INANDA

ADVANCED PROGRAMME MATHEMATICS

PRELIMINARY EXAMINATION
PAPER 1: CALCULUS and ALGEBRA

GRADE 12

1 SEPTEMBER 2021

EXAMINER: MRS S RICHARD **MODERATOR: MRS C KENNEDY**	**MARKS: 200** **TIME: 2 hours**

NAME: Memo

Please put a cross next to your teacher's name:

Mrs Kennedy	Mrs Richard

INSTRUCTIONS:

- ✓ This paper consists of 27 pages and a separate 4-page formula sheet. Please check that your paper is complete.
- ✓ Please answer all questions on the Question Paper.
- ✓ You may use an approved non-programmable, non-graphics calculator unless otherwise stated. PLEASE ENSURE YOUR CALCULATOR IS IN **RADIAN** MODE.
- ✓ Round answers to 2 decimal places, unless stated otherwise.
- ✓ It is in your interest to show all your working details.
- ✓ Work neatly. Do NOT answer in pencil.
- ✓ Diagrams are not drawn to scale.

QUESTION	Q1 [42]	Q2 [14]	Q3 [11]	Q4 [22]	Q5 [12]	Q6 [27]	Q7 [7]	Q8 [7]	Q9 [14]	Q10 [6]	Q11 [28]	Q12 [10]	TOTAL [200]
LEARNER'S MARKS													

QUESTION 1

a) Solve for x, without using a calculator and showing all working:

i) $\left|\frac{3}{x-1}\right| = 12$ (4)

✓ 2 cases

$x - 1 > 0$

$\frac{3}{x-1} = 12$ ✓m

$3 = 12(x-1)$

$3 = 12x - 12$

$15 = 12x$

$\frac{5}{4} = x$ ✓a

$x - 1 < 0$

$\frac{3}{x-1} = -12$

$3 = -12(x-1)$

$3 = -12x + 12$

$-9 = -12x$

$\frac{3}{4} = x$ ✓a

ii) $\log(2x+1) - \log(x-1) = 1$ State restrictions where necessary. (5)

$\log\left(\frac{2x+1}{x-1}\right) = 1$ ✓ log law

$10^1 = \frac{(2x+1)}{(x-1)}$ ✓m

$10x - 10 = 2x + 1$

$8x = 11$

$x = \frac{11}{8}$ ✓a

$2x + 1 > 0$

$2x > -1$

$x > -\frac{1}{2}$ ✓a

$x - 1 > 0$

$x > 1$ ✓a

iii) $\ln(e^{2x} - 20) = x$ (give your answer in terms of *ln a*, given *a* is a constant) (6)

$e^{x} = e^{2x} - 20$ ✓ m

$0 = e^{2x} - e^{x} - 20$ ✓ a

$0 = (e^{x} + 4)(e^{x} - 5)$ ✓ a

$e^{x} \neq -4$ ✓ ca or $e^{x} = 5$ ✓ ca

$\ln 5 = x$ ✓ ca

b) The following formula models the number of years (t), from now, in terms of the number of people (P) that stay in a town at time t:

$$t = 100\ \ln\left(\frac{4}{3} - \frac{P}{60000}\right)$$

i) Determine how many people initially live in the town when t = 0.
(Without the use of a calculator and showing all working out.) (5)

$$0 = 100\ \ln\left(\frac{4}{3} - \frac{P}{60\,000}\right) \quad \checkmark M$$

*

$$0 = \ln\left(\frac{80000 - P}{60\,000}\right) \quad \checkmark \text{LCD}$$

$$0 = \ln(80000 - P) - \ln 60\,000 \quad \checkmark \text{ log law.}$$

$$\ln 60\,000 = \ln(80\,000 - P) \quad \checkmark M$$

$$60\,000 = 80\,000 - P$$

$$P = 20\,000 \quad \checkmark a$$

* or $e^0 = \frac{4}{3} - \frac{P}{60000}$ ✓M

$1 = {}^4/_3 - {}^P/_{60000}$ a

$$\frac{P}{60000} = \frac{1}{3} \quad \checkmark M$$

$$P = \frac{60\,000}{3} = 20\,000 \quad \checkmark a$$

ii) As a result of migration to the cities, the town's population is decreasing. Calculate after how many years (to the nearest year) there will be no residents left in the town. (3)

$$P = 0$$

$$t = 100\ \ln \frac{4}{3} \quad \checkmark a$$

$$t = 28{,}7682\ldots \quad \checkmark a$$

$$= 29 \text{ years} \quad \checkmark a$$

iii) Change the subject of the formula to P, hence write the formula as P = . . (5)

$$\frac{t}{100} = \ln\left(\frac{4}{3} - \frac{P}{60\,000}\right) \checkmark a$$

$$e^{\frac{t}{100}} \checkmark^{m} = \frac{4}{3} - \frac{P}{60\,000} \checkmark a$$

$$\frac{P}{60\,000} = \frac{4}{3} - e^{\frac{t}{100}} \checkmark a$$

$$P = 60000\left(\frac{4}{3} - e^{\frac{t}{100}}\right) \checkmark ca$$

$$= 80\,000 - 60\,000\, e^{\frac{t}{100}}$$

iv) Hence, or otherwise, determine the initial rate at which the population decreases (that is when t = 0 years). (5)

$$\frac{dP}{dt} \checkmark^{m} = -60\,000 \checkmark^{a}\, e^{\frac{t}{100}} \checkmark^{a} \cdot \frac{1}{100} \checkmark a$$

$$= -600\, e^{\frac{t}{100}}$$

at $t = 0$

$$\frac{dP}{dt} = -600 \checkmark^{ca} \text{ people / year}$$

c) Given $f(x) = (x-1)\ln(x-1)$ for $x > 1$ and $g(x) = e^x + 1$

i) Show that $f \circ g(x) = x.e^x$ (4)

M

$$f(g(x)) = (e^x + 1 - 1)\ln(e^x + 1 - 1)$$

$$= e^x \ln e^x$$ a

$$= x e^x \ln e$$ log law

$$= x e^x$$ given

ii) Hence solve for x if $f \circ g(x) = 2x$ (5)

$$x e^x = 2x$$ M

$$x(e^x - 2) = 0$$ a

$x \neq 0$ a or $e^x = 2$ a

$x > 1$ $\ln 2 = x$ a

$x = 0,69$

[42]

QUESTION 2 Note $i = \sqrt{-1}$

a) Determine an equation in the form: $x^4 + ax^3 + bx^2 + cx + d = 0$ given that $x = 1 - \sqrt{2}$ and $x = 3 - i$ are roots of the equation. (8)

$\therefore\ x = 1 + \sqrt{2}$ ✓a and $x = 3 + i$ ✓a are also roots

$x = 1 - \sqrt{2}$
$x = 1 + \sqrt{2}$

ADD 2

$(1-\sqrt{2})(1+\sqrt{2})$ MULTIPLY
$= 1 - 2 = -1$

✓a ✓a

$\therefore\ x^2 - 2x - 1$ is a factor

$x = 3 - i$
$x = 3 + i$

ADD 6

$(3-i)(3+i)$ MULTIPLY
$= 9 - i^2$
$= 9 - (-1)$
$= 10$

✓a ✓a

$\therefore\ x^2 - 6x + 10$ is a factor

$\therefore\ (x^2 - 2x - 1)(x^2 - 6x + 10)$

$$= x^4 - 6x^3 + 10x^2$$
$$-2x^3 + 12x^2 - 20x$$
$$-x^2 + 6x - 10$$

$$= x^4 - 8x^3 + 21x^2 - 14x - 10$$

✓ca ✓ca

b) Determine the values of a and b, where a and b are real numbers that satisfy the equation: $\frac{a+2i}{1-3i} \times bi = -7 - i$ (6)

$abi + 2bi^2 = (-7-i)(1-3i)$ ✓M ✓M

$abi - 2b = -7 + 21i - i + 3i^2$

$= -7 + 20i - 3$

$abi - 2b = -10 + 20i$ ✓a ✓a

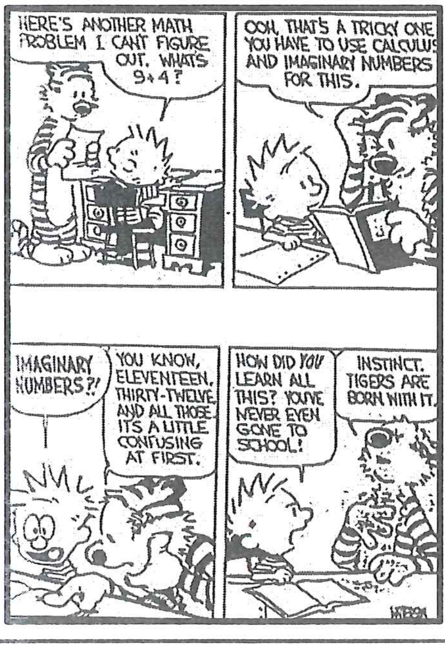
$\therefore ab = 20$

$-2b = -10$

$\therefore b = 5$ ✓ca

$ab = 20$

$\therefore a = 4$ ✓ca



[14]

QUESTION 3

Prove by Mathematical induction that $8^n - 7n + 6$ is divisible by 7 for all $n \in \mathbb{N}$ (11)

Prove true for $n = 1$

ie. $8^1 - 7(1) + 6$ M

$= 8 - 7 + 6$

$= 7$ $\therefore$ div by 7

$\therefore$ true for $n = 1$ shown

M words

Assume true for $n = k$ M

ie $8^k - 7k + 6 = 7p$ for $p \in \mathbb{Z}$ a

$8^k = 7p + 7k - 6$ M

Prove true for $n = k + 1$ M

ie. $8^{k+1} - 7(k+1) + 6$

$= 8^k . 8 - 7k - 7 + 6$

a

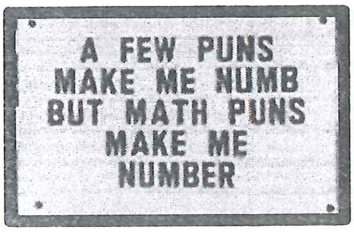
$= (7p + 7k - 6)(8) - 7k - 1$

$= 56p + 56k - 48 - 7k - 1$

$= 56p + 49k - 49$ a

$= 7(8p + 7k - 7)$ $\therefore$ div by 7

$\therefore$ True for $n = 1$, true for $n = k + 1$ iff true for $n = k$ $\therefore$ true for all $n \in \mathbb{N}$ by PMI M M



[11]

QUESTION 4

a) Determine the derivative of $f(x)=\dfrac{1}{\sqrt{3x-2}}$ by first principles. (8)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \left(\frac{1}{\sqrt{3(x+h)-2}} - \frac{1}{\sqrt{3x-2}} \right) \times \frac{1}{h} \quad \checkmark m \quad \checkmark a$$

$$= \lim_{h\to 0} \frac{\sqrt{3x-2} - \sqrt{3(x+h)-2}}{h\left(\sqrt{3(x+h)-2}\right)\left(\sqrt{3x-2}\right)} \times \frac{\sqrt{3x-2}+\sqrt{3(x+h)-2}}{\sqrt{3x-2}+\sqrt{3(x+h)-2}} \quad \checkmark m$$

$$= \lim_{h\to 0} \frac{(3x-2) - (3(x+h)-2)}{h\left(\sqrt{3(x+h)-2}\right)\left(\sqrt{3x-2}\right)\left(\sqrt{3x-2}+\sqrt{3(x+h)-2}\right)} \quad \checkmark a \quad \checkmark a$$

$$= \lim_{h\to 0} \frac{3x-2-3x-3h+2}{h\left(\sqrt{3(x+h)}-2\right)\left(\sqrt{3x-2}\right)\left(\sqrt{3x-2}+\sqrt{3(x+h)-2}\right)}$$

$$= \lim_{h\to 0} \frac{-3\cancel{h}}{\cancel{h}\left(\sqrt{3(x+h)-2}\right)\left(\sqrt{3x-2}\right)\left(\sqrt{3x-2}+\sqrt{3(x+h)-2}\right)} \quad \checkmark m$$

$$= \frac{-3}{\sqrt{3x-2}\sqrt{3x-2}\left(\sqrt{3x-2}+\sqrt{3x-2}\right)} \quad \checkmark a$$

$$= \frac{-3}{(3x-2)\,2\sqrt{3x-2}} \quad \checkmark a$$

b) Determine: $D_x\left[\left(\dfrac{3x-1}{2x+5}\right)^5\right]$ (6)

$$= 5\left(\frac{3x-1}{2x+5}\right)^4 \cdot \frac{3(2x+5)-(3x-1)(2)}{(2x+5)^2}$$

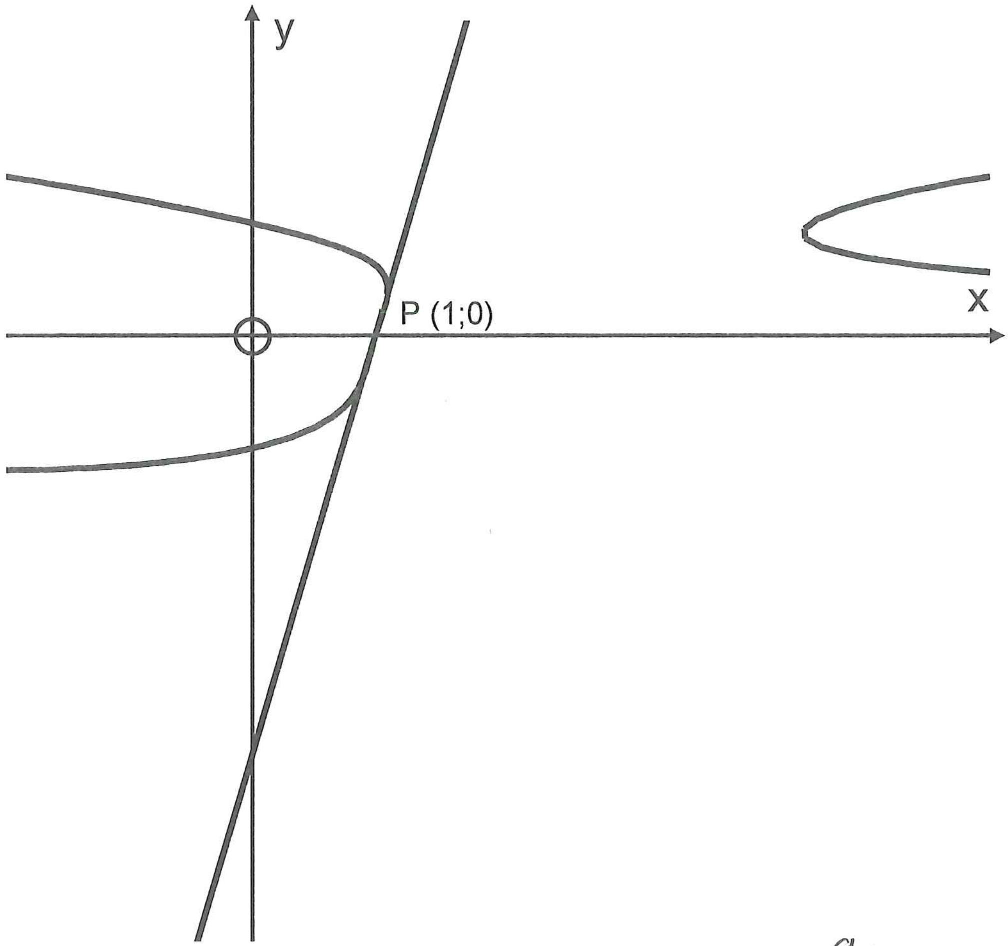
✓a ✓a ✓a ✓a ✓a

$$= 5\left(\frac{3x-1}{2x+5}\right)^4 \frac{6x+15-6x+2}{(2x+5)^2}$$

$$= 5\left(\frac{3x-1}{2x+5}\right)^4 \cdot \frac{17}{(2x+5)^2} \quad ✓ca$$

$$= \frac{85(3x-1)^4}{(2x+5)^6}$$

c) Determine the gradient of the tangent to the curve $3y^4 + 4x - x^2 \sin y - 4 = 0$ at the point P (1; 0) (8)



$$12y^3 \frac{dy}{dx} + 4 - \left(2x \sin y + x^2 \cos y \frac{dy}{dx}\right) + 0 = 0$$

$$12y^3 \frac{dy}{dx} + 4 - 2x \sin y - x^2 \cos y \frac{dy}{dx} = 0$$

$$0 + 4 - 2(1) \sin 0 - 1 \cos 0 \frac{dy}{dx} = 0$$

$$4 - 1 \frac{dy}{dx} = 0$$

$$4 = \frac{dy}{dx}$$

[22]

QUESTION 5

The function f(x) is defined as follows:

$$f(x) = \begin{cases} \dfrac{a}{x} & \text{if } x \geq 2 \\ b - 2x & \text{if } x < 2 \end{cases}$$

Determine the values of a and b if f(x) is **differentiable** at x = 2.

Justify your answer, using correct notation of limits. (12)

For continuity:

✓notation

$\lim\limits_{x \to 2^-} b - 2x = b - 4$ ✓a

notation

$\lim\limits_{x \to 2^+}$ ✓ $\frac{a}{x} = \frac{a}{2}$ ✓a $\therefore$ $b - 4 = \frac{a}{2}$

$2b - 8 = a$ ✓ca

$f(2) = \frac{a}{2}$ ✓m

For differentiability:

$\lim\limits_{x \to 2^-} -2 = -2$ ✓a

$\lim\limits_{x \to 2^+} -ax^{-2} = -a(2)^{-2}$ ✓a

$= -\frac{a}{4} = -2$ ✓m

$\therefore -a = -8$

$a = 8$ ✓ca

$\therefore 2b - 8 = a$

$2b - 8 = 8$ ✓ca

$2b = 16$

$b = 8$ ✓ca

[12]

QUESTION 6

Given the function $f(x)=\dfrac{x^3+4x^2+x-6}{x^2-1}$

a) Determine the coordinates of the stationary points. (9)

$$f'(x)=\frac{(3x^2+8x+1)(x^2-1)-(x^3+4x^2+x-6)(2x)}{(x^2-1)^2}$$

$$0 = 3x^4-3x^2+8x^3-8x+x^2-1-2x^4-8x^3$$
$$-2x^2+12x$$

$$0 = x^4-4x^2+4x-1$$

$$0=(x-1)(x^3+x^2-3x+1)$$

$\therefore\ x\neq 1$ or $x=-2{,}41$ or $x=0{,}41$

removable discontinuity. (mark on sketch)

$y=0{,}17$ or $y=5{,}83$

$\therefore\ (-2{,}41\,;\,0{,}17)$ A $\quad (0{,}41\,;\,5{,}83)$ B

b) Determine the intercepts with the axes. (4)

y-intercepts $x=0$ $y=6$ ✓

x-intercepts $0 = x^3 + 4x^2 + x - 6$

$0 = (x-1)(x+2)(x+3)$ ✓

$\therefore$ $x \neq 1$ ✓ or $x = -2$ or $x = -3$ ✓

c) Determine the equations of any asymptotes. (4)

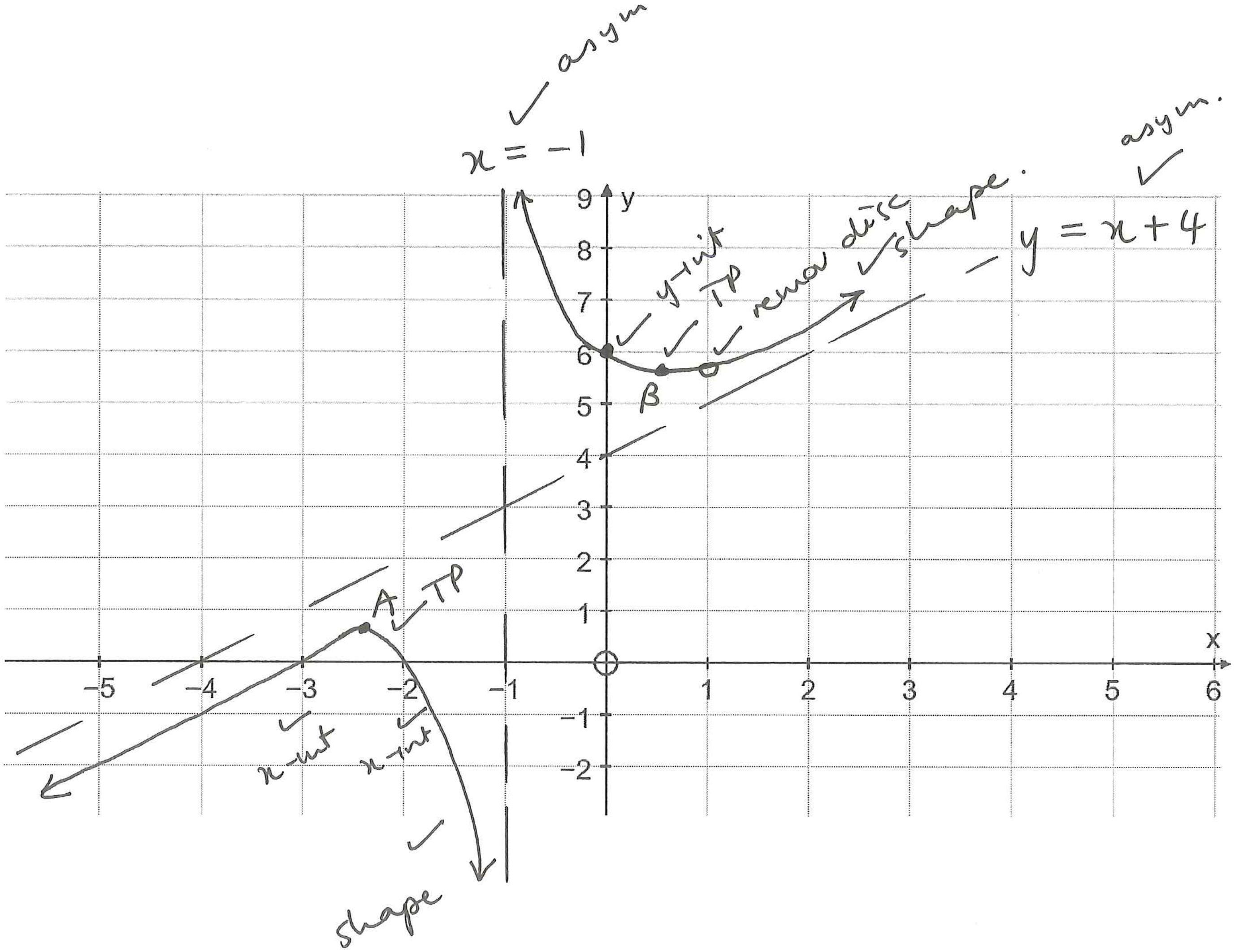
$x = -1$ ✓ vertical asymptote ✓

$$\begin{array}{r|l} & x + 4 \\ x^2 - 1 & x^3 + 4x^2 + x - 6 \\ & x^3 - x \\ \hline & 4x^2 + 2x \end{array}$$

$\therefore$ $y = x + 4$ ✓ is an oblique ✓ asymptote

No horizontal asymptotes.

d) Sketch the graph of f(x) on the given axes. (10)

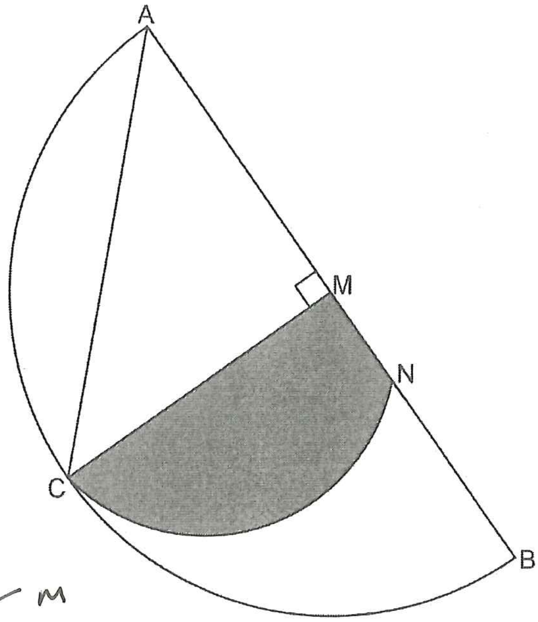


[27]

QUESTION 7

ABC is a semi-circle with centre M.
ANC is a sector with centre A and corresponding arc NC.

AM = 15cm, $A\widehat{M}C = \frac{\pi}{2}$ radians and $M\widehat{A}C = \beta$



a) Give a reason why $\beta = \frac{\pi}{4}$ radians. (1)

AM = MC radii ✓ M

$\therefore \beta = \frac{\pi}{4}$ equal <'s ; equal sides.

b) Determine the area of the shaded region MNC. (6)

$AC^2 = 15^2 + 15^2$ Pyth

$= 450$ ✓ M

$AC = \sqrt{450}$ ✓ a

$= 15\sqrt{2} = 21,21\ldots$

Area shaded = sector − Δ ✓ M

$= \frac{1}{2} r^2 \theta - \frac{1}{2} bh$

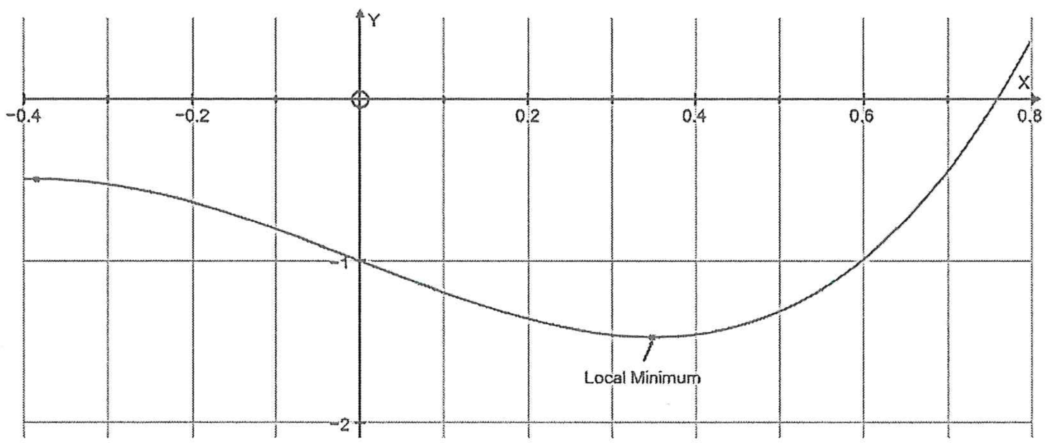
$= \frac{1}{2}(15\sqrt{2})^2\left(\frac{\pi}{4}\right) - \frac{1}{2}(15)(15)$ ✓ a ✓ a

$= 64,21 \text{ cm}^2$ ✓ ca

[7]

QUESTION 8

A portion of the graph of $f(x) = x^4 + 5x^3 - 2x - 1$ is shown below:



Use the **Newton-Raphson Method** with an initial approximation of 1 to determine the **x-coordinate of the local minimum** shown above. Give your answer to 4 decimal places. (7)

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

local min $\therefore$ $f'(x) = 4x^3 + 15x^2 - 2 = 0$ ✓ a

$f''(x) = 12x^2 + 30x$ ✓ a

$x_0 = 1$

$$x_1 = x_0 - \frac{4(x_0)^3 + 15(x_0)^2 - 2}{12(x_0)^2 + 30(x_0)}$$ ✓ a

$= 0{,}5952 \ldots$ ✓ a

$x_2 = 0{,}40715\ldots$ ✓ ca

$x_3 = 0{,}3538\ldots$ ✓ ca

$x_4 = 0{,}34928\ldots$

$\vdots$

$\therefore \quad x = 0{,}3492$ ✓ ca

[7]

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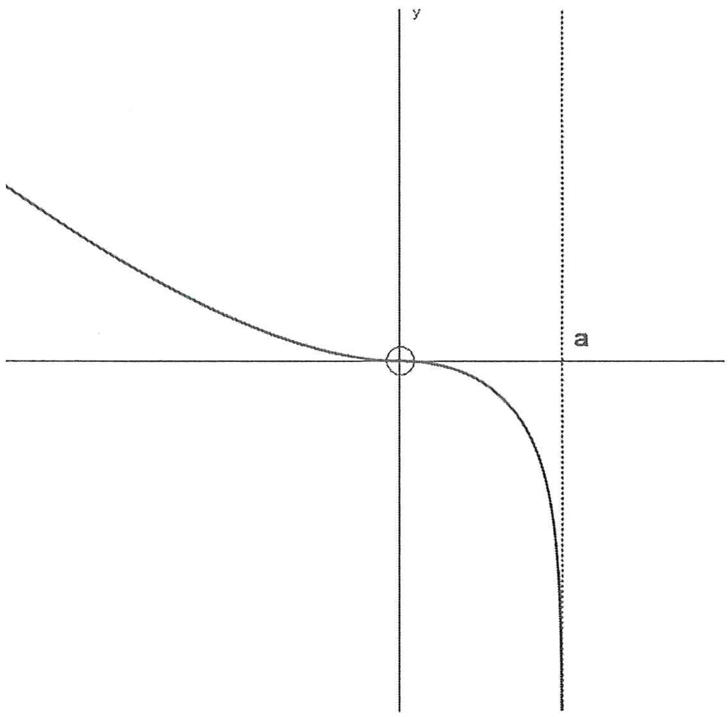
QUESTION 9

The graph of the function $f(x) = |x|\ln(1-x)$, $x < a$ is shown below.



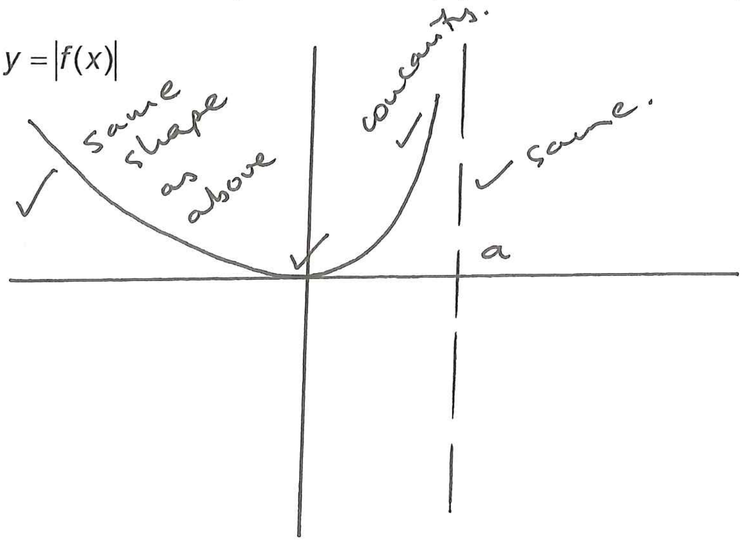
a) Write down the value of a. (2)

$a = 1$ ✓✓

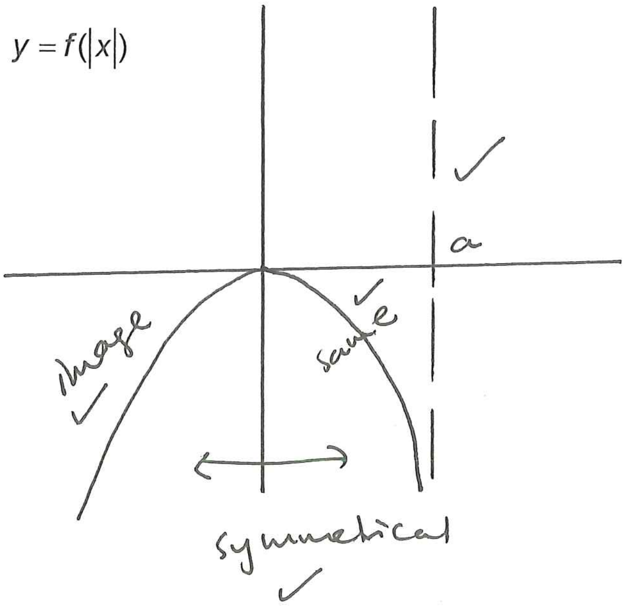
b) Sketch the following graphs.

You do not need to work out any values – simply show how the shape changes.

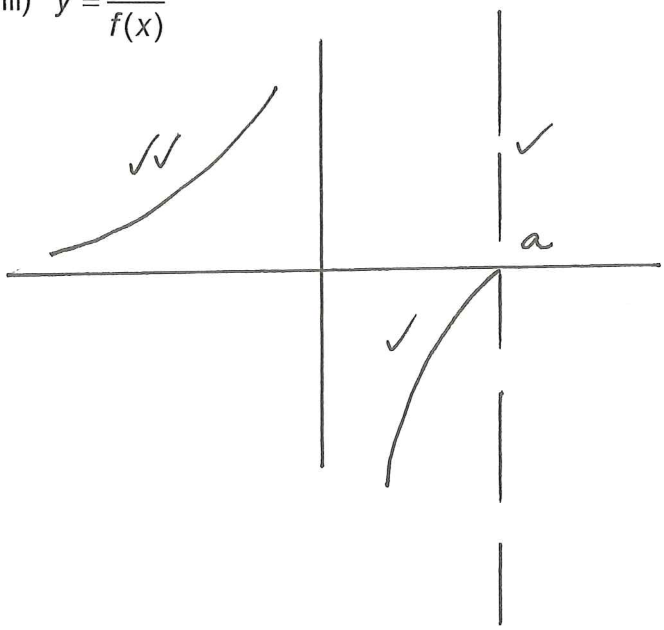
i) $y = |f(x)|$ (4)



ii) $y = f(|x|)$ (4)



iii) $y = \dfrac{1}{f(x)}$ (4)



[14]

QUESTION 10

Consider $\int_a^b f(x)dx = \lim_{n\to\infty} \frac{3}{n}\sum_{i=1}^{n}\left[\left(2+\frac{3i}{n}\right)^2 - 2\left(2+\frac{3i}{n}\right)+2\right]$

a) Determine the values of a and b. (2)

$b - a = 3$ ✓ a

$a = 2$

$b - 2 = 3$

$b = 5$ ✓ a

b) Write down the function $f(x)$ (2)

$f(x) = x^2 - 2x + 2$ ✓a ✓a

c) Calculate the area enclosed by the graph of f, the x-axis and the lines x = a and x = b. (2)

$\int_2^5 x^2 - 2x + 2 \, dx$ ✓ ca

in calculator

$= 24$ ✓ ca

[6]

QUESTION 11

a) Determine the following integrals:

i) $\int x^2\sqrt{5x^3-13}\,dx$ (6)

let $5x^3-13 = u$ ✓ m

$\frac{du}{dx} = 15x^2$

$\frac{1}{15}\,du = x^2\,dx$ ✓ a

$= \frac{1}{15}\int u^{\frac{1}{2}}\,du$ ✓ m

$= \frac{1}{15}\,u^{3/2}\cdot\frac{2}{3} + c$ ✓ a ✓ a

$= \frac{2}{45}\left(5x^3-13\right)^{3/2} + c$ ✓ ca

-1 if no $+c$

ii) $\int x\cos 3x\,dx$ (using integration by parts) (8)

$f(x) = x$ ✓ m

$f'(x) = 1$ ✓ a

$g'(x) = \cos 3x$ ✓ m

$g(x) = \int \cos 3x\,dx$

$= \frac{1}{3}\sin 3x$ ✓ a

$\int x\cos 3x\,dx = x\left(\frac{1}{3}\sin 3x\right) - \int (1)\left(\frac{1}{3}\sin 3x\right)dx$ ✓ ca ✓ ca

$= \frac{1}{3}x\sin 3x - \frac{1}{3}\int \sin 3x\,dx$

$= \frac{1}{3}x\sin 3x + \frac{1}{3}\cdot\frac{1}{3}\cos 3x + c$ ✓ a ✓ a

$= \frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x + c$

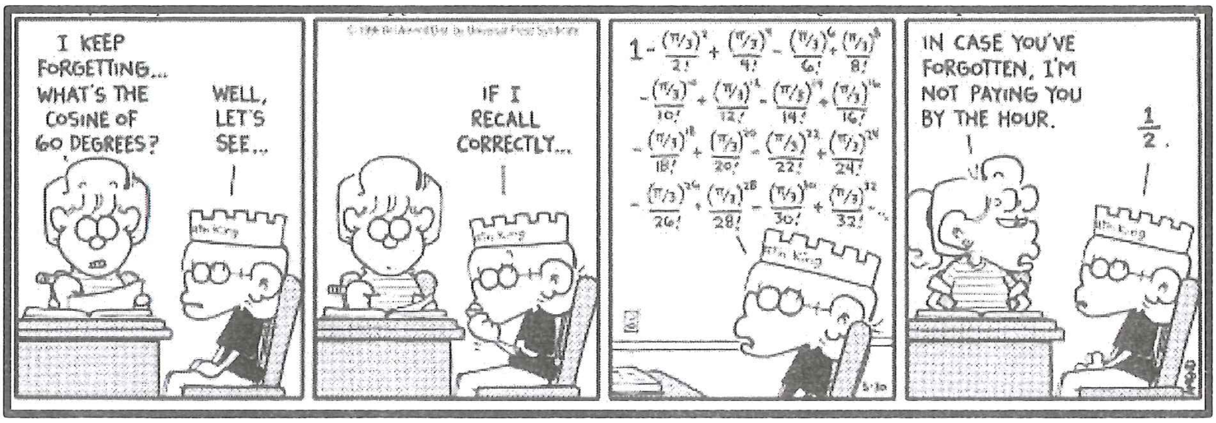
iii) $\int \cot^3 x.\text{cosec}^2 x\, dx$ (6)

$= -\int u^3\, du$ ✓ca   let $\cot x = u$ ✓m

$\therefore \dfrac{du}{dx} = -\text{cosec}^2 x$ ✓a

$= -\dfrac{u^4}{4} + c$ ✓a ✓a

$= -\dfrac{\cot^4 x}{4} + c$ ✓ca



b) Given $f(x)=\dfrac{-x-11}{x^2+x-2}$

i) Decompose f(x) into partial fractions. (4)

$$\frac{-x-11}{(x-1)(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x+2)} \quad \checkmark m$$

$$-x-11 = A(x+2) + B(x-1) \quad \checkmark a$$

$x=1$ $\quad -12 = A(3) + 0$

$$\therefore A = -4 \quad \checkmark ca$$

$x=-2$ $\quad -9 = -3B$

$$\therefore B = 3 \quad \checkmark ca$$

$$\therefore \frac{-x-11}{x^2+x-2} = \frac{-4}{x-1} + \frac{3}{x+2}$$

ii) Hence determine $\int \dfrac{-x-11}{x^2+x-2}dx$ (4)

$$\int \frac{-4}{x-1} + \frac{3}{x+2}\,dx \quad \checkmark ca$$

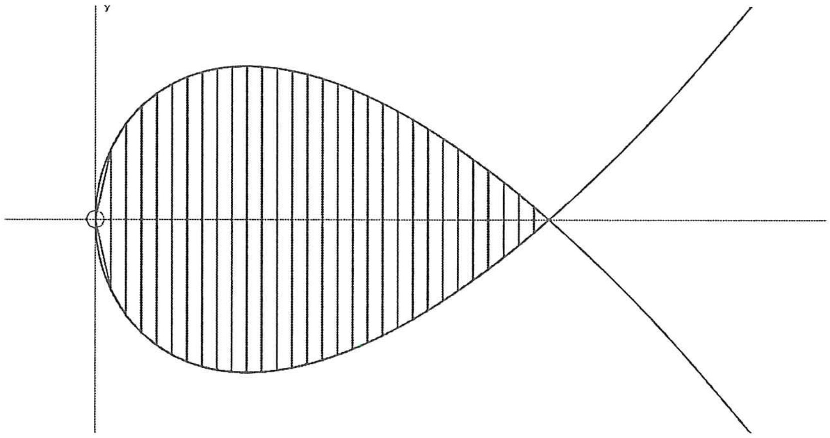
$$= -4\int \frac{1}{x-1}\,dx + 3\int \frac{1}{x+2}\,dx \quad \checkmark m$$

$$= -4\ln|x-1| + 3\ln|x+2| + c \quad \checkmark ca \quad \checkmark ca$$

[28]

QUESTION 12

The loop $y^2 = x(a-x)^2$ is shown.



The shaded region is rotated about the x-axis.

Determine the volume of the solid formed by this rotation, in terms of a. (10)

$$\text{Vol} = \pi \int y^2\,dx$$

✓ m

$$= \pi \int_0^a x(a-x)^2\,dx$$

✓ ✓ a

$$= \pi \int_0^a x(a^2 - 2ax + x^2)\,dx$$

✓ ca

$$= \pi \int_0^a a^2x - 2ax^2 + x^3\,dx$$

✓ ✓ ✓ a

$$= \pi \left[\frac{a^2x^2}{2} - \frac{2ax^3}{3} + \frac{x^4}{4} \right]_0^a$$

$$= \frac{\pi a^2 a^2}{2} - \pi \frac{2a.a^3}{3} + \frac{\pi a^4}{4}$$

✓ m subs

$$= \frac{\pi a^4}{2} - \frac{2\pi a^4}{3} + \frac{\pi a^4}{4}$$

✓ a

$$= \frac{\pi a^4}{12}$$

✓ ca

[10]

[Total: 200 marks]

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