## QUESTION 1

1.1 Solve for $x \in R$ if $|x^2 - x| = 42$ (5)

$x^2 - x = 42$ v $x^2 - x = -42$

$x^2 - x - 42 = 0$ ✓ $x^2 - x + 42 = 0$ ✓

$x = 7$ ✓ $v$ $x = -6$ ✓ $no\ solution\ in\ R$ ✓

1.2 Given: $e^x + 12e^{-x} = 8$ and give your answer in the form:

$x = lna$ and $x = lnb$ (7)

$Let\ e^x = k$

$k + \frac{12}{k} - 8 = 0$ ✓

$k^2 - 8k + 12 = 0$

$k = 6$ ✓ v $k = 2$ ✓

$\therefore e^x = 6$ ✓ $e^x = 2$ ✓

$x = ln6$ ✓ $x = ln2$ ✓

1.3 If $\frac{m}{2-4i} = \frac{1}{3} + ni$, calculate the values of $m$ and $n$ (6)

$\frac{m}{2-4i} \times \frac{2+4i}{2+4i}$ ✓

$= \frac{2m+4mi}{20}$ ✓

$\therefore \frac{2m}{20} = \frac{1}{3}$ ✓ $\frac{4\left(\frac{10}{3}\right)}{20} = n$ ✓

$m = \frac{10}{3}$ ✓ $n = \frac{2}{3}$ ✓

Or

LCD: $3(2 - 4i)$

$3m = 2 - 4i + 6ni - 12ni^2$

$3m = 2 - 4i + 6ni + 12n$

$0i = -4i + 6ni$ $3m = 2 + 12\left(\frac{2}{3}\right)$

$n = \frac{2}{3}$ $m = \frac{10}{3}$

1.4 $f(x) = x^4 + x^3 - 2x^2 + 2x + 4$

Solve for $f(x) = 0$ if $x \in ℂ$ and it is given that $f(1 - i) = 0$. (10)

$x = 1 - i$ is a solution $\quad \therefore x = 1 + i$ is also a solution ✓

$x^2 - [(1 - i) + (1 + i)]x + (1 - i)(1 + i) = 0$ ✓✓

$x^2 - 2x + (1 - i^2) = 0$ ✓

$x^2 - 2x + 2 = 0$ ✓

$\therefore x^4 + x^3 - 2x^2 + 2x + 4 = (x^2 - 2x + 2)(x^2 + 3x + 2) = 0$ by inspection ✓✓✓

$\therefore x = 1 \pm i \quad \vee \quad x = -2 \quad \vee \quad x = -1$ ✓✓

1.5 $f(x) = 1 - 2e^{-2x}$

Determine $f^{-1}(x)$ in the form $y = \cdots$ and provide the domain. (6)

$y = 1 - 2e^{-2x}$

Inverse:

$x = 1 - 2e^{-2y}$ ✓

$x - 1 = -2e^{-2y}$ ✓

$\frac{1 - x}{2} = e^{-2y}$ ✓

$\therefore -2y = ln\left[\frac{1-x}{2}\right]$ ✓

$\therefore y = -\frac{1}{2} ln\left[\frac{1-x}{2}\right]$ ✓; $\quad x < 1$ ✓

**[34]**

## QUESTION 2

Prove by Mathematical Induction that, for all $n \in N$,

$$\sum_{r=1}^{n} \frac{5-4r}{5^r} = \frac{n}{5^n}$$

$\frac{1}{5} + \left(-\frac{3}{25}\right) + \left(-\frac{7}{125}\right) + \cdots \ldots + \frac{5-4n}{5^n} = \frac{n}{5^n}$ ✓

- Prove true for $n = 1$

  $LHS = \frac{1}{5} \quad RHS = \frac{1}{5^1} = \frac{1}{5}$ ✓

  $\therefore formula\ true\ for\ n = 1$ ✓

- Assume formula true for $n = k$

  $\frac{1}{5} + \left(-\frac{3}{25}\right) + \left(-\frac{7}{125}\right) + \cdots \ldots + \frac{5-4k}{5^k} = \frac{k}{5^k}$ ✓ ✓

- Prove true for $n = k + 1$

  **Proposed formula:** $\frac{k+1}{5^{k+1}}$ ✓

  Add next term to both sides:

  $\frac{1}{5} + \left(-\frac{3}{25}\right) + \left(-\frac{7}{125}\right) + \cdots \ldots + \frac{5-4k}{5^k} + \frac{5-4(k+1)}{5^{k+1}} = \frac{k}{5^k} + \frac{5-4(k+1)}{5^{k+1}}$ ✓ ✓

  $= \frac{5k+5-4k-4)}{5.5^k}$ ✓

  $= \frac{k+1}{5^{k+1}}$ ✓

  ✓

  Or $= \frac{k.5^{k+1}+5^k(5-4(k+1))}{5^k.5^{k+1}}$

  $= \frac{k.5^{k+1}+5^k(1-4k)}{5^k.5^{k+1}}$

  $= \frac{5^k(5k+1-4k)}{5^k.5^{k+1}}$

  $= \frac{5^k(k+1)}{5^k.5^{k+1}}$

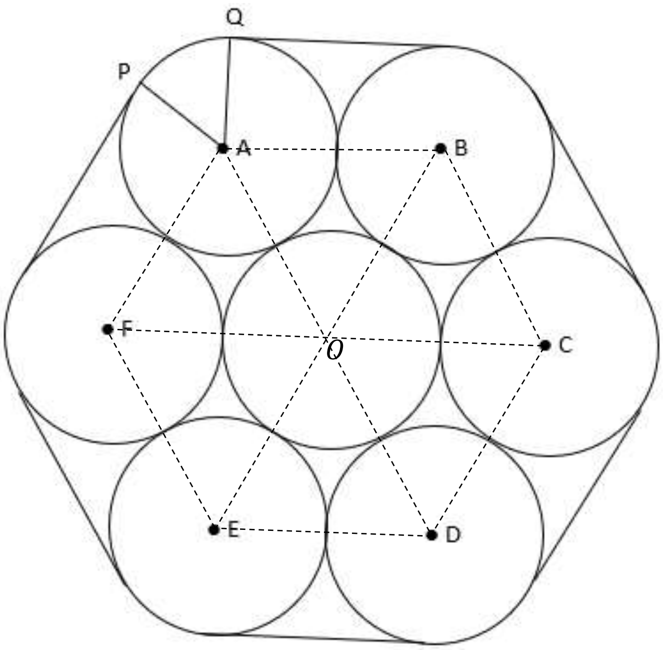
  $= \frac{k+1}{5^{k+1}}$

- This is the proposed formula, so by M.I. the statement is true for all $n \in N$

**[11]**

## QUESTION 3

The diagram shows a cross-section of seven cylindrical pipes, each of radius $20cm$, held together by a thin rope which is wrapped tightly around the pipes. The centres of the six outer pipes are $A, B, C, D, E$ and $F$. Points $P$ and $Q$ are situated where straight sections of the rope meet the pipe with centre $A$.




3.1 Given that $P\hat{A}Q = \frac{1}{3}\pi$ radians, find arc $PQ$, and hence the length of the rope in terms of $\pi$. (4)

$PQ = r\theta$

$= 20 \times \frac{\pi}{3}$

$= \frac{20\pi}{3}$ ✓

$\therefore Length\ of\ rope = 6PQ + 6(2 \times r)$

$= 6\left(\frac{20\pi}{3}\right)$ ✓ $+ 6(40)$ ✓

$= (40\pi + 240)cm$ ✓

3.2 Prove that the area of the hexagon $ABCDEF$ is $2400\sqrt{3}\ cm^2$. (5)

$Area = 6 \times A_{\Delta ABO}$

$= 6 \times \frac{1}{2}(40)(40)sin\frac{\pi}{3}$ ✓ ✓ ✓

$A\hat{O}B = \frac{2\pi}{6} = \frac{\pi}{3}$

$= 2400\frac{\sqrt{3}}{2}$ ✓

$= 2400\sqrt{3}\ cm^2$ ✓

Or $h = \sqrt{40^2 - 20^2}$

$= 20\sqrt{3}$

$Area = 6 \times \frac{40 \times 20\sqrt{3}}{2}$

$= 2400\sqrt{3}\ cm^2$

3.3 Find the area of sector $PAQ$, and hence the area of the complete region enclosed by the rope. Give your final answer to two decimal places. (7)

$A_{PAQ} = \frac{1}{2}r^2\theta$

$= \frac{1}{2}(20)^2 \times \frac{\pi}{3}$ ✓

$= \frac{200\pi}{3}$ ✓

$Area = (6 \times PAQ) + (6 \times AQRB) + (ABCDEF)$

$= \left(6 \times \frac{200\pi}{3}\right) + (6 \times 40 \times 20) + (2400\sqrt{3})$ ✓ ✓ ✓ ✓

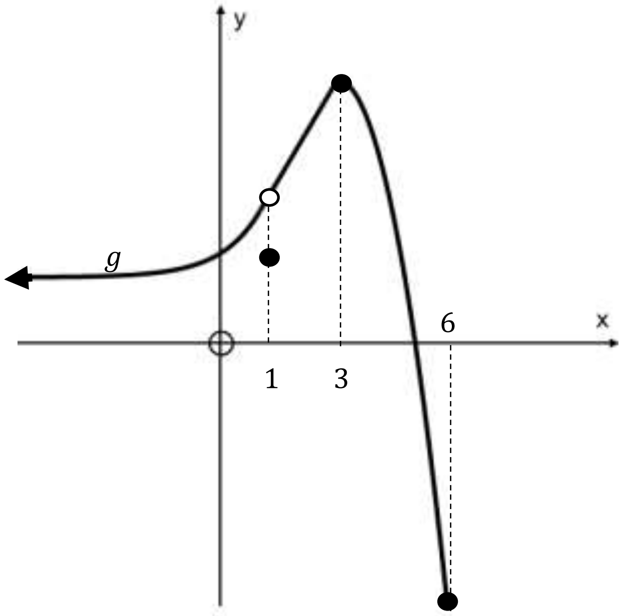
$= 400\pi + 4800 + 2400\sqrt{3}$

$= 10\ 213{,}56\ cm^2$ ✓

**[16]**

## QUESTION 4

Given: $g(x) = \begin{cases} e^{x-1} + a & if \quad x < 1 \\ 1{,}5 & if \quad x = 1 \\ x + 1 & if \quad 1 < x \leq 3 \\ -x^2 + 6x - 5 & if \quad 3 < x \leq 6 \end{cases}$




4.1 Determine the value of $a$ such that the limit at $x = 1$ exists. (4)

$$\lim_{x\to 1^-}(e^{x-1} + a) = \lim_{x\to 1^+}(x + 1) \quad \checkmark \checkmark$$

$\therefore 1 + a = 2$ ✓

$a = 1$ ✓

4.2 Using the diagram, identify the type of discontinuity that exists at $x = 1$. Give a reason. (2)

Removable discontinuity ✓

$\lim_{x \to 1^-} g(x) = \lim_{x \to 1^+} g(x) \neq g(1)$ ✓

4.3 Given that the graph of $g$ is continuous at $x = 3$, determine whether $g$ is also differentiable at $x = 3$? Show all working. (4)

$\lim_{x \to 3^-} g'(x)$

$= \lim_{x \to 3^-} (1)$

$= 1$ ✓

$\lim_{x \to 3^+} g'(x)$

$= \lim_{x \to 3^+} (-2x + 6)$

$= -2(3) + 6$

$= 0$ ✓

$\therefore \lim_{x \to 3^-} g'(x) \neq \lim_{x \to 3^+} g'(x)$ ✓

$\therefore g(x) is\ not\ differentiable\ at\ x = 3$ ✓

**[10]**

## QUESTION 5

5.1 Determine $f'(x)$ by first principles if $f(x) = \sqrt{x+3}$ (7)

$f(x) = \sqrt{x+3}$ $\qquad f(x+h) = \sqrt{x+h+3}$

$$\therefore f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h+3} - \sqrt{x+3}}{h} \times \frac{\sqrt{x+h+3} + \sqrt{x+3}}{\sqrt{x+h+3} + \sqrt{x+3}} \checkmark \checkmark$$

$$= \lim_{h \to 0} \frac{x+h+3-(x+3)}{h\left(\sqrt{x+h+3} + \sqrt{x+3}\right)} \checkmark \checkmark$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h+3}+\sqrt{x+3}} \checkmark \checkmark$$

$$= \frac{1}{2\sqrt{x+3}} \checkmark$$

5.2 Find $\frac{dy}{dx}$ if $y = \frac{sec2x}{x^2}$ , without simplifying. (5)

$$\frac{dy}{dx} = \frac{\checkmark 2sec2xtan2x \,.\, \checkmark x^2 - \checkmark 2x.\checkmark sec2x}{x^4 \checkmark}$$

5.3 Determine $D_x\left[ln\left(\frac{x-4}{4x-1}\right)\right]$ and simplify your answer. (5)

$$D_x\left[ln\left(\frac{x-4}{4x-1}\right)\right]$$

$$= \frac{4x-1}{x-4} \checkmark \times \frac{(4x-1) \checkmark -4(x-4)}{(4x-1)^2 \checkmark} \checkmark$$

$$= \frac{1}{x-4} \times \frac{15}{4x-1}$$

$$= \frac{15}{(x-4)(4x-1)} \checkmark$$

Or

$$D_x[ln(x-4) - ln(4x-1)]$$

$$= \frac{1}{x-4} - \frac{4}{4x-1}$$

$$= \frac{4x-1-(4x-16)}{(x-4)(4x-1)}$$

$$= \frac{15}{(x-4)(4x-1)}$$

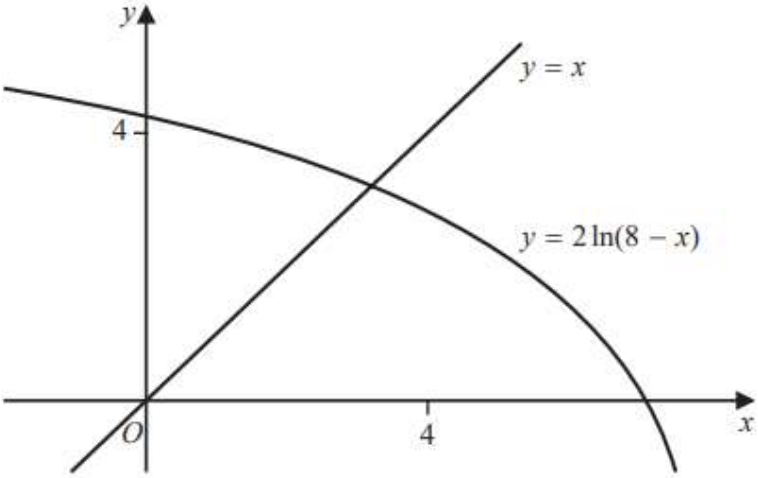
5.4 Given: $f(x) = cosx$ and $g(x) = \frac{2}{e^{2x}}$ , determine $\frac{d}{dx}\left(f(g(x)\right)$. (5)

$$f(g(x)) = cos(2e^{-2x}) \checkmark$$

$$\frac{d}{dx}\left(f(g(x))\right) = -sin(\checkmark 2\checkmark e^{-2x}).(-4\checkmark e^{-2x})$$

$$= 4e^{-2x}.sin(2e^{-2x}) \checkmark$$

5.5 The curve with equation $y = 2\ln(8-x)$ meets the line $y = x$ at a single point, $x = \alpha$




(a) Show that $3 < \alpha < 4$ (4)

| | |
|---|---|
| $2\,ln(8-3) - 3 = 0{,}227\ldots\ldots > 0$ ✓<br>$2\,ln(8-4) - 4 = -1{,}227\ldots\ldots < 0$ ✓<br><br>$\therefore$ *the function* $y = 2\,ln(8-x) - x$ *goes from above to below the* $x-$*axis and* ✓ *is continuous* ✓ $\therefore 3 < \alpha < 4$ | Or<br>$y = 2\,ln(8-3) = 3{,}2188\ldots$ and $y = 3$<br>$\therefore y = 2\ln(8-x)$ ***lies above*** $y = x$<br>$y = 2\,ln(8-4) = 2{,}772\ldots$ and $y = 4$<br>$\therefore y = 2\ln(8-x)$ ***lies below*** $y = x$<br>and graphs are continuous $\therefore 3 < \alpha < 4$ |

(b) Use Newton's method to determine the $x-$ coordinate of this point of intersection to 4 decimal places. (6)

Let $f(x) = 2\,ln(8-x) - x$

$\therefore f'(x) = \frac{-2}{8-x} - 1$ ✓

$a_{n+1} = a_n - \dfrac{2\ln(8-a_n) - a_n}{\frac{-2}{8-a_n} - 1}$ ✓ ✓

$Let\ a_0 = 3{,}5$ ✓

$a_1 = 3{,}159495\ldots\ldots$

$a_2 = 3{,}15563\ldots\ldots$ ✓

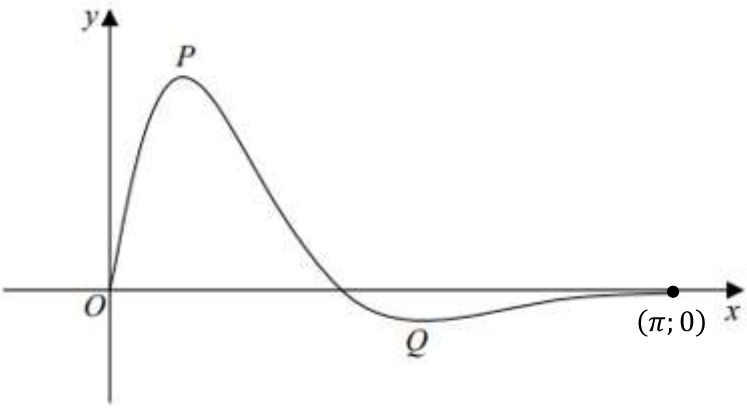
$a_3 = 3{,}15563\ldots\ldots$ $\therefore \alpha = 3{,}1556$ ✓

**[32]**

## QUESTION 6

6.1 Below is the sketch of the curve with equation $y = f(x)$, where

$$f(x) = \frac{4sin2x}{e^{\sqrt{2}x-1}} \ ; 0 \leq x \leq \pi$$




The curve has a maximum turning point at $P$ and a minimum turning point at $Q$.

Show that the $x$ –coordinate of $P$ and $Q$ can be found by solving the equation:

$tan2x = \sqrt{2}$ (Do not solve this equation)

$$f'(x) = \frac{8cos2x.\left(e^{\sqrt{2}x-1}\right) - 4sin2x\left(\sqrt{2}\ e^{\sqrt{2}x-1}\right)}{\left(e^{\sqrt{2}x-1}\right)^2} \checkmark\checkmark\checkmark\checkmark\checkmark$$

$$0 = \frac{4\ e^{\sqrt{2}x-1}(2cos2x - \sqrt{2}\ sin2x)}{\left(e^{\sqrt{2}x-1}\right)^2} \checkmark\checkmark$$

$0 = 2cos2x - \sqrt{2}\ sin2x$ ✓

$0 = 2 - \sqrt{2}\ tan2x$

$tan2x = \frac{2}{\sqrt{2}}$ ✓

$tan2x = \sqrt{2}$ ✓

Or $0 = \frac{4\ (2cos2x - \sqrt{2}\ sin2x)}{e^{\sqrt{2}x-1}}$

$0 = 8cos2x - 4\sqrt{2}\ sin2x$

$\frac{8cos2x}{cos2x} = \frac{4\sqrt{2}\ sin2x}{cos2x}$

$8 = 4\sqrt{2}\ tan2x$

$tan2x = \frac{8}{4\sqrt{2}}$ ✓

$tan2x = \sqrt{2}$

(10)

6.2 Below are the graphs of four rational functions.
Below are five equations representing rational functions:

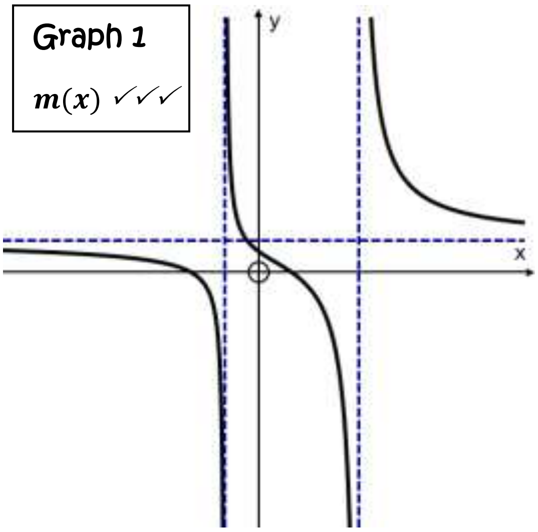
$$f(x) = \frac{x^2+x-2}{x^2-x-6} = \frac{(x-1)(x+2)}{(x+2)(x-3)} \qquad g(x) = \frac{x^2+2x+1}{x-1}$$

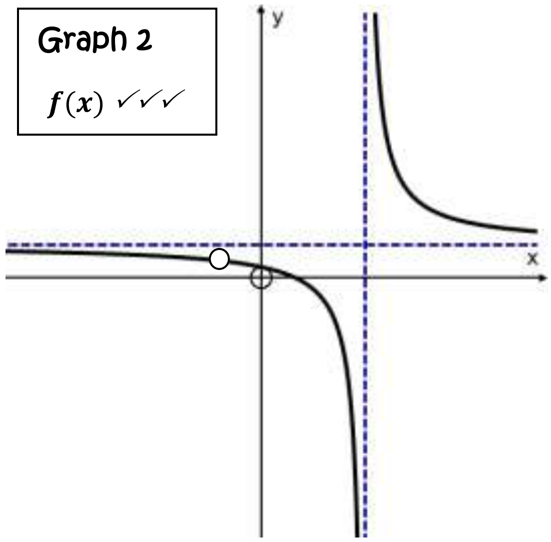
$$k(x) = \frac{x^2-x-2}{2-x} \qquad m(x) = \frac{x^2+x-2}{x^2-2x-3}$$

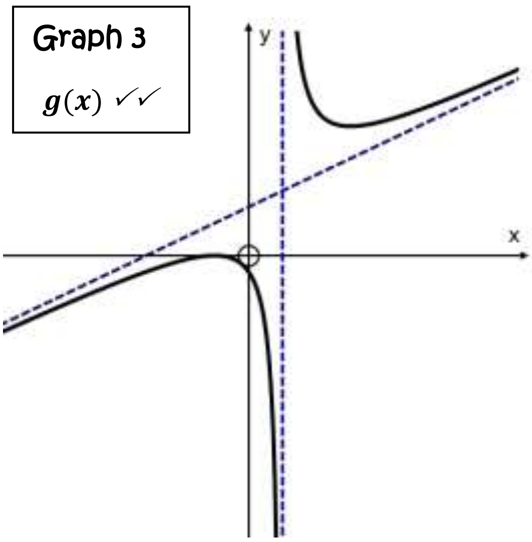
$$n(x) = \frac{x^3+4x^2+x-6}{x^2-1}$$

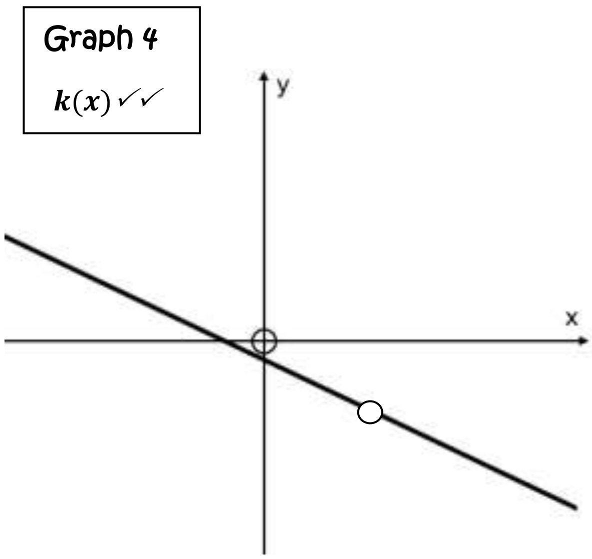
(a) Fill in the name of the correct function in the boxes:

For example: Graph 6 $p(x)$ (10)



(b) Show algebraically that **Graph 2** is always decreasing. (8)

$f(x) = \dfrac{x-1}{x-3}\ ; x \neq 2$ ✓

$f'(x) = \dfrac{1(x-3)-(x-1)1}{(x-3)^2}$ ✓ ✓ ✓

$= \dfrac{-2}{(x-3)^2} < 0$ ✓ ✓

$\therefore f'(x)$ *will always be* $-$ *ve* ✓ *and* $\therefore$ *graph* 2 *will always be decreasing* ✓

**[28]**

## QUESTION 7

7.1 The area under the curve of $f(x)$ from $x = 2$ to $x = b$ can be calculated by constructing $n$ rectangles of equal width and then using a Riemann sum as follows:

$$\boldsymbol{Area = \frac{3}{n}\sum_{i=1}^{n}\left[-\left(2+\frac{3i}{n}\right)^2 + 5\left(2+\frac{3i}{n}\right)\right]}$$
$$\boldsymbol{= \frac{27}{2} - \frac{9}{n} - \frac{9}{2n^2}}$$

(a) What is the function $f(x)$? (2)

$f(x) = -x^2 + 5x$ ✓ ✓

(b) Determine the value of $b$. (2)

$b - 2 = 3$ ✓

$b = 5$ ✓

(c) Calculate the approximate area if the region is divided into 3 rectangles. (2)

$A = \frac{27}{2} - \frac{9}{3} - \frac{9}{2(3)^2}$ ✓

$= 10\ units^2$ ✓

(d) Is the estimated area in (c) an over-estimation or an under-estimation of the exact area? Explain. (3)

$Under - estimation$

If $n = 10, A = 12{,}555$ ✓

✓ ✓

$\therefore$ *Area getting bigger and closer to the exact area as n increases.*

(e) Determine $\int_2^b f(x)dx$, using the Riemann sum given above. (2)

$$\int_2^5 f(x)dx = \lim_{n\to\infty}\left(\frac{27}{2} - \frac{9}{n} - \frac{9}{2n^2}\right) \checkmark$$

$$= \frac{27}{2} \quad \checkmark$$

(f) Calculate the value of $\int_2^b [-f(x)]dx$ and explain your answer. (2)

$$\int_2^5 -f(x)dx = -\frac{27}{2} \quad \checkmark$$

$Reflection\ about\ x - axis \checkmark \therefore Negative\ as\ area\ lies\ below\ x - axis$

7.2 Determine the following without a calculator and showing all working:

(a) $\int 3x(x^2+4)^5 dx$ (do not use substitution) (5)

$\int 3x(x^2+4)^5 dx$

$= \frac{3}{2}\int 2x(x^2+4)^5 dx \quad \checkmark \quad \checkmark \quad \checkmark$

$= \frac{1}{4}(x^2 \overset{\checkmark}{+} 4)^6 + c \quad \checkmark$

(b) $\int 2x\sqrt{x+2}\,dx$ (use a $u$ −substitution) (9)

Let $u = x + 2 \quad \therefore \quad x = u - 2$ ✓

$\therefore \frac{du}{dx} = 1 \qquad \therefore du = dx$ ✓

$\int 2(u-2).u^{\frac{1}{2}}\,du$ ✓

$= \int \left(2u^{\frac{3}{2}} - 4u^{\frac{1}{2}}\right)\,du$ ✓ ✓

$= \frac{4}{5}u^{\frac{5}{2}} - \frac{8}{3}u^{\frac{3}{2}} + c$ ✓ ✓

$= \frac{4}{5}(x+2)^{\frac{5}{2}} - \frac{8}{3}(x+2)^{\frac{3}{2}} + c$ ✓ ✓

(c) (i) Decompose $\frac{3x+3}{x^2+x-2}$ into its partial fractions. (7)

$\frac{3x+3}{(x-1)(x+2)}$ ✓ $= \frac{A}{x-1} + \frac{B}{x+2}$ ✓

$A = \frac{3(1)+3}{(1)+2} = 2$ ✓✓

$B = \frac{3(-2)+3}{(-2)-1} = 1$ ✓✓

$\frac{3x+3}{x^2+x-2} = \frac{2}{x-1} + \frac{1}{x+2}$ ✓

(ii) Hence determine: $\int \frac{3x+3}{x^2+x-2} dx$ (4)

$\int \frac{2}{x-1} + \frac{1}{x+2} dx$ ✓

$= 2ln|x-1| + ln|x+2| + c$ ✓✓✓

(d) $\int \frac{1}{1+cot^2 x} dx$ (6)

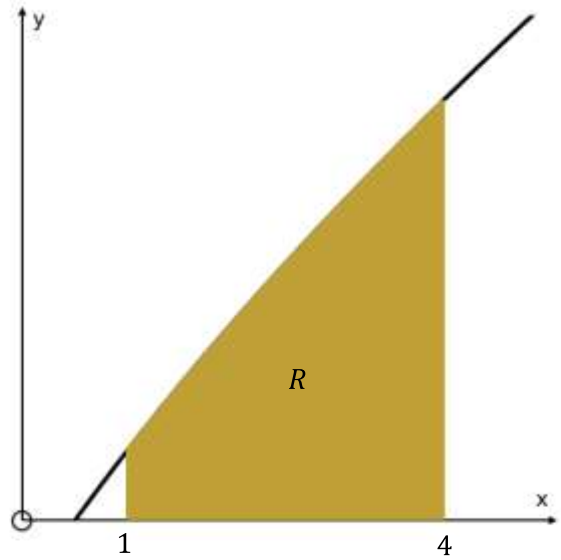
$= \int \frac{1}{cosec^2 x} dx$ ✓

$= \int sin^2 x \, dx$ ✓

$= \int \left(\frac{1}{2} - \frac{1}{2}cos2x\right) dx$ ✓✓

$= \frac{x}{2} - \frac{1}{4}sin2x + c$ ✓✓

7.3 The sketch below shows a part of the curve with equation $y = x^{\frac{1}{2}} \ln 2x$.
The finite region $R$ is bounded by the curve, the $x$ –axis and the lines $x = 1$ and $x = 4$.




(a) Khanyisile started off correctly working out the indefinite integral

$\int x^{\frac{1}{2}} ln\, 2x \;\; dx$ using integration by parts.

Complete her work:

$$\int x^{\frac{1}{2}} ln\, 2x \;\; dx$$

Let $f(x) = ln\, 2x$ $\quad\therefore f'(x) =$ $\frac{2}{2x} = \frac{1}{x}$ ✓

$g'(x) =$ $x^{\frac{1}{2}}$ ✓ $\quad\therefore g(x) =$ $\frac{2}{3}x^{\frac{3}{2}}$ ✓

$\therefore \int x^{\frac{1}{2}} ln\, 2x \;\; dx =$

$$\frac{2}{3}x^{\frac{3}{2}}.\, ln\, 2x - \int \frac{2}{3}x^{\frac{3}{2}}.\frac{1}{x}\, dx$$ ✓ ✓

$$= \frac{2}{3}x^{\frac{3}{2}}.\, ln\, 2x - \int \frac{2}{3}x^{\frac{1}{2}}\, dx$$ ✓

$$= \frac{2}{3}x^{\frac{3}{2}}.\, ln\, 2x - \frac{2}{3}.\frac{2}{3}\, x^{\frac{3}{2}} + c$$ ✓ ✓

$$= \frac{2}{3}x^{\frac{3}{2}}.\, ln\, 2x - \frac{4}{9}\, x^{\frac{3}{2}} + c$$ ✓

(9)

(b) Using your integral in (a), determine the exact area of $R$, giving your answer in the form:

$a\,ln\,2 + b$, where $a$ and $b$ are exact constants. (8)

$A = \left[ \frac{2}{3}x^{\frac{3}{2}}.\, ln\,2x - \frac{4}{9}\, x^{\frac{3}{2}} \right]_1^4$ ✓

$= \left[\frac{2}{3}(4)^{\frac{3}{2}}.\, ln\,2(4) - \frac{4}{9}\,(4)^{\frac{3}{2}}\right] - \left[\frac{2}{3}(1)^{\frac{3}{2}}.\, ln\,2(1) - \frac{4}{9}\,(1)^{\frac{3}{2}}\right]$ ✓ ✓

$= \left(\frac{16}{3}\,ln8 - \frac{32}{9}\right) - \left(\frac{2}{3}\,ln2 - \frac{4}{9}\right)$ ✓ ✓

$= \frac{16}{3}\,ln2^3 - \frac{32}{9} - \frac{2}{3}\,ln2 + \frac{4}{9}$

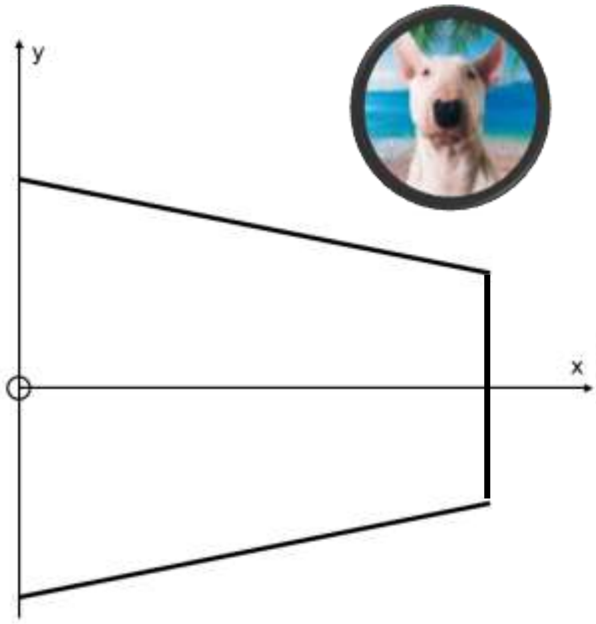
$= 16ln2 - \frac{32}{9} - \frac{2}{3}\,ln2 + \frac{4}{9}$ ✓

$= \frac{46}{3}\,ln2 - \frac{28}{9}$ ✓ ✓

**[61]**

## QUESTION 8

Mrs Vermeulen's bull terrier had a minor operation. He had to wear a head piece, shaped as a truncated cone, so that he would not lick his stitches.
She measures the diameter of the top and bottom circles and finds them to be $22cm$ and $40cm$ respectively. The vertical height of the head piece is $25cm$.





Using the diagram on the cartesian plane above as a guide, Mrs Vermeulen wants you to determine the integral for the **volume** of her dog's head piece in terms of $\pi$. Do not work out the value of the integral.

$m = \frac{20-11}{0-25}$ ✓✓

$= -\frac{9}{25}$ ✓

$\therefore y = -\frac{9}{25}x + 20$ ✓

$V = \pi \int_0^{25} \left(-\frac{9}{25}x + 20\right)^2 dx$ ✓✓✓✓

(8)

**[8]**

**Total: 200 marks**

## Extra working space