GR 12 APM

JULY 2012 (Memo)

Question 1

① $\underline{n=1:}$ $3^1 + 3^{(1)+1} + 3^{(1)+2}$
$= 3 + 9 + 27$
$= 39$
which is divisible by 13.

② $\underline{\text{Assume}}$: $3^k + 3^{k+1} + 3^{k+2}$
$= 13r, \quad r \in \mathbb{N}$

③ $\underline{\text{Prove}}$: $3^{k+1} + 3^{(k+1)+1} + 3^{(k+1)+2}$

$= 3^k.3 + 3^{k+1}.3 + 3^{k+2}.3$
$= 3(3^k + 3^{k+1} + 3^{k+2})$
$= 3(13r)$
$= 39r$
which is divisible by 13

$\therefore$ $3^n + 3^{n+1} + 3^{n+2}$ is divisible by 13 for all natural values of n.

[10]

Question 2

a) $x^3 + 2x^2 + 4x + 8 = 0$
$\therefore x^2(x+2) + 4(x+2) = 0$
$\therefore (x+2)(x^2 + 4) = 0$
$\therefore \underline{x = -2}$ or $\underline{x = \pm 2i}$

[5]

b) (i) $(a+bi)(c-i) = 5+i$ ①
$\therefore ac - bi^2 + bci - ai = 5+i$
$\therefore ac + b + (bc-a)i = 5+i$

$a+bi + c-i = 3+2i$ ②
$\therefore a+c + (b-1)i = 3+2i$

NOW equate real and imaginary part
$\therefore ac+b = 5$ ③
$bc - a = 1$ ④
$a + c = 3$ ⑤
$b - 1 = 2$ ⑥
$\therefore \underline{b = 3}$

$\underline{\text{Sub into ④}}$
$3c - a = 1$
and $a + c = 3$ ⑤

$\underline{④ + ⑤:}$ $4c = 4$
$\therefore \underline{c = 1}$

$\underline{\text{Sub back into}}$ ④: $3(1) - a = 1$
$\therefore \underline{2 = a}$

(8)

c) (i) $\lim_{x\to 0} \frac{5\sin 2x}{2\sin 2x\cos 2x}$

$= \lim_{x\to 0} \frac{5}{2\cos 2x}$

$= 5/2$ (6)

(ii) $\lim_{x\to\infty} \frac{x}{\sqrt{x^2-x}}$

$= \lim_{x\to\infty} \frac{1}{\sqrt{1-\frac{1}{x}}}$

$= 1$ (8)

d) i) (1) $|x-1| - 4 = 0$

If $(x-1) \geq 0$, $x \geq 1$

$x-1 = 4$

$\therefore x = 5$

If $(x-1) < 0$, $x < 1$

$x - 1 = -4$

$\therefore x = -3$ (3)

(2) $(1 ; -4)$ (1)

(ii) $|x-1| - 4 = 2x + 3$

If $(x-1) \geq 0$, $x \geq 1$

$x - 1 - 4 = 2x + 3$

$-8 \neq x$

If $(x-1) < 0$, $x < 1$

$-(x-1) - 4 = 2x + 3$

$\therefore -x + 1 - 4 = 2x + 3$

$\therefore -6 = 3x$

$\therefore -2 = x$

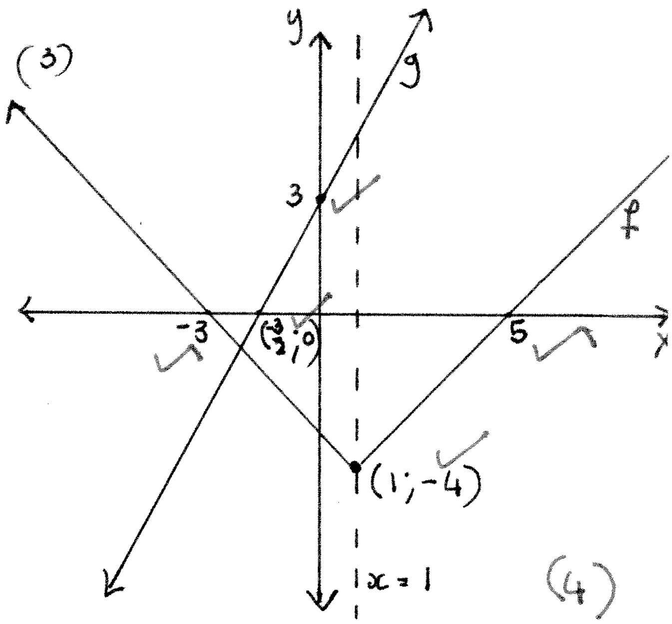
$y = 2(-2) + 3$

$= -1$

$(-2 ; -1)$ (5)

iii) $x < -2$ (2)

(3)



(4)

Question 3

a) $\ln 45$

$= \ln 9 \times 5$ ✓

$= \ln 9 + \ln 5$

$= 2\ln 3$ ✓ $+ \ln 5$ ✓

$= 2(1,10) + 1,161$

$= 2,20 + 1,161$

$= \underline{3,361}$ ✓✓ (5)

b) $4\ln x - \frac{1}{2}\ln y + \ln z$

$= \ln x^4 - \ln\sqrt{y} + \ln z$

$= \underline{\ln \frac{x^4 . z}{\sqrt{y}}}$ ✓✓ (2)

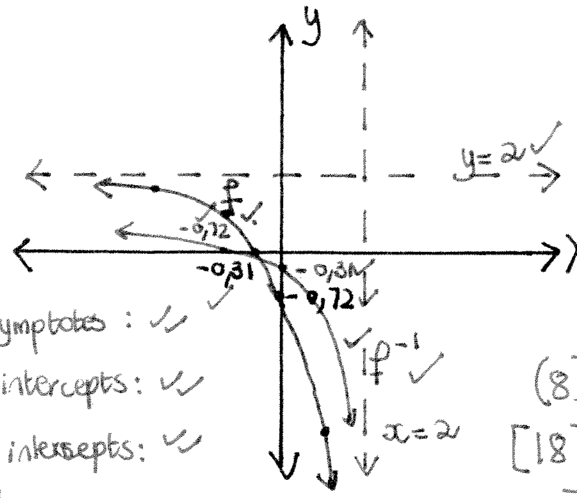
c) $34 = e^{10k}$

$\therefore \ln 34$ ✓ $= 10k \ln e$ ✓

$\therefore \frac{\ln 34}{10 . \ln e} = k$

$\therefore \underline{0,35 = k}$ ✓ (3)

d)



Asymptotes : ✓✓

X-intercepts: ✓✓

Y-intersepts: ✓✓

Shape: ✓✓

(8)

[18]

X-intercept: $e^{x+1} = 2$

$\therefore (x+1)\ln e = \ln 2$

$\therefore x = \frac{\ln 2}{\ln e} - 1$

$\therefore x = -0,31$

Question 4

a) Area of $\triangle$ − Area of sector ✓✓

$= \frac{1}{2}(10)(10)\sin 0,8 - \frac{1}{2}(6)^2 0,8$ ✓✓

$= \underline{21,47 \text{ cm}^2}$ ✓ (5)

b) $\hat{C} = \frac{\pi - 0,8}{2} = 1,17$ radians ✓

$\frac{CD}{\sin 0,8} = \frac{10}{\sin 1,17}$ ✓✓

$\therefore CD = 7,79$ cm ✓

$AB = 6 \times 0,8 = 4,8$ ✓

Perimeter $= 2(4) + 7,79 + 4,8$

$= \underline{20,59 \text{ cm}}$ ✓ (6)

[11]

Question 5

$LHS = \frac{1 - \sin x + 1 + \sin x}{(1+\sin x)(1-\sin x)}$ ✓✓✓

$= \frac{2}{1-\sin^2 x}$ ✓

$= \frac{2}{\cos^2 x}$

$= 2\sec^2 x$ ✓

$= RHS$

(5)

Question 6

a) (i) $\frac{dy}{dx} = 4(3x^2+1)^3(6x)$ ✓✓✓ (3)

$= \underline{24x(3x^2+1)^3}$

(ii) $f'(x) = x^3 . \cos x + 3x^2 \sin x + 10 \sin 2x$ ✓✓✓

(3)

iii) $\frac{dy}{dx} = \frac{2x^2 \cdot \sec 2x \tan 2x - 2x \cdot \sec 2x}{(x^2)^2}$ (3)

$= \frac{2x \sec 2x(x \tan 2x - 1)}{x^4}$

$= \frac{2 \sec 2x (x \tan 2x - 1)}{x^3}$

b) i) $\lim_{x \to -2^-} 3x+7$ $\lim_{x \to -2^+} x^2 - 3$

$= 1$ $= 1$

$\therefore \lim_{x \to -2} f(x) = 1$ (5)

(ii) $f(-2) = 0$

$\therefore f(-2) \neq \lim_{x \to -2} f(x)$

$\therefore$ Removable discontinuity at $x = -2$ (3)

(iii) NOT differentiable at $x = -2$, as it is NOT continuous. (2)

[19]

Question 8

a) y-intercept: $y = -8$

SP: $f'(x) = 0$

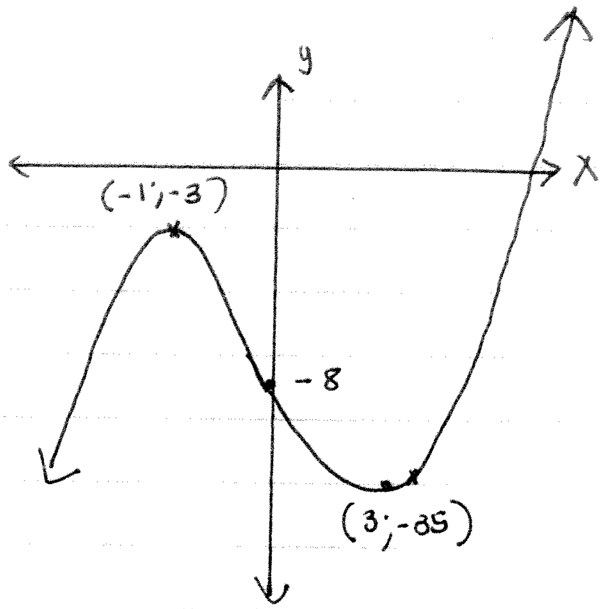
$3x^2 - 6x - 9 = 0$

$\therefore x^2 - 2x - 3 = 0$

$\therefore (x-3)(x+1) = 0$

$\therefore x = 3$ or $x = -1$

$y = -35$ $y = -3$



b) Guess $a_1 = 5$

$f'(x) = 3x^2 - 6x - 9$

$a_2 = a_1 - \frac{(a_1)^3 - 3(a_1)^2 - 9(a_1) - 8}{3(a_1)^2 - 6(a_1) - 9}$

$= 5,0811$ [4]

Question 7

i) $3x - 1 = 0$

$x = 1/3$

$\therefore \ a = 1/3$ (1)

ii) $f'(x) = 0$

$\therefore \dfrac{(3x-1)(2x) - x^2(3)}{(3x-1)^2} = 0$

$\therefore 6x^2 - 2x - 3x^2 = 0$

$\therefore 3x^2 - 2x = 0$

$\therefore x(3x - 2) = 0$

$\therefore x = 0$ or $x = \frac{2}{3}$ (8)

iii) $m = 0$ (1)

(iv) $f\left(\frac{2}{3}\right) = \dfrac{(2/3)^2}{3(2/3) - 1}$

$= 4/9$

$\therefore \ 0 < k < \frac{4}{9}$ (4)

(v)

$$\begin{array}{r|l} & \frac{x}{3} + \frac{1}{9} \\ \hline 3x - 1 & x^2 + 0x + 0 \\ & x^2 - \frac{1}{3}x \\ \hline & \quad \frac{1}{3}x + 0 \\ & \quad \frac{1}{3}x - \frac{1}{9} \\ \hline & \qquad \frac{1}{9} \end{array}$$

$f(x) = \dfrac{\left(\frac{x}{3} + \frac{1}{9}\right)(3x-1) + \frac{1}{9}}{(3x-1)}$

$= \dfrac{x}{3} + \dfrac{1}{9} + \dfrac{1/9}{(3x-1)}$

$\therefore \ y = \dfrac{x}{3} + \dfrac{1}{9}$ (8)

OR

$m = \lim\limits_{x \to \infty} \dfrac{f(x)}{x}$

$= \lim\limits_{x \to \infty} \dfrac{x^2}{3x^2 - x}$

$= \lim\limits_{x \to \infty} \dfrac{1}{3 - 1/x}$

$= \dfrac{1}{3}$

$c = \lim\limits_{x \to \infty} f(x) - \frac{1}{3}x$

$= \lim\limits_{x \to \infty} \dfrac{x^2}{3x - 1} - \dfrac{x}{3}$

$= \lim\limits_{x \to \infty} \dfrac{3x^2 - x(3x-1)}{3(3x-1)}$

$= \lim\limits_{x \to \infty} \dfrac{x}{9x - 3}$

$= \lim\limits_{x \to \infty} \dfrac{1}{9 - 3/x}$

$= 1/9$

$y = \frac{1}{3}x + \frac{1}{9}$ (8)

[22]

Question 9

a) $\frac{dy}{dx} = 0$

$28y^3\left(\frac{dy}{dx}\right) + 3x^2y + x^3\left(\frac{dy}{dx}\right) + 1 = 0$

$\therefore 28y^3\left(\frac{dy}{dx}\right) + x^3\left(\frac{dy}{dx}\right) = -3x^2y - 1$

$\therefore \frac{dy}{dx} = \frac{-3x^2y - 1}{28y^3 + x^3}$

$m = \frac{-3(-2)^2(1) - 1}{28(1)^3 + (-2)^3}$

$= \frac{-13}{20}$ $(-0,65)$

(8)

b) $y = (1+2x)^{-1}$

$y^1 = -1(1+2x)^{-2}\ (2)$

$y^2 = -1.-2.(1+2x)^{-3}(2)(2)$

$y^3 = -1.-2.-3(1+2x)^{-4}(2)(2)(2)$

$\therefore y^n = (-1)^n(n!)(1+2x)^{-(n+1)}.(2)^n$

(7)

[15]

Question 10

a) $\int \frac{1}{\sqrt{2-3x}}\,dx$

Let $u = 2-3x$

$\frac{du}{dx} = -3$

$\therefore du = -3\,dx$

$\therefore -\frac{1}{3}du = dx$

$= \int u^{-1/2}\left(-\frac{1}{3}du\right)$

$= -\frac{1}{3}\int u^{-1/2}\,du$

$= -\frac{1}{3}\left(2u^{1/2}\right)$

$= -\frac{2}{3}\sqrt{2-3x} + c$

(6)

b) $\int \sin x.(\cos x)^2\,dx$

Let $u = \cos x$

$\frac{du}{dx} = -\sin x$

$\therefore du = -\sin x\,dx$

$\therefore -du = \sin x\,dx$

$= \int u^2(-du)$

$= -\left(\frac{1}{3}u^3\right)$

$= -\frac{\cos^3 x}{3} + c$

(6)

c) $\int(\sec^2 x + 2\sec x.\tan x + \tan^2 x)\,dx$

$= \int(\sec^2 x + 2\sec x\tan x + \sec^2 x - 1)\,dx$

$= \int(2\sec^2 x + 2\sec x\tan x - 1)\,dx$

$= 2\tan x + 2\sec x - x + c$

(5)

d) ① Rectangle width:

$$\frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}$$

② Rectangle height:

- $x_i = a + i\left(\frac{b-a}{n}\right)$
  $= 1 + \frac{2i}{n}$

- $f(x_i) = 10 - \left(1 + \frac{2i}{n}\right)^2$
  $= 10 - \left(1 + \frac{4i}{n} + \frac{4i^2}{n^2}\right)$
  $= 9 - \frac{4i}{n} - \frac{4i^2}{n^2}$

③ Area

$$= \lim_{n\to\infty}\left[\frac{b-a}{n}\sum_{i=1}^{n} f(x_i)\right]$$

$$= \lim_{n\to\infty}\left[\frac{2}{n}\sum_{i=1}^{n}\left(9 - \frac{4i}{n} - \frac{4i^2}{n^2}\right)\right]$$

$$= \lim_{n\to\infty}\left[\frac{2}{n}\left(9n - \frac{4}{n}\left(\frac{n^2}{2}+\frac{n}{2}\right) - \frac{4}{n^2}\left(\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}\right)\right)\right]$$

$$= \lim_{n\to\infty}\left[\frac{2}{n}\left(9n - \frac{4n}{2} - \frac{4}{2} - \frac{4n}{3} - \frac{4}{2} - \frac{4}{6n}\right)\right]$$

$$= \lim_{n\to\infty}\left[18 - 4 - \frac{4}{n} - \frac{8}{3} - \frac{4}{n} - \frac{4}{3n^2}\right]$$

$$= \frac{34}{3} \text{ units}^2$$

(12)

[29]

Question 11

a) $\int_0^3 2x\sqrt{9-x^2}\, dx$

$$= \int_0^3 2x\left(9-x^2\right)^{1/2} dx$$

$$= \left[-\frac{2}{3}\left(9-x^2\right)^{3/2}\right]_0^3$$

$$= \left[-\frac{2}{3}\left(9-3^2\right)^{3/2}\right] - \left[-\frac{2}{3}\left(9-0^2\right)^{3/2}\right]$$

$$= 0 + \frac{2}{3}\left(3^2\right)^{3/2}$$

$$= 18 \text{ units}^2$$

(8)

b) $f'(x) = 0$

$\therefore\ 2\sqrt{9-x^2} + 2x\left(\frac{1}{2}\times\left(9-x^2\right)^{-1/2}(-2x)\right) = 0$

$\therefore\ 2\sqrt{9-x^2} - \frac{2x^2}{\sqrt{9-x^2}} = 0$

$\therefore\ 2\left(9-x^2\right) - 2x^2 = 0$

$\therefore\ 18 - 2x^2 - 2x^2 = 0$

$\therefore\ x^2 = \frac{18}{4}$

$\therefore\ x = \pm\frac{3}{\sqrt{2}}$

and $y = 2\left(\frac{3}{\sqrt{2}}\right)\sqrt{9-\left(\frac{3}{\sqrt{2}}\right)^2}$

$= \frac{6}{\sqrt{2}}\sqrt{\frac{9}{2}}$

$= \frac{18}{2}$

$= 9$

(8)

$\therefore\ A\left(\frac{3}{\sqrt{2}}\ ;\ 9\right)$

c) $V = \pi\int_0^{3/\sqrt{2}} y^2\, dx$

$$= \pi\int_0^{3/\sqrt{2}}\left(2x\sqrt{9-x^2}\right)^2 dx$$

$$= \pi\int_0^{3/\sqrt{2}} 4x^2\left(9-x^2\right) dx$$

$$= \pi\int_0^{3/\sqrt{2}}\left(36x^2 - 4x^4\right) dx$$

$$= \pi\left[12x^3 - \frac{4}{5}x^5\right]_0^{3/\sqrt{2}}$$

$$= \pi\left[\left(12\left(\frac{3}{\sqrt{2}}\right)^3 - \frac{4}{5}\left(\frac{3}{\sqrt{2}}\right)^5\right) - (0)\right]$$

$$= \pi\left[\frac{12}{1}\times\frac{27}{2\sqrt{2}} - \frac{4}{5}\times\frac{243}{4\sqrt{2}}\right]$$

$$= \pi\left[\frac{162}{\sqrt{2}} - \frac{243}{5\sqrt{2}}\right]$$

$$= 251{,}91 \text{ units}^3$$

(9)