GRADE 12 EXAMINATION
NOVEMBER 2010

# ADVANCED PROGRAMME MATHEMATICS

Time: 3 hours 300 marks

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**PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY**

1. This question paper consists of 17 pages, an Answer Sheet of 1 page and an Information Booklet of 4 pages (i – iv). Please check that your question paper is complete.

2. This question paper consists of FOUR Modules:

   **MODULE 1: CALCULUS AND ALGEBRA (200 marks) is compulsory.**

   Choose **ONE** of the **THREE** Optional Modules:

   **MODULE 2: STATISTICS (100 marks)** OR
   **MODULE 3: FINANCE AND MODELLING (100 marks)** OR
   **MODULE 4: MATRICES AND GRAPH THEORY (100 marks)**

3. Non-programmable and non-graphical calculators may be used, unless otherwise indicated.

4. All necessary calculations must be clearly shown and writing should be legible.

5. Diagrams have not been drawn to scale.

6. Write all your answers in the answer book provided.

7. Round off your answers to two decimal digits, unless otherwise indicated.

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## MODULE 1 CALCULUS AND ALGEBRA

### QUESTION 1

Prove by using mathematical induction that

$$1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \ldots + n\left(\frac{1}{2}\right)^{n-1} = 4 - \frac{n+2}{2^{n-1}}, \quad \text{for all} \quad n \in \mathrm{N}$$ (15)

**[15]**

### QUESTION 2

Newton's law of cooling for a liquid, in this case a cup of soup, is given by the equation

$$\mathrm{T} = \mathrm{T}_s + (\mathrm{T}_0 - \mathrm{T}_s)\mathrm{e}^{-kt}$$

where

- $t$ is the time in minutes
- $k$ is a constant for the specific fluid
- $\mathrm{T}$ is the temperature in °C at any given time $t$
- $\mathrm{T}_0$ is the initial temperature in °C (the value of $\mathrm{T}$ at $t = 0$)
- $\mathrm{T}_s$ is the surrounding temperature in °C

2.1 A cup of soup cooled from 90° C to 60° C in 10 minutes in a room where the temperature was 20° C. Show, by solving $k$ and showing all working, that $k = -\frac{1}{10}\ \ell n\left(\frac{4}{7}\right)$. (8)

2.2 Determine the temperature of the soup after 15 minutes (from the beginning). **Give the answer correct to the nearest whole number.** (3)

2.3 Give the equation of the asymptote of the graph of $\mathrm{T}$. (2)

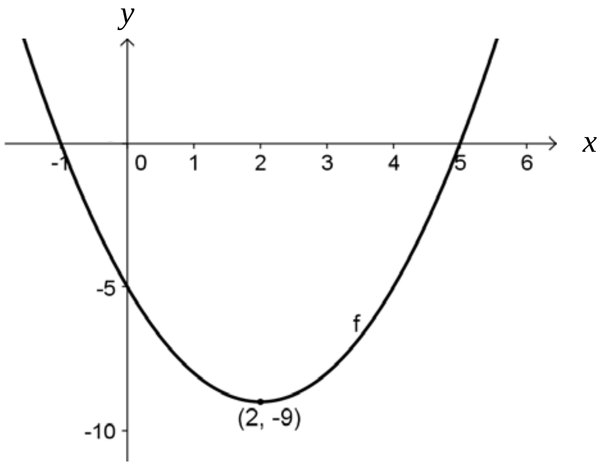
2.4 Explain the meaning of this asymptote in real life terms. (2)

**[15]**

## QUESTION 3

3.1 Determine all the Real solutions to the equation $|x|^2 - 4|x| - 5 = 0$. You may not use a calculator and must show all your working. (8)

3.2 The given sketch shows the graph of $y = f(x) = x^2 - 4x - 5$.




Using the information on the sketch, draw on **separate sets** of axes, showing all intercepts on the axes and stationary points, the graphs of:

(a) $y = |f(x)|$ (4)

(b) $y = f(|x|)$ (You may use your answers from 3.1) (7)

**[19]**

## QUESTION 4

Given $p(x) = x^3 + ax^2 + bx - 6$ with a zero at $x = 1 + i$.

4.1 Determine the values of $a$ and $b$. (7)

4.2 Give the Real root of the equation, $p(x) = 0$. (2)

4.3 Hence, or otherwise, solve for $x$ if $x^3 - 5x^2 + 8x - 6 > 0$. (2)

**[11]**

**QUESTION 5**

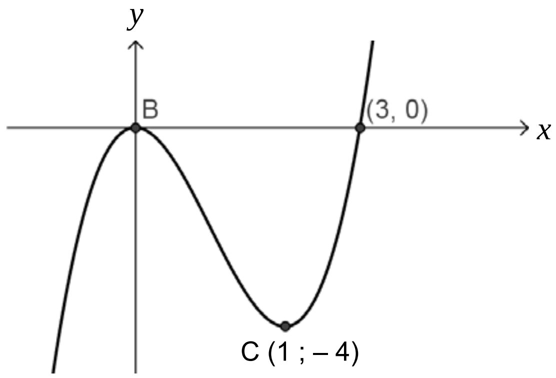
5.1 $g(x) = \begin{cases} \dfrac{(x-3)(x+1)}{x-3} & \text{if } x < -1 \\ x^2+1 & \text{if } x \geq -1 \end{cases}$

Determine, with algebraic motivation, whether $g$ is continuous at the following points, and state the type of discontinuity if applicable:

(a) $x = 3$ (4)

(b) $x = -1$ (5)

5.2 The following sketch shows the graph of $y = f'(x)$. This graph cuts the $x$-axis at (3; 0) and B(0; 0), and it has stationary points at B and at C(1; -4).




The graph of $f$ has two stationary points. Give the $x$-values of these points **and** state, with motivation, the nature of these points. (Remember that the sketch represents $f'(x)$.) (7)

**[16]**

**QUESTION 6**

6.1 If $f(x) = \dfrac{1}{1-2x}$, determine $f'(x)$ from **first principles**. (8)

6.2 Determine $\dfrac{dy}{dx}$ if $y = \sin y . \sin x$. (8)

6.3 If $y = \tan x$,

(a) determine $y''$ (5)

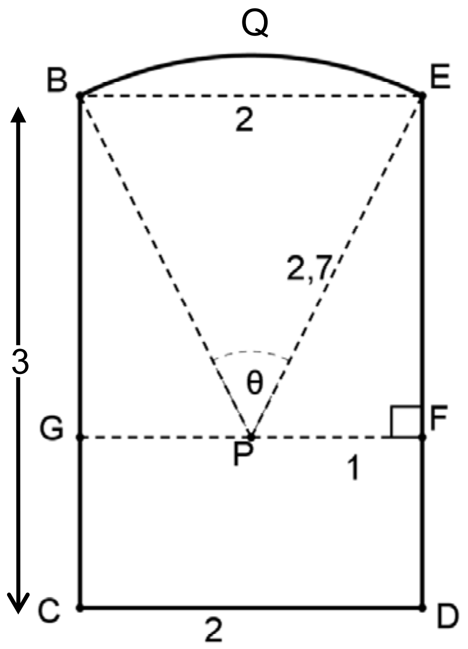
(b) prove that $y'' - 2y = 2y^3$ (5)

**[26]**

**QUESTION 7**

Pray-More Church needs to replace a window in the shape of a rectangle with an arched top. This top is an arc of a circle with centre at P and radius 2,7 m, as seen in the diagram below. BC = 3 m, CD = 2 m. P is the midpoint of FG.

(The diagram is not drawn to scale).




7.1 Calculate the value of θ in radians. (Hint: use the cosine rule in ΔBPE). Give the answer **correct to 3 decimal digits**. (5)

7.2 Calculate the area of the window. (9)

**[14]**

**QUESTION 8**

8.1 Determine the following integrals:

(a) $\int \frac{1}{\sqrt{2-3x}}\,dx$ (5)

(b) $\int x\sin x\,dx$ (5)

(c) $\int x.\cos\left(x^2\right)dx$ (5)

8.2 Use a **Riemann Sum** to determine $\int_0^2 \left(x^2+x\right)dx$. (13)

**[28]**

**QUESTION 9**

$f(x)=\dfrac{3(x-6)(x-1)}{(x-2)(x+1)}$. $f$ has no stationary points, but it has a point of inflection at $x\approx 0,8$.

9.1 Calculate the approximate $y$-value of the point of inflection. (2)

9.2 Write down all intercepts with the axes and equations of the horizontal and vertical asymptotes. (7)

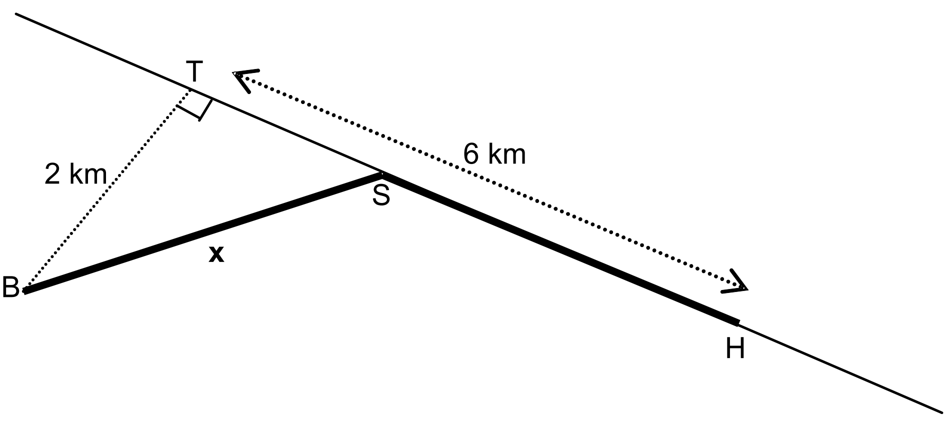
9.3 The graph of $f$ cuts the horizontal asymptote at a point. Calculate the $x$-value of this point. (5)

9.4 Hence, draw a sketch graph of $f$, showing all intercepts, asymptotes, the inflection point as well as the point calculated in 9.3. (9)

**[23]**

**QUESTION 10**

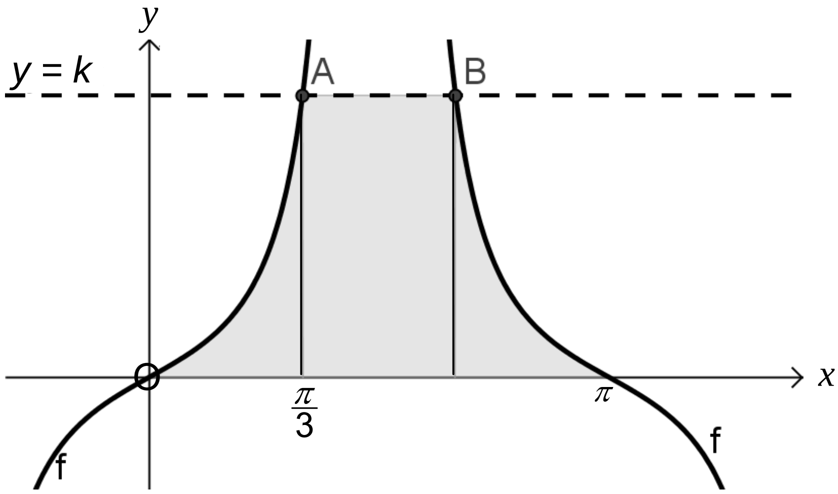
10.1 A man in a rowing boat at B is 2 km from the nearest point on the beach, T, as illustrated in the diagram below. He wants to reach his house, H that is 6 km from T. He will row to S and then walk to H. He rows at 3 km/h and walks at 5 km/h. TSH is a straight line.




(a) If BS = $x$ km show that the distance he must walk is given by $\text{SH}=6-\sqrt{x^2-4}$ km. (2)

(b) Determine the value of $x$ so that he will reach his house in the shortest possible time. (Hint: time $=\dfrac{\text{distance}}{\text{speed}}$) (10)

10.2 The sketch shows the graph of $f(x) = \tan x.\sec x$, which cuts the $x$-axis at 0 and at π, and the line $y = k$, which cuts $f$ at $A\left(\frac{\pi}{3}; k\right)$ and at B.




(a) Determine the coordinates of A and B. Give the answer in surd form where necessary. (4)

(b) Write down an expression that can be used to obtain the area of the shaded region, i.e. the region between $f$, the line $y = k$ and the $x$-axis. (4)

(c) Hence, calculate the value of this area. (6)

10.3 The graph of $f$ rotates around the $x-$axis. The volume of the solid of rotation formed between $x = 0$ and $x = a$ is equal to $\frac{\pi}{3}$. Determine the value of $a$.

(You may assume that $0 < a < \frac{\pi}{3}$). (7)

**[33]**

**Total for Module 1: 200 marks**

## MODULE 2 STATISTICS

**This section consists of 5 questions. All answers must be rounded to four decimal places.**

### QUESTION 1

Records from a call centre switchboard showed that in 2008 the times it took for calls to be answered were normally distributed with a mean of 27 seconds and a standard deviation of 2,5 seconds.

1.1 Calculate an interval in which 90% of all calls were answered in 2008. (3)

In 2009 a random sample of 20 calls was taken and the time taken for each call to be answered is recorded below:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 21 | 26 | 29 | 18 | 35 | 22 | 25 | 29 | 27 | 28 |
| 30 | 29 | 25 | 26 | 27 | 19 | 24 | 30 | 32 | 21 |

1.2 Calculate the 95% confidence interval for the mean time it took for a call to be answered in 2009. (8)

1.3 Based on the data, the call centre manager claims that the time it takes for a call to be answered has been reduced in 2009. Clearly state this claim as a hypothesis and determine whether at the 5% significance level there has been a reduction in the time it took for a call to be answered. (10)

**[21]**

### QUESTION 2

2.1 If the letters of the word PROBABILITY are arranged randomly in a row, find the probability that the two I's are separated. (6)

2.2 If a diagonal of a polygon is defined to be a line joining any two non-adjacent vertices, how many diagonals are there in a fifteen-sided polygon? (4)

**[10]**

## QUESTION 3

A pupil loses his original bivariate data set but finds the following recorded on a piece of paper:

- $\sum x_i = 124$
- $\overline{x} = 10\frac{1}{3}$
- $\sum x_i^2 = 1578$
- $\sum y_i = 124$
- $\sum x_i y_i = 1045$
- $\sum y_i^2 = 1568$

3.1 Show that there are 12 data points in the data set. (2)

3.2 Determine $\overline{y}$. (2)

3.3 Show that the least squares regression line of $y$ on $x$ in the form $y = a + bx$ is $y = 18,5649 - 0,7966x$. (8)

3.4 Use this regression line to estimate the $y$-value if $x = 9$. (2)

3.5 Calculate the correlation coefficient by using $r = b \times \frac{S_x}{S_y} = \frac{S_{xy}}{S_x . S_y}$

Given that: $S_x^2 = \frac{\sum x_i^2}{n} - \overline{x}^2$ and $S_y^2 = \frac{\sum y_i^2}{n} - \overline{y}^2$ and $S_{xy} = \frac{\sum x_i . y_i}{n} - \overline{x}.\overline{y}$ (4)

3.6 Describe the correlation between the variables. (2)

**[20]**

## QUESTION 4

4.1 The probability mass function of the discrete random variable *X*, is given by

$$\mathrm{P}(\mathrm{X}=x)=a\left(\frac{2}{3}\right)^{x} \quad \text{for all} \quad x\in\mathbb{N}_0.$$

(a) Find $P(x \le 1)$ in terms of $a$. (3)

(b) Find the value of the constant $a$. (5)

4.2 Assuming that a couple are equally likely to have a boy or a girl baby, find the probability that in a family of seven children there will be at least 3 girls. (10)

4.3 In a batch of 50 computers, 20 have a virus. If 5 computers are selected at random, what is the probability that two of them have a virus? (8)

4.4 A sack contains a very large number of green and blue marbles in the ratio 1:2. How many marbles must be randomly selected so that the probability of at least one marble being green is greater than 0,975? (10)

**[36]**

## QUESTION 5

The two events A and B are such that $\mathrm{P(A)} = 0{,}5$, $\mathrm{P(B)} = 0{,}3$ and $\mathrm{P}(\mathrm{A}|\mathrm{B}) = 0{,}1$.

5.1 Prove that the probability that both events occur is 0,03. (2)

5.2 Calculate the probability that:

(a) at least one of the events occur. (2)

(b) exactly one of the events occurs. (4)

(c) B occurs given A has occurred. (2)

5.3 Are events A and B independent? Support your answer using calculations. (3)

**[13]**

**Total for Module 2: 100 marks**

## MODULE 3 FINANCE AND MODELLING

**This section consists of 5 questions.**

### QUESTION 1

Part of a spreadsheet used to calculate terms in a second order difference equation is shown below. The formula for calculating successive terms is given as $T_n = aT_{n-1} + bT_{n-2}$.

| | |
|---|---|
| $T_2$ | 100,00 |
| $T_3$ | 1125,50 |
| $T_4$ | 1233,63 |
| $T_5$ | 2393,43 |

1.1 Determine the values of $a$ and $b$ (correct to 3 decimal places). (7)

1.2 Hence, determine $T_1$. (3)
**[10]**

### QUESTION 2

2.1 Xolani bought a flat for R485 000. He paid a 10% deposit then took out a mortgage bond over a 20 year period for the balance of the cost. Initially the interest was 11% per annum compounded monthly.

(a) Calculate his monthly bond repayments. (7)

After exactly 3 years, the bank increased the interest rate to 13% per annum compounded monthly.

(b) Calculate his new monthly repayments. (12)

2.2 Sam is saving for her retirement using a retirement annuity. She wishes to be able to withdraw R20 000 per month in perpetuity (for as long as she lives) from the month immediately after her retirement date which will be her $65^{th}$ birthday. She wishes to only live off the interest of her investment so that her daughter can inherit the capital. The interest that she will earn on her living annuity after her $65^{th}$ birthday is 12% per annum compounded monthly.

(a) Explain why she needs to accumulate R2 000 000 in her annuity by the time she retires. Support you explanation with appropriate calculations. (4)

(b) She makes her first payment into her retirement annuity on her $37^{th}$ birthday and her final payment on her $65^{th}$ birthday. The retirement annuity earns 9,8% per annum compounded monthly. Determine the required monthly contribution to her retirement annuity. (8)
**[31]**

## QUESTION 3

Kate invests R250 each month, on the first day of the month, into a savings account earning 7,5% per annum compounded monthly. Each year she increases her monthly investment by 10%.

3.1 If her first payment is on 1 January 2010, how much will she have in her account on 31 December 2012? (10)

3.2 Unfortunately, she could not make payments from June to October inclusive in 2010. Calculate by how much this will reduce her account balance on 31 December 2012. (8)

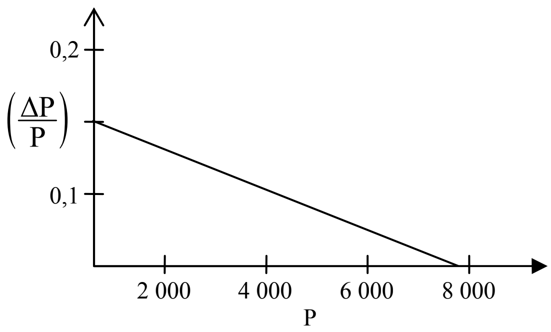
**[18]**

## QUESTION 4

In 1898 when the Kruger National Park was proclaimed a Game Reserve, the elephant population ($P$) had been hunted close to extinction. In 1905 ten elephants were recorded in the park but since then conservation efforts have ensured the growth of the elephant population. From the late 1960s until the 1990s culling operations were used to control the elephant population because the park managers decided that 7 500 elephants was the carrying capacity of the park. The intrinsic annual growth rate of an elephant population is 0,15. Use a logistic model to answer the following questions.

4.1 Estimate how many elephants were in the park in 1920. (8)

4.2 Below is a graph showing the relationship between the growth rate $\left(\frac{\Delta P}{P}\right)$ and the population (P).




Give the exact values of the $x$ and $y$ intercepts and state what they represent. (5)

4.3 Find the equation of the linear relationship in Question 4.2. (4)

4.4 Calculate:

(a) the growth rate when the population was 3 000. (3)

(b) the population when the growth rate is 0,05. (3)

**[23]**

**QUESTION 5**

In the Arctic the lemming ($L_n$) is the main prey of the stoat ($S_n$). A Predator-Prey model has been used to model the interaction between these two species on an island. The period between iterations is 1 year. The equations used to model the population interactions are:

$$L_{n+1} = L_n + 0,8L_n\left(1 - \frac{L_n}{10\,000}\right) - 0,02L_n.S_n$$

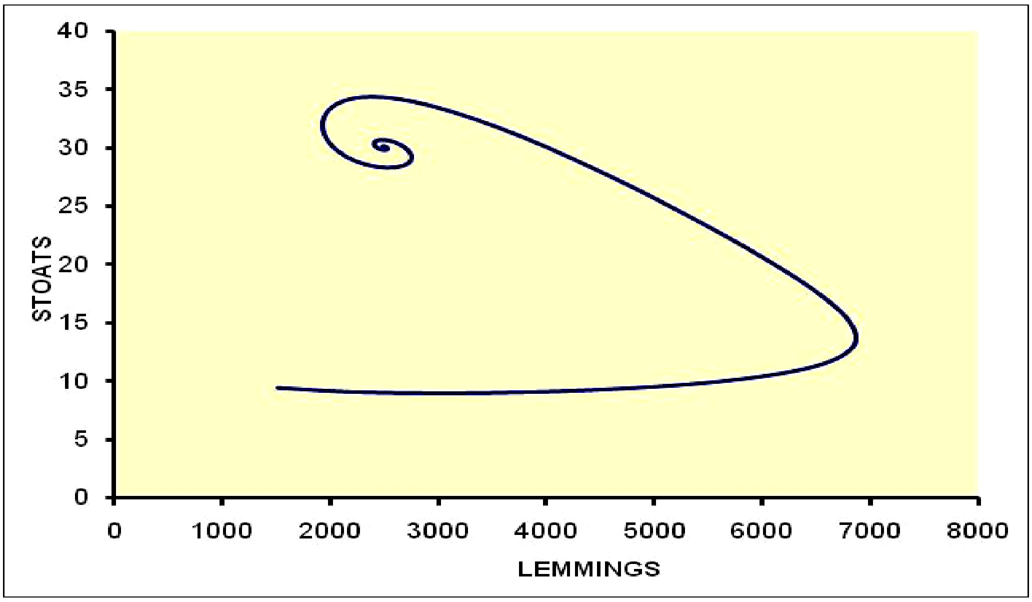
$$S_{n+1} = S_n + 0,00004L_n.S_n - 0,1S_n$$

5.1 What is the average life span of a stoat? (3)

5.2 What is the intrinsic growth rate of the lemming population? (1)

5.3 What is the carrying capacity of lemmings on the island? (1)

The graph below shows the populations of the lemmings and stoats.




5.4 Use the graph to estimate the initial population of both species. (2)

5.5 Indicate on the graph where the lemming population is decreasing and the stoat population is increasing. (Use the graph provided on the Answer Sheet.) (2)

5.6 If the stoat population equation was changed to

$$S_{n+1} = S_n + 0,00006L_n.S_n - 0,1S_n$$

determine the new stable point of the two populations. (9)

**[18]**

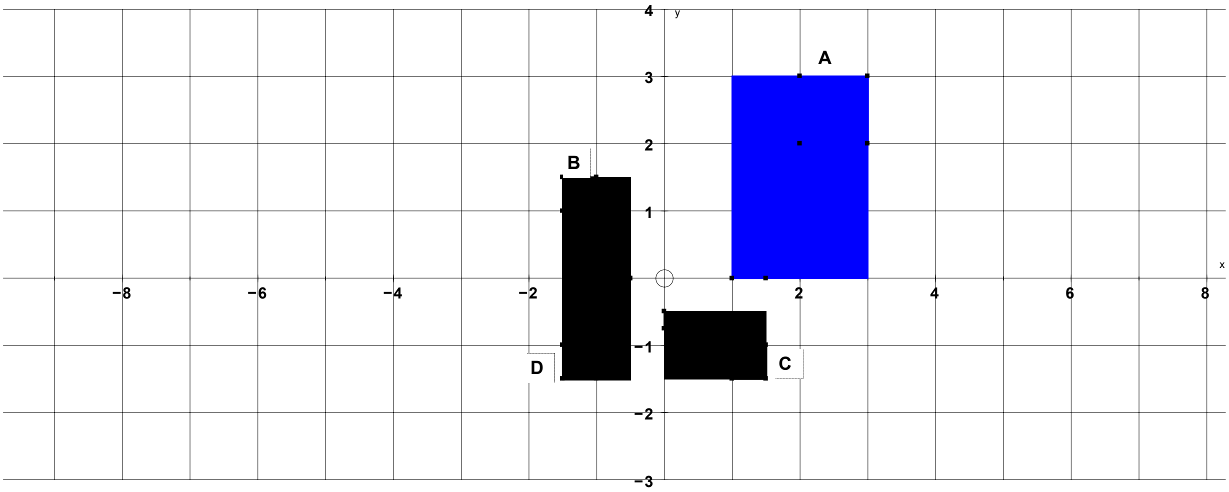
**Total for Module 3: 100 marks**

## MODULE 4 GRAPH THEORY AND MATRICES

**This module consists of 7 questions.**

### QUESTION 1

Below is a diagram representing four similar shapes A, B, C and D.




Mat P = $\begin{pmatrix} 0,5 & 0 \\ 0 & 0,5 \end{pmatrix}$ Mat Q = $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ Mat R = $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ Mat S = $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Use matrices P, Q, R and S to give the matrix product that produces each of the following transformations. You may write your answers using the letters P, Q, R and S.

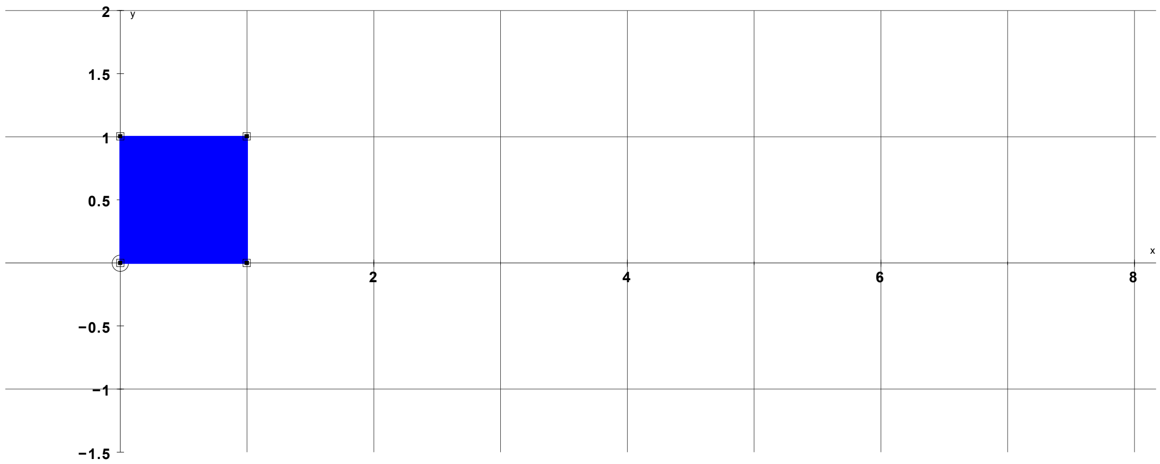
1.1 Shape A onto shape B. (4)

1.2 Shape B onto shape C. (2)

1.3 Shape A onto shape D. (4)

**[10]**

**QUESTION 2**

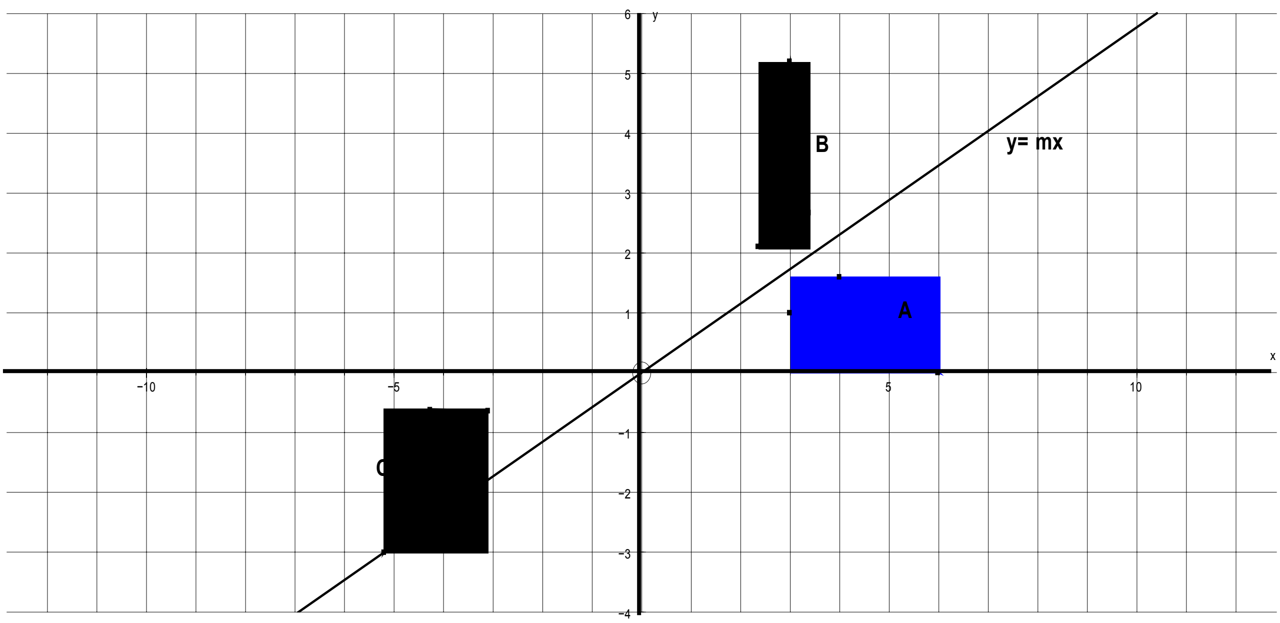



2.1 On the axes provided on the Answer Sheet, draw the above square after it has been sheared by a factor of 3 and then stretched by a factor of 2 with both transformations parallel to the *x*-axis and with the y-axis as the invariant line. (6)

2.2 Would the resultant shape be the same if the order of the transformations in Question 2.1 were reversed? (2)

**[8]**

**QUESTION 3**




In the above diagram, triangle B is a reflection of triangle A in the line $y = \frac{x}{\sqrt{3}}$.

Determine:

3.1 The angle of inclination of the straight line. (2)

3.2 The matrix that maps A onto B. (3)

3.3 The matrix that maps B onto C. (5)

**[10]**

## QUESTION 4

Three equations are given:

$$x + 2y - z = 8$$
$$-x + y + bz = -5$$
$$2x + 3y + z = 11$$

4.1 Find a value for $b$ so that there is no unique solution to the system of simultaneous equations. (6)

4.2 If the above system of equations was represented as the matrix equation $AX = Y$ and $b = 4$:

(a) Find $A^{-1}$. (10)

(b) Hence, or otherwise, solve the system of simultaneous equations. (4)

**[20]**

## QUESTION 5

The matrix below gives the distance (in metres) between each of the six security cameras in one of the 2010 World Cup stadiums.

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| A | | 60 | 100 | 110 | 80 | 70 |
| B | 60 | | 70 | 80 | 90 | 85 |
| C | 100 | 70 | | 50 | 65 | 105 |
| D | 110 | 80 | 50 | | 60 | 40 |
| E | 80 | 90 | 65 | 60 | | 40 |
| F | 70 | 85 | 105 | 40 | 40 | |

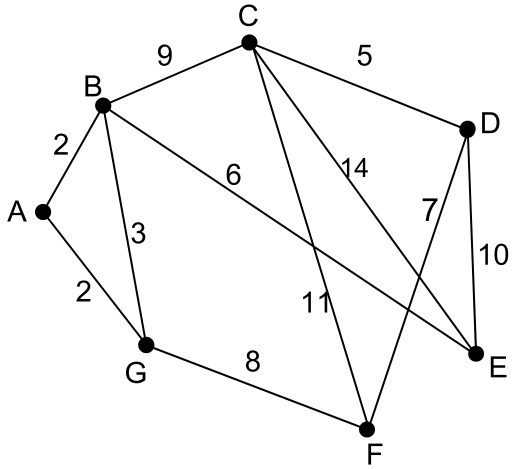
5.1 A minimum spanning tree of cabling needs to be laid between the cameras. Starting at D, use Prim's algorithm to construct a minimum spanning tree. (8)

5.2 The surveillance team needs to check each of the cameras on a hourly basis. Find an upper bound for the shortest route they can travel if the security office is at A. (8)

**[16]**

**QUESTION 6**

Below is a graph representing the distance along tar roads between various farms.




6.1 The road maintenance department needs to check each road for potholes. Which roads should they travel twice in order to find an Eulerian circuit of minimum length? (8)

6.2 There is a dirt road between GD which is 7 km long. If the road maintenance department used this road as part of the Eulerian circuit, determine how much longer or shorter their new circuit would be than the circuit in Question 6.1. (10)

6.3 The road maintenance department could start their checking at D and finish at E, but must still travel every road at least once. What would be the total distance they need to travel? (6)

**[24]**

**QUESTION 7**

How many non-isomorphic Eulerian circuits can be drawn on 6 vertices? The graphs must be simple and connected.

Show all your working to support your answer. **[12]**

**Total for Module 4: 100 marks**

**Total: 300 marks**