St John's College

UPPER V

Advanced Programme Mathematics

July 2011 **Time: 3 hours** **Marks: 300**

Examiner: Mr G Evans
Moderator: Mrs K Jacobs

INSTRUCTIONS AND INFORMATION

Read the following instructions carefully

1. This paper consists of 9 pages, including an information sheet. Please make sure your paper is complete.
2. Read the questions carefully.
3. Plan your time carefully – approx 2 hours Section A and 1 hour Section B.
4. Answer all the questions.
5. Number your answers exactly as the questions are numbered.
6. You may use an approved non-programmable and non-graphical calculator, unless otherwise stated.
7. Round off your answers to one decimal digit, where appropriate.
8. All the necessary and reasonable working details must be clearly shown.
9. It is in your own interest to write legibly and to present your work neatly.

Section A – Calculus and Algebra (200 marks)

Question 1

(a) Solve the inequality: $\frac{5x}{x+1} \leq 4$ (6)

(b) Angie solves a quadratic equation using the formula, and gets the result:

$$x = \frac{-2 \pm \sqrt{m-8k}}{2}$$

(1) Write down the value of m. (1)
(2) What is the coefficient of x^2 in the original equation? (2)
(3) For which values of k will the equation produce solutions with an imaginary part. (3)

(c) If $(a+3i)(4-i) = (11+bi)$, find the values of a and b respectively. (8)

(d) It is given that:

$$|\ln x| - 2 = \frac{p}{|\ln x|}$$

If $x = e^3$ is one solution, then find the value of p and hence any other solutions. (10)

[30]

Question 2

(a) Sketch the graph: $f(x) = |2^x - 6|$ (6)

(b) Hence, or otherwise, solve the inequality: $|2^x - 6| < 2$ (6)

[12]

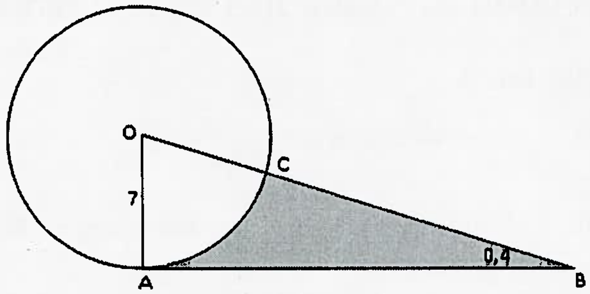
Question 3

Prove, by induction, that $5^{n+1} + 2^n$ is a multiple of 3 for all $n \in N$

[12]

Question 4

AB is a tangent to the circle centre O.
$A\hat{B}O = 0{,}4$ radians. OA = 7 cm.



(a) Write down the size of $A\hat{O}C$ (2)

(b) Find the area of the shaded region. (9)

[11]

Question 5

Determine the limits:

(a) $\lim_{x \to 6} \frac{\sqrt{4x+1} - \sqrt{3x+7}}{x-6}$ (8)

(b) $\lim_{x \to 0} \frac{\sin 4x \cos x + \sin x \cos 4x}{x}$ (6)

[14]

Question 6

(a) Consider the function f defined below:

$$f(x) = \begin{cases} 2^x + a & \text{if} \quad x < 3 \\ 2b - 3a & \text{if} \quad x = 3 \\ \frac{b}{x-2} & \text{if} \quad x > 3 \end{cases}$$

Given that the function is continuous at $x = 3$, determine the values of a and b respectively. (11)

(b) Consider the following functions and write in each case:
(1) the type of discontinuity
(2) the equation of any asymptotes

$f(x) = \frac{x^2 - 9}{x - 3}$ $\qquad g(x) = \frac{x^2 - 9}{x^2 - 4}$ $\qquad h(x) = \frac{x^2 + 1}{x}$

(16)

[27]

Question 7

It is required to solve: $\tan x = 1 - x^2$ using Newton's method.

(a) By drawing a suitable rough sketch, show that a reasonable first approximation is $x_1 = 0,7$ (4)

(b) Hence solve for the nearest solution, correct to 5 decimal places (8)

(c) Does the initial guess in Newton's method need to be close to the actual answer?
With the aid of a sketch, explain why and/or why not? (5)

[17]

Question 8

(a) Find $\frac{dy}{dx}$ if $y = \sqrt{2x+1}.\tan\left(\sqrt{2x+1}\right)$ Simplify your answer. (10)

(b) Determine the equation of the tangent to the curve $y^2 + xy = 8$ at the point (-2; 4). (11)

[21]

Question 9

Cars are passing a particular point on a bridge. For the purposes of this problem we will assume that each car is 3,5 metres long and that there is a distance of d metres between each car. The cars are all travelling at v km/h.

(a) Show that the number of cars passing through the point on the bridge each hour (i.e. the *flow rate*) is given by:

$$F = \frac{1000v}{d + 3,5}$$

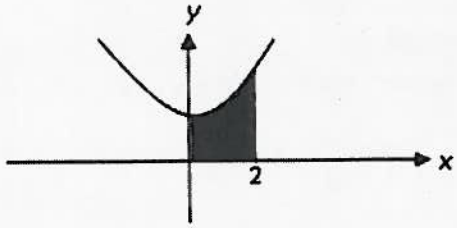
(4)

(b) To ensure safe driving, we insist that cars keep a safe following distance, represented by the value d. Let us assume that a car travelling at v km/h requires a following distance of $d = 0,006v^2$.

Now find the velocity that maximises the flow rate of traffic over the bridge. (10)

[14]

Question 10



Use Riemann Sums to determine the shaded area, bounded by the curve $y = 2x^2 + 1$ the axes and the line $x = 2$.

[10]

Question 11

Integrate the following functions:

(a) $\int x.\cos(5x^2)\,dx$ (4)

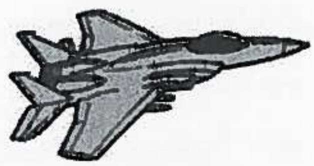
(b) $\int \frac{1}{x^2\left(1+\frac{1}{x}\right)^3}dx$ (8)

(c) Given the general formula: $\int \frac{1}{a^2 - x^2}dx = \frac{1}{2a}\ln\frac{a+x}{a-x} + C$

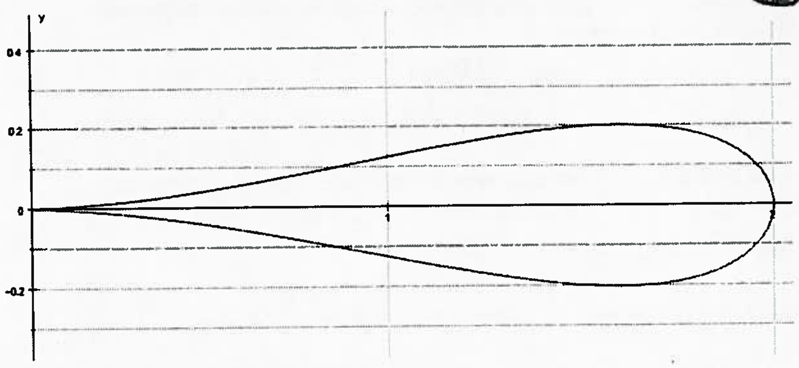
find, without the use of a calculator, the value of: $\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{1}{9-4x^2}\,dx$ (8)

[20]

Question 12



The fuel tank on the wing of a jet is formed by rotating the region bounded by the curve $y = \frac{1}{8}x^2.\sqrt{2-x}$ about the x-axis between $x = 0$ and $x = 2$, where the units are measured in metres.



Determine the volume of the fuel tank. **[12]**

Section B – Finance (100 marks)

Question 1

An investment of P now is worth four times as much in 8 years' time. Calculate:

(a) The effective annual interest rate. (8)

(b) The nominal annual rate if compounding occurs quarterly. (5)

[13]

Question 2

- Sophie deposits a lump sum of R5 000 in an account giving 7,25% interest per annum compounded monthly.
- After 3 years she adds a further R10 000 to the account.
- During the sixth year she makes 12 **monthly** deposits of R1 500, starting at the end of the first month.
- At the end of the 6^{th} year the interest rate increases to 9,25% per annum.
- She withdraws R8 000 after 7 years

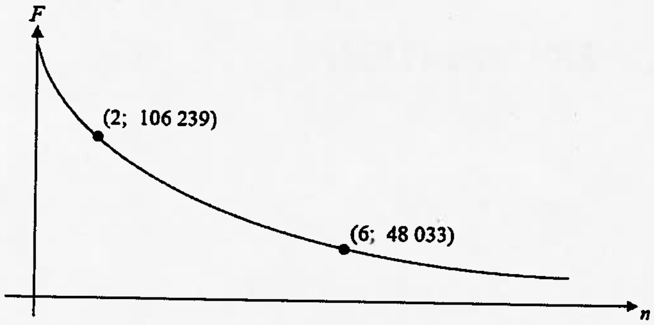
(a) Draw a time-line to represent the information given over a ten-year time period. (8)

(b) Determine the amount that can be withdrawn after a total of ten years. (16)

[24]

Question 3

The depreciation of a vehicle is represented by the graph below where F is the current value of the car (to the nearest rand) and n is the age of the car in years.



(a) What is the significance of the y-intercept of the graph? (2)

(b) According to this model, will the value of the vehicle ever become 0. Explain. (3)

(c) Calculate: (1) the annual rate of depreciation (11)

(2) the original value of the car. (4)

[20]

Question 4

A loan is taken to assist in the purchase of a town house property. The interest on the outstanding balance is 13% per annum compounded monthly.

(a) If the interest payable for the first month is R780, show that the amount of the original loan was R72 000. (4)

(b) If the loan is to be repaid over a period of 20 years, calculate the monthly payment. (9)

(c) It is possible to repay the loan over a longer period. How would you advise a client wishing to extend the repayment time of the loan? What are the advantages and pitfalls? (6)

[19]

Question 5

(a) An iterative formula for calculating the value of a sinking fund is:

$$F_{n+1} = 1,01 \times F_n + 250 \qquad \text{where: } F_0 = 0$$

(1) Write down F_1, F_2 and F_3. (6)

(2) Find the effective annual interest rate. (8)

(b) Solve for a and b if the sequence -3; 10; 8; -6; -11; .... is generated by the recursive formula: (10)

$$T_n = \frac{T_{n-1}}{a} + b.T_{n-2}$$

[24]

END OF EXAMINATION

INFORMATION SHEET

Algebra

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$|x| = \begin{cases} x & if \quad x \geq 0 \\ -x & if \quad x < 0 \end{cases}$$

$$\sum_{i=1}^{n} 1 = n$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}$$

$z = a + bi$ $\qquad$ $z^* = a - bi$

$\ell n A + \ell n B = \ell n (AB)$ $\qquad$ $\ell n A - \ell n B = \ell n \left(\frac{A}{B}\right)$

$\ell n A^n = n \, \ell n A$ $\qquad$ $\log_a x = \frac{\log_b x}{\log_b a}$

Calculus

$$Area = \lim_{n \to \infty} \left(\frac{b-a}{n}\right) \sum_{i=1}^{n} f(x_i)$$

$$\int_a^b x^n dx = \left[\frac{x^{n+1}}{n+1}\right]_a^b$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\int f'(g(x)).g'(x)\, dx = f(g(x)) + c$$

$$\int f(x).g'(x) dx = f(x).g(x) - \int g(x).f'(x)\, dx + c$$

$$x_{r+1} = x_r - \frac{f(x_r)}{f'(x_r)}$$

$$V = \pi \int_a^b y^2 dx$$

Function	Derivative
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x.\tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x.\cot x$
$f(g(x))$	$f'(g(x)).g'(x)$
$f(x).g(x)$	$g(x).f'(x)+f(x).g'(x)$
$\dfrac{f(x)}{g(x)}$	$\dfrac{g(x).f'(x)-f(x).g'(x)}{[g(x)]^2}$

$A=\frac{1}{2}r^2\theta$ $\quad$ $s=r\theta$

In $\triangle ABC$: $\quad \dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$

$a^2=b^2+c^2-2bc.\cos A$

$Area=\frac{1}{2}ab.\sin C$

$\sin^2 A+\cos^2 A=1$ $\quad$ $1+\tan^2 A=\sec^2 A$ $\quad$ $1+\cot^2 A=\operatorname{cosec}^2 A$

$\sin(A\pm B)=\sin A.\cos B\pm\cos A\sin B$ $\quad$ $\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$

$\sin 2A=2\sin A\cos A$ $\quad$ $\cos 2A=\cos^2 A-\sin^2 A$

$\sin A.\cos B=\frac{1}{2}[\sin(A+B)+\sin(A-B)]$

$\sin A.\sin B=\frac{1}{2}[\cos(A-B)-\cos(A+B)]$

$\cos A.\cos B=\frac{1}{2}[\cos(A-B)+\cos(A+B)]$

Finance & Modelling

$F=P(1+in)$ $\quad$ $F=P(1-in)$ $\quad$ $F=P(1+i)^n$ $\quad$ $F=P(1-i)^n$

$F=x\left[\dfrac{(1+i)^n-1}{i}\right]$ $\quad$ $P=x\left[\dfrac{1-(1+i)^{-n}}{i}\right]$ $\quad$ $r_{eff}=\left(1+\dfrac{r}{k}\right)^k-1$