Advanced Programme Mathematics

Grade 12

Prelims 2009

Notes

1. Calculators in radian mode.
2. Calculators in **RADIAN MODE**.
3. Answer Section B in a separate answer book. Answer books must be in same order as question paper. Therefore, leave sufficient space (min. 1 page per whole question) when leaving a question for later.
4. Following on from the last point, there are 3 hours to complete **300 marks** (ignoring any bonus marks).
5. Unless otherwise stated, answers to be given to 2 decimal places.
6. You should have 3 pages of formulae plus a Normal Distribution table. These are the IEB formulae sheets you will get at the end of the year, so if the formula you are looking for is not here, either you should know this formula, or you are looking for the wrong one!
7. Read the long questions carefully, there are hints and directions that make seemingly impossible questions possible.

Section A: Algebra and Calculus (200)

1. Factorisation:

   1.1. State the Complex Conjugate Root Theorem and its Real number counterpart, The Conjugate Surd Theorem. (5)

   1.2. Explain why the two theorems in 1.1 can be derived directly from the formula for finding the roots of a general quadratic function. (5)

   1.3. Given $f(x) = x^6 - 14x^5 + 84x^4 - 304x^3 + 681x^2 - 866x + 130$

   1.3.1. If $(3 - 2\sqrt{2})$, $(3 + 2\sqrt{2})$, and $(1 - 3i)$ are all roots of $f(x)$, show that $g(x) = x^4 - 8x^3 + 23x^2 - 62x + 10$ is a factor of $f(x)$. (8)

   1.3.2. Use 1.3.1. to factorise $f(x)$ fully for $x \in C$. (12)

[30]

---

2. Solve for $x \in R$ in the following:

   2.1. $\log_3 x^2 - 6\log_x 9 = 2$ (6)

   2.2. $2^{3x}3^{2x} = 100$ (6)

[12]

3. In this question we look at a crude version of Radiocarbon dating, which is a method for dating dead organic materials, such as old wood, which contain Carbon. C_{14} is an unstable carbon isotope which occurs in small quantities in all living material. When the object dies this C_{14} undergoes radioactive decay according to the following equation:

$$C(t) = C_0 e^{-pt}$$

where t is in years, $C(t)$ is the number of C_{14} in the object at time t, C_0 is the number of C_{14} when the object is alive ($t = 0$), and p is a factor determined by the half-life, t_h, of C_{14} which is the time taken for the amount of C_{14} to halve, i.e.

$$C(t + t_h) = \frac{1}{2}C(t).$$

When a specimen is found, for example a fossilized piece of wood in Newlands Forest, the amount of C_{14} in this piece is compared to that in a similar live specimen, and the above equation is then used to date the dead piece of wood.

3.1. The half-life of C_{14} has been experimentally found as 5730 years. Use this figure to calculate the value of p in the above equation, give your answer in scientific notation rounded off to 3 significant figures. (6)

3.2. If the number of C_{14} in the hypothetical fossilized piece of wood found in Newlands Forest is 10^6 today, and the number in a similar live specimen is 10^7, calculate the following (use your rounded off answer to 3.1):

3.2.1. The number of C_{14} that will be left in 1000 years. (3)

3.2.2. How old is the fossilized piece of wood? Round off to the nearest year. (5)

[14]

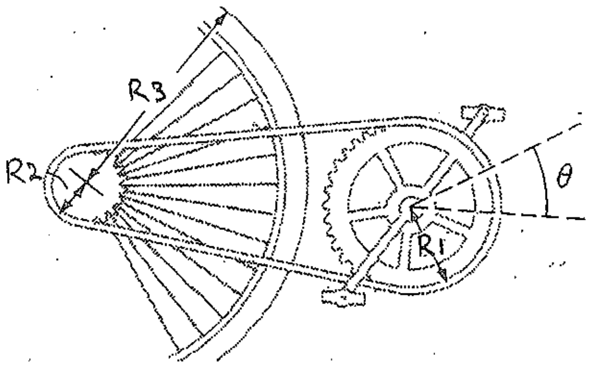
4. Use Mathematical Induction to prove for $n \in N$

$10^{n+1} + 3.10^{n} + 5$ is divisible by 9 (12)

[12]

5. We know that in any wheel, the arclength, S, is a function of the angle, θ, which subtends that arc. Therefore, if we know the angular velocity, $\frac{d\theta}{dt}$ rad.s^{-1}, we can calculate the ground speed of this wheel as $\frac{dS}{dt} = \frac{dS}{d\theta}\frac{d\theta}{dt}$.

In the diagram below, the pedal wheel, chain and back wheel of a bicycle are shown. The radius in cm of the pedal wheel (which is attached to the chain) is R_1, the radius of the chain wheel attached to the back wheel is R_2, and the radius of the back wheel is R_3. The angular velocity of the pedal wheel is $\frac{d\theta}{dt}$ rad.s^{-1}.



5.1. Show that the chain is moving at $R_1 \frac{d\theta}{dt}$ cm.s^{-1}. (2)

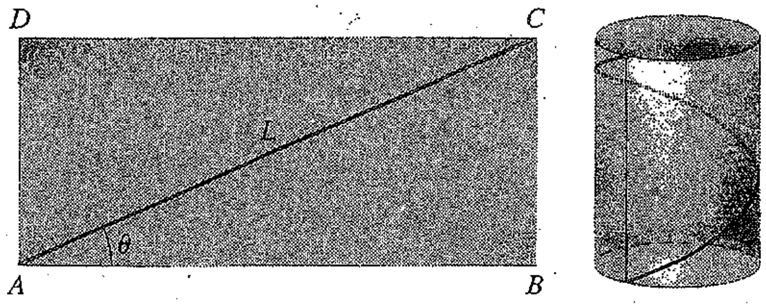
5.2. Show that the angular velocity of the back wheel is given by $\frac{d\phi}{dt} = \frac{R_1}{R_2}\frac{d\theta}{dt}$ rad.s^{-1}. (4)

5.3. Now show that the ground speed of the bicycle is $\frac{R_1 R_3}{R_2}\frac{d\theta}{dt}$ cm.s^{-1}. (2)

5.4. Hence find the speed, in km/hr, you'd be travelling if you were riding this bike and were pedalling at 2 revolutions per second, when R_1, R_2 and R_3 are 13, 5, and 36cm respectively. (4)

[12]

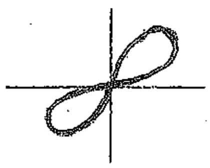
6. In the diagram below, a cylinder is formed by joining the corners of ABCD, which is a rectangular piece of elastic material. A straight piece of wire, with a fixed length, L, joins AC, with $C\hat{A}B = \theta$. Find θ which maximizes the volume of the resulting cylinder, and give this volume. You do not need to prove that this is a maximum. (15)



[15]

7. The Lemniscate of Bernoulli is a modification of an ellipse, which results in a shape like the infinity symbol. The following is one version of this function, with its associated (rough) graph given:

$(x^2 + y^2)^2 = 4xy$



Find the equation of the tangent to this curve at the point (1; 1) (10)

[10]

8. Find the following limits, if they exist:

8.1. $\lim\limits_{x\to\infty} \dfrac{\tan 3x \sin 2x}{x^2}$ (5)

8.2. $\lim\limits_{x\to\infty} \dfrac{5x^2 - 3x + \sqrt{9x^4 - 2x^3}}{3x^2 - 7x + 2}$ (5)

[10]

9. The following equation is given:

$$3\tan\left(x-\frac{\pi}{6}\right)=5-6\cos\left(3x-\frac{\pi}{4}\right)$$

9.1. Show that a root of this equation lies in $x \in [1{,}65;1{,}75]$ (3)

9.2. Use Newton's method, with $x_1 = 1{,}65$ as a first approximation, to find this root correct to 5 decimal places. (12)

[15]

---

10. Given $f(x)=\dfrac{x^2-2x+1}{x+1}$

10.1. Find the x and y intercepts. (3)

10.2. Find all turning points, showing clearly whether they are local maximum(s) or minimum(s). (10)

10.3. Give the equation of all asymptotes. (6)

10.4. Give the domain of $f(x)$, and show what happens to $f(x)$ at the edges of this domain. (4)

10.5. Do a neat sketch of the graph of $f(x)$, showing all relevant information calculated above. (7)

[30]

---

11. Given $f(x) = -2x^2 + 8$

11.1. Use a right Riemann Sum to calculate the area between $f(x)$ and the x-axis between $x = -2$ and $x = 2$. (15)

11.2. Check your result using the Fundamental Theorem of Calculus. (3)

[18]

---

12. A region is bounded by the graphs of $f(x) = \sin x + 1$, and $g(x) = \cos x$, for $x \in \left[0; \frac{\pi}{2}\right]$.

12.1. Draw a rough sketch of this region. (3)

12.2. Find the volume resulting from rotating this region about the x-axis. (12)

[15]

---

13. Determine the following integrals:

13.1. $\int x.\arctan x.dx$ (15)

13.2. $\int_1^2 \frac{20x - 42x^2}{\sqrt{7x^3 - 5x^2 + 3}} dx$ (5)

[20]

End of Section A: Total available marks 213

Section B: Statistics (100)

1. The probability distribution of the average daily earnings of a South African busker (street musician) can be approximated by a Normal distribution with a mean of R20 and standard deviation of R3.

   1.1. Find the probability that a busker chosen at random will have an average daily income of less than R17 or more than R26. (6)

   1.2. A member of the local band Goldfish, claims that if buskers sing this band's songs, they'll improve their average daily earnings.

   1.2.1. Give suitable null and alternate hypotheses, H_0 and H_1, to test this claim. (3)

   1.2.2. In the good 'ol tradition that local is lekker, a random sample of 36 Cape Town buskers take on the above experiment, singing only Goldfish songs for a week. It is found that, for this week, the average daily earnings of this sample group was R22.
   Use this sample to test the claim at a 1% level of significance. (6)

   1.2.3. How confident are you with your conclusion for 1.2.2? Why? (2)

[17]

2. The following table shows the average number of annual traffic incidents (including speeding, accidents, drunk driving, etc.) by age:

Age	18	19	20	22	25	30	35
Incidents	13	15	14	11	8	5	3

2.1. Which row should be assigned to *x*, the independent variable? (2)

2.2. Use your calculator to find the correlation coefficient, r, for this data. (4)

2.3. Determine the equation of the least squares regression line for this data. You may use the fact that this line passes through the point $(\bar{x};\bar{y})$, as well as the following information:

$\bar{x} = 24{,}14$; $\bar{y} = 9{,}86$; $\sum x = 169$; $\sum y = 69$; $\sum x^2 = 4319$;
$\sum y^2 = 809$; $\sum xy = 1496$ (10)

2.4. Use your answer in 2.3 to give the expected number of traffic incidents a 28 year old will have in a year. (2)

2.5. Use your answer in 2.2 to comment on the confidence you have in your answer for 2.4. (3)

[21]

3. Confidence intervals:

   3.1. By what factor should the sample size be increased in order to halve the width of a 95% confidence interval? (3)

   3.2. Was your answer above affected by the fact that it was a 95% confidence interval? Justify your answer. (2)

   3.3. A housing complex contains 8000 units. A random sample of 100 units yields the average number of cars per unit as 1,6 with a standard deviation of 0,8. Give a 95% confidence interval for the total number of cars in this housing complex. (10)

[15]

---

4. The famous Birthday Problem is a fine example of how the complement of an event can be used to turn a seemingly complicated problem into a relatively easy one. Use this hint wisely when answering the following two problems relating to common birthdays.

   4.1. There are 23 people in a room. What is the probability that at least 2 of them share a common birthday? (6)

   4.2. How many people should you invite to your party to be at least 50% sure of one of them sharing your birthday. (6)

[12]

---

5. {The probability of getting event B given that event A has happened, is called a conditional probability, and is calculated from the equation found in your formula sheet: $P(B \mid A) = \dfrac{P(B \cap A)}{P(A)}$.}

A study was done in the 1950's in England and Wales which looked at the movement of males up or down the social class system in successive generations. Dividing the classes into upper (U), middle (M), and lower (L), they found the probability of the son being in a certain class given that their father was in a certain class.

The following table (called a transition matrix) shows these probabilities, where the rows depict the class of the father (e.g. U1 for an upper class father), and the columns depict the class of the son (e.g. M2 for a middle class son), and the figures in each cell give the probability of a son being in that column if the father was in that row. (e.g. $P(U2|U1) = 0{,}45$)

	U2	M2	L2
U1	0,45	0,48	0,07
M1	0,05	0,70	0,25
L1	0,01	0,50	0,49

5.1. In one generation, the distribution of the fathers is 10% in U, 40% in M, and 50% in L. Find the probability that a son of one of these fathers chosen at random will be in M. (10)

5.2. If a son chosen at random is in M, find the probability that his father was in L. (10)

[20]

6. The Poisson Process is a well known discrete probability distribution which can be used as a good approximation for the Binomial Distribution when the number of trials, n, is large, and the probability of success in each trial is small.

   The probability mass function for a Poisson Process is:

$$P(X=k)=\begin{cases}\dfrac{\lambda^k}{k!}e^{-\lambda} & \text{k} = 0,1,2,3,\ldots \\ 0 & \text{otherwise}\end{cases}$$

   where λ is a necessary parameter for the Poisson Process, specified per unit of time. So if $\lambda = 0{,}5$ per minute, then, for a question covering a period of 5 minutes, we would use $\lambda = 2{,}5$.

   One of the many applications of the Poisson Process has been in the analysis of telephone exchanges, like a call centre. The number of calls coming into the centre during a unit of time may be modeled as a Poisson variable, if the centre services a large number of clients who act more or less independently.

   Let's say we have such a call centre, and we model it with a Poisson Process with parameter $\lambda = 0{,}5$ per minute. Find the probability that the centre will receive 2 or more telephone calls within a 5 minute period. (10)

[10]

---

7. The p.d.f. of a continuous random variable, X, is given as:

$$f(x)=\begin{cases}cx^2 & 0 \le \text{x} \le 1 \\ 0 & \text{elsewhere}\end{cases}$$

   7.1. Find the value of c. (5)

   7.2. Find $P(0{,}1 < X < 0{,}5)$. (5)

[10]

End of Section B: Total available marks 105

END OF PAPER!!!

GRD 12 AP Maths Prelims '09

1.1. Complex Conjugate Root Theorem: if $a+ib$ is a root of $f(x)$, then so is $a-ib$; where $a, b \in \mathbb{R}$ and $i=\sqrt{-1}$

Conjugate Surd Theorem: if $\alpha+\beta\sqrt{\gamma}$ is a root of $f(x)$, then so is $\alpha-\beta\sqrt{\gamma}$, where $\alpha, \beta, \gamma \in \mathbb{R}$.

1.2. for $f(x) = ax^2+bx+c$ $\quad x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$= \frac{-b}{2a} \pm \frac{1}{2a}\sqrt{b^2-4ac}$$

now if $b^2-4ac<0$ we have a complex root and another which is just the first's complex conjugate.

if $b^2-4ac>0$ we see our two roots are in the form $\alpha \pm \beta\sqrt{\gamma}$ and $\alpha, \beta, \gamma \in \mathbb{R}$.

3.1 from 1.1: if $1-3i$ is a root, so is $1+3i$. And if $3-2\sqrt{2}$, $3+2\sqrt{2}$, $1-3i$, $1+3i$ are roots of $f(x)$ then

$[x-(3-2\sqrt{2})][x-(3+2\sqrt{2})][x-(1-3i)][x-(1+3i)]$ is a factor of $f(x)$

$$= [(x-3)+2\sqrt{2}][(x-3)-2\sqrt{2}][(x-1)+3i][(x-1)-3i]$$
$$= [(x-3)^2-(2\sqrt{2})^2][(x-1)^2-(3i)^2]$$
$$= (x^2-6x+9-8)(x^2-2x+1+9) \quad (\because i^2=-1)$$
$$= (x^2-6x+1)(x^2-2x+10)$$
$$= x^4-2x^3+10x^2-6x^3+12x^2-60x+x^2-2x+10$$
$$= x^4-8x^3+23x^2-62x+10$$

and in prelims by

1.3.2.

$$\begin{array}{r|l} & x^2 - 6x + 13 \\ \hline x^4 - 8x^3 + 23x^2 - 62x + 10 & x^6 - 14x^5 + 84x^4 - 304x^3 + 681x^2 - 866x + 130 \\ & \underline{x^6 - 8x^5 + 23x^4 - 62x^3 + 10x^2} \\ & -6x^5 + 61x^4 - 242x^3 + 671x^2 - 866x \\ & \underline{-6x^5 + 48x^4 - 138x^3 + 372x^2 - 60x} \\ & 13x^4 - 104x^3 + 299x^2 - 806x + 130 \\ & \underline{13x^4 - 104x^3 + 299x^2 - 806x + 130} \end{array}$$

($g(x)$ is a factor $\therefore$ must get no remainder)

the roots of $x^2-6x+13$ are given by $x = \dfrac{6 \pm \sqrt{36-52}}{2}$
$= 3 \pm 2i$

$\therefore f(x) = (x-(3-2\sqrt{2}))(x-(3+2\sqrt{2}))(x-(1-3i))(x-(1+3i))(x-(3-2i))(x-(3+2i))$

.1 $\log_3 x^2 - 6\log_x 3^2 = 2$

$\therefore 2\log_3 x - 12\left(\dfrac{1}{\log_3 x}\right) = 2$

$\therefore 2(\log_3 x)^2 - 12 = 2\log_3 x$

let $k = \log_3 x$ $\quad \therefore 2k^2 - 2k - 12 = 0$

$\therefore (k-3)(2k+4) = 0$

$\therefore \log_3 x = 3$ or $\log_3 x = -2$

$\therefore x = 3^3 = 27$ $\quad \therefore x = 3^{-2} = \frac{1}{9}$

.2 $2^{3x} \cdot 3^{2x} = 100$ $\therefore (2^3 \cdot 3^2)^x = 100$

$\therefore \log(72)^x = \log 100$ (log both sides)

$\therefore x = \dfrac{2}{\log 72}$

$= 1,08$

Grd 12 prelims 09

.1) $C(t+t_h) = \frac{1}{2} C(t)$ (t_h = half-life)

$\therefore C_0 e^{-p(t+t_h)} = \frac{1}{2} C_0 e^{-pt}$

$\therefore e^{-pt_h} = \frac{1}{2}$

$\therefore -pt_h = \ln(\frac{1}{2})$ $\therefore p = \frac{\ln(\frac{1}{2})}{-5730} = 1{,}21 \times 10^{-4}$

.2.1) $C(1000) = 10^6 e^{-p(1000)}$

$= 886\,033{,}96$

$\therefore$ 886034 C_{14} in 1000 years

3.2.2) $C_0 = 10^7$ $C(t) = 10^6$

$\therefore 10^6 = 10^7 e^{-pt}$

$\therefore t = \frac{\ln(0{,}1)}{-p} = 19030$ yrs

$n=1$: $10^2 + 3.10 + 5 = 135 = 9(15)$ $\therefore$ true for $n=1$.

assume true for some $k \in \mathbb{N}$ i.e. $10^{k+1} + 3.10^k + 5 = 9p$ $p \in \mathbb{N}$

RTP $10^{k+2} + 3.10^{k+1} + 5 = 9m$ $m \in \mathbb{N}$

$10^{k+2} + 3.10^{k+1} + 5 = 10.10^{k+1} + 3.10.10^k + 10.5 - 10.5 + 5$

$= 10(10^{k+1} + 3.10^k + 5) - 50 + 5$

$= 10(9p) - 9(5)$ (by assumptn

$= 9(10p - 5)$

which is clearly divisible by 9 $(10p - 5 = m)$

$\therefore$ If this is true for some $k \in \mathbb{N}$ then it is true for the next number, $k+1$, but it is true for $k=1$, $\therefore$ it is true for all $n \in \mathbb{N}$ ($\forall n \in \mathbb{N}$) by the principle of mathematical induction.

5.1) $S_1 = R_1\theta$ is arc length of chain wheel subtended by θ i.e. it is length the chain will move if the pedals move by an angle of θ.

$$\therefore \frac{dS_1}{d\theta} = R_1 \qquad \therefore \frac{dS_1}{dt} = R_1\frac{d\theta}{dt} \text{ cm.s}^{-1}$$

5.2) If the chain is moving round the pedal chain wheel at $\frac{dS_1}{dt}$ cm.s^{-1} then it is moving at same speed round the back chain wheel (same chain $\therefore$ same speed)

$$\text{i.e. } \frac{dS_2}{dt} = \frac{dS_1}{dt} = R_1\frac{d\theta}{dt}$$

but $\frac{dS_2}{dt} = R_2\frac{d\phi}{dt}$ where ϕ is θ's equivalent in the back wheel.

$$\therefore R_2\frac{d\phi}{dt} = R_1\frac{d\theta}{dt}$$

$$\therefore \frac{d\phi}{dt} = \frac{R_1}{R_2}\frac{d\theta}{dt} \text{ rad.s}^{-1}$$

5.3) Ground Speed given by $\frac{dS_3}{dt}$:

$$\frac{dS_3}{dt} = R_3\frac{d\phi}{dt} = \frac{R_3R_1}{R_2}\frac{d\theta}{dt} \text{ cm.s}^{-1}$$

5.4) $\frac{d\theta}{dt} = 2\text{rev.s}^{-1} = 2(2\pi)\text{ rad.s}^{-1}$

$$\therefore \frac{dS_3}{dt} = \frac{(36)(13)}{5}(4\pi) = 1176,21 \text{ cm.s}^{-1}$$

$$= \left(\frac{3600}{1\times10^5}\right)(1176,21) \text{ km/hr}$$

$$= 42,34 \text{ km/hr}$$

6) $AB = L\cos\theta$
$CB = L\sin\theta$

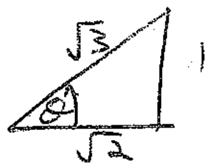
height of cylinder $= CB$
circumference of base, $2\pi r = AB$
$\therefore r = \frac{L\cos\theta}{2\pi}$

$$V(\theta) = \pi r^2 CB = \pi\left(\frac{L^2\cos^2\theta}{4\pi^2}\right)(L\sin\theta)$$
$$= \frac{L^3}{4\pi}\cos^2\theta\sin\theta$$
$$\therefore V'(\theta) = \frac{L^3}{4\pi}\left(2\cos\theta(-\sin\theta)\sin\theta + \cos^2\theta\cos\theta\right)$$
$$= \frac{L^3}{4\pi}\left(\cos^3\theta - 2\cos\theta\sin^2\theta\right)$$

for min/max $V'(\theta) = 0$ $\therefore \cos\theta(\cos^2\theta - 2\sin^2\theta) = 0$

$\therefore \cos\theta = 0$
$\therefore \theta = \frac{\pi}{2}$
clearly min.

or $\cos^2\theta = 2\sin^2\theta$
$\therefore \tan^2\theta = \frac{1}{2}$
$\therefore \theta = \arctan\left(\frac{1}{\sqrt{2}}\right)$
$= 0{,}615$



$$\text{max: } V(0{,}615) = \frac{L^3}{4\pi}\left(\left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2\left(\frac{1}{\sqrt{3}}\right)\right)$$
$$= \left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{6\pi}\right)L^3$$

---

7) $\frac{d}{dx}$ both sides: $2(x^2+y^2)(2x+2yy') = 4y + 4xy'$
$$\therefore y'(4x^2y + 4y^3 - 4x) = 4y - 4xy^2 - 4x^3$$
$$\therefore y' = \frac{y - xy^2 - x^3}{x^2y + y^3 - x}$$
$$\therefore \text{at } (1;1) \quad y' = \frac{1-(1)(1)^2-(1)^3}{(1)^2(1)+(1)^3-(1)} = -1$$

$\therefore$ equation of tangent passing through $(1;1)$: $1 = (-1)(1) + c$
$\therefore c = 2$

$\therefore$ tangent: $\underline{y = -x + 2}$

.1) $\lim_{x\to0} \frac{\tan 3x \sin 2x}{x^2} = \lim_{x\to0} \left(\frac{1}{\cos 3x}\right)\left(\frac{\sin 3x}{x}\right)\left(\frac{\sin 2x}{x}\right)$

$= \left(\lim_{x\to0} \frac{1}{\cos 3x}\right)\left(\lim_{x\to0} \frac{3\sin 3x}{3x}\right)\left(\lim_{x\to0} \frac{2\sin 2x}{2x}\right)$

$= (1)(3)(2)$

$= \underline{6}$

.2) $\lim_{x\to\infty} \frac{5x^2 - 3x + \sqrt{9x^4 - 2x^3}}{3x^2 - 7x + 2}$

$\div$ by highest power, i.e. x^2 : $= \lim_{x\to\infty} \frac{5 - \frac{3}{x} + \sqrt{9 - \frac{2}{x}}}{3 - \frac{7}{x} + \frac{2}{x^2}}$

$= \frac{5 - 0 + \sqrt{9 - 0}}{3 - 0 + 0} = \underline{\frac{8}{3}}$

) 9.1.) $f(\theta) = 3\tan(x - \frac{\pi}{6}) - 5 + 6\cos(3x - \frac{\pi}{4})$

assuming $f(\theta)$ is continuous on $x \in [1,65 ; 1,75]$ if there is a sign change over this interval then there is a root in this interval:

$f(1,65) = -1,82 \quad f(1,75) = 1,89$

there is a sign change $\therefore$ there is a root in $x \in [1,65 ; 1,75]$

2) $f'(\theta) = 3\sec^2(x - \frac{\pi}{6}) - 18\sin(3x - \frac{\pi}{4})$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$\therefore \theta_{n+1} = \theta_n - \frac{3\tan(\theta_n - \frac{\pi}{6}) - 5 + 6\cos(3\theta_n - \frac{\pi}{4})}{3\sec^2(\theta_n - \frac{\pi}{6}) - 18\sin(3\theta_n - \frac{\pi}{4})}$

$x_1 = 1,65 \quad \therefore x_2 = 1,707738$

$\therefore x_3 = 1,703167 \quad \therefore x_4 = 1,703133 \quad \therefore x_5 = 1,703133$

$\therefore \quad x = \underline{1,70313}$

0 $\quad f(x) = \dfrac{(x-1)^2}{x+1}$

10.1) y-int at $(0;1)$ $\quad$ x-int: $(x-1)^2 = 0 \therefore$ touches at $(1;0)$

0.2) $f'(x) = \dfrac{(2x-2)(x+1) - (x^2 - 2x+1)(1)}{(x+1)^2}$

at turning points $f'(x) = 0$ &

$\therefore 2x^2 - 2 - x^2 + 2x - 1 = 0$

$\therefore x^2 + 2x - 3 = 0$

$\therefore (x+3)(x-1) = 0 \quad \therefore x = -3$ or $x = 1$

$f(-3) = -8 \qquad f(1) = 0$

$f''(x) = \dfrac{(2x+2)(x+1)^2 - (x^2+2x-3)(2x+2)}{(x+1)^4}$

$\therefore f''(-3) = -1 < 0 \quad \therefore$ local max at $(-3;-8)$

$f''(1) = 1 > 0 \quad \therefore$ local min at $(1;0)$

10.3. vertical asymptote: $x = -1$ $\quad$ (div by zero in $f(x)$)

in $f(x)$, degree of numerator > degree of denominator

$\therefore$ there will be an oblique asymptote:

$$\begin{array}{r|l} & x - 3 \\ x+1 & x^2 - 2x + 1 \\ & \underline{x^2 + x} \\ & \quad -3x + 1 \\ & \quad \underline{-3x - 3} \\ & \qquad 4 \end{array}$$

$\therefore f(x) = x - 3 + \dfrac{4}{x+1}$

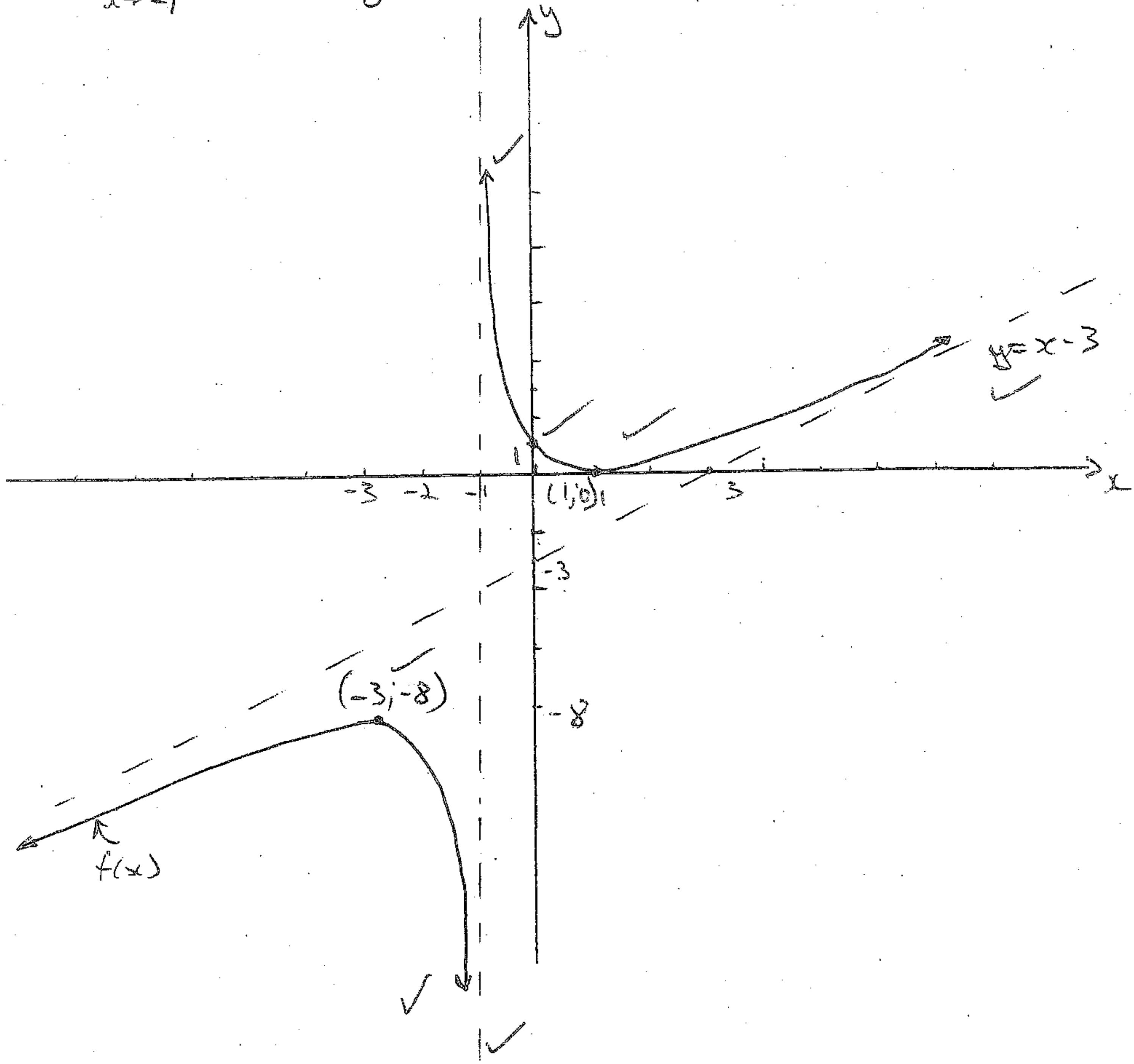
$\therefore y = x - 3$ is oblique asymptote

... terms of

?.4 Domain of f : $x \in (-\infty;-1) \cup (-1;\infty)$

$\lim_{x \to \pm\infty} f(x) = \pm\infty$ (along oblique asymptote)

$\lim_{x \to -1^-} f(x) = \frac{+}{0^-} = -\infty$ $\lim_{x \to -1^+} f(x) = \frac{+}{0^+} =$



Ctt'd 12 Prelims '09

1.1) $\int_{-2}^{2}(-2x^2+8)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x$

$\Delta x = \frac{2-(-2)}{n} = \frac{4}{n}$

$x_i = -2 + i\frac{4}{n}$

$\therefore f(x_i) = -2(-2+i\tfrac{4}{n})^2+8$

$= -2\left(4 - \frac{16i}{n} + \frac{16i^2}{n^2}\right)+8$

$= \frac{32i}{n} - \frac{32i^2}{n^2}$

$\therefore \sum_{i=1}^{n} f(x_i)\Delta x = \frac{4}{n}\sum_{i=1}^{n}\left(\frac{32i}{n} - \frac{32i^2}{n^2}\right)$

$= \sum_{i=1}^{n}\frac{128}{n^2}i - \sum_{i=1}^{n}\frac{128}{n^3}i^2$

$= \frac{128}{n^2}\left(\frac{n^2}{2}+\frac{n}{2}\right) - \frac{128}{n^3}\left(\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}\right)$

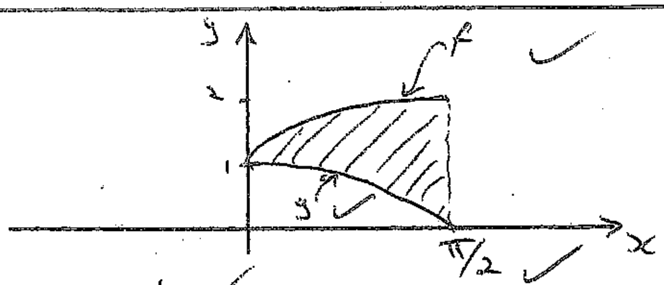
$= 64 + \frac{64}{n} - \frac{128}{3} - \frac{64}{n} - \frac{64}{3n^2}$

$\therefore \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = 64 - \frac{128}{3} = \frac{64}{3}$

.2) $\int_{-2}^{2}(-2x^2+8)\,dx = \left[-\frac{2}{3}x^3+8x\right]_{-2}^{2} = \left[-\frac{16}{3}+16\right]-\left[\frac{16}{3}-16\right]$

$= \frac{64}{3}$

.1)



2.2) $V = \pi\int_{0}^{\pi/2}\left[(f(x))^2 - (g(x))^2\right]dx$

$= \pi\int_{0}^{\pi/2}(\sin^2x + 2\sin x + 1 - \cos^2x)\,dx$

$= \pi\int_{0}^{\pi/2}(2\sin x + 1 - \cos 2x)\,dx$ $\left(-(\cos^2x - \sin^2x) = -\cos 2x\right)$

$= \pi\left[-2\cos x + x - \frac{1}{2}\sin 2x\right]_{0}^{\pi/2}$

$= \pi\left[\left(0+\frac{\pi}{2}-0\right) - (-2+0-0)\right] = \frac{\pi^2}{2} + 2\pi$ units^3

1) $\int x \arctan x \, dx$

1st substitute out arcfunction

let $\theta = \arctan x$

$\therefore \tan\theta = x \quad \therefore \frac{dx}{d\theta} = \sec^2\theta$

$\therefore dx = \sec^2\theta \, d\theta$

$\therefore = \int \theta \tan\theta \sec^2\theta \, d\theta$

2nd by parts:

let $u = \theta \quad \therefore du = d\theta$

and $dv = \tan\theta \sec^2\theta \, d\theta \quad \therefore v = \frac{1}{2}\sec^2\theta$

$\therefore = (\theta)(\frac{1}{2}\sec^2\theta) - \frac{1}{2}\int \sec^2\theta \, d\theta$

$= \frac{1}{2}\theta\sec^2\theta - \frac{1}{2}\tan\theta + C$

$= \frac{1}{2}\theta(1+\tan^2\theta) - \frac{1}{2}\tan\theta + C$

substitute back for $\theta = f(x)$:

$= \frac{1}{2}\arctan x(1+x^2) - \frac{1}{2}x + C$

2) $\int_1^2 \frac{20x - 42x^2}{\sqrt{7x^3 - 5x^2 + 3}} \, dx$

let $u = 7x^3 - 5x^2 + 3$

$\therefore \frac{du}{dx} = 21x^2 - 10x$

$\therefore -2\frac{du}{dx} = 20x - 42x^2$

$\therefore \cancel{dx = \frac{-2du}{20x - 42x}}$

$\therefore -2du = (20x - 42x^2)dx$

when $x = 2 \quad u = 7(8) - 5(4) + 3$
$= 39$

" $x = 1 \quad u = 7 - 5 + 3 = 5$

$\therefore = \int_5^{39} \frac{-2}{\sqrt{u}} \, du$

$= \left[-4\sqrt{u}\right]_5^{39} = -4(\sqrt{39} - \sqrt{5})$

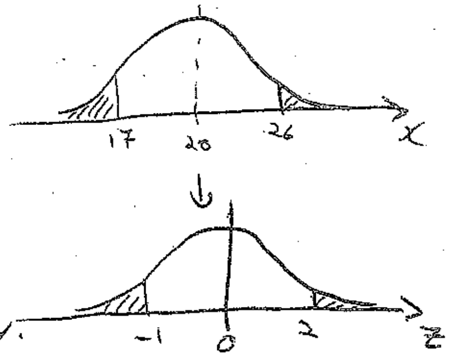
$= -16,04$

Section B

1) 1.1 $P[(X<17)\cup(X>26)]$ where $X \sim N(20;3^2)$

$= P\left[\left(Z<\frac{17-20}{3}\right)\cup\left(Z>\frac{26-20}{3}\right)\right]$

$= P(Z<-1) + P(Z>2)$



$=(0,5 - H(1)) + (0,5 - H(2))$

where the first bracket is obtained thru symmetry.

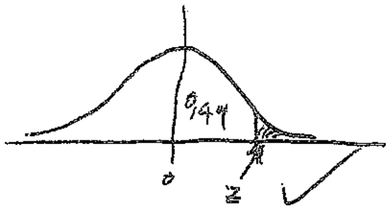
$=(0,5-0,3413) + (0,5-0,4772) = 0,1815$

$= 18,15\%$

2) $H_0: \mu = R20$

$H_1: \mu > R20$

3)



critical value: $H(z) = 0,49$

$\therefore z = 2,33$

Test statistic $= \dfrac{22-20}{\frac{3}{\sqrt{36}}} = 4$

4 is clearly above critical value of 2,33 $\therefore$ we can reject H_0 in favour of our alternate hypothesis, H_1. i.e. there is enough evidence to support Goldfish's claim.

4. Very confident: not only is our test statistic well above our critical value, but this critical value is set at a 1% significance level, which means we'll only, or we should only exceed it 1% of the time.

Grd 12 Prelims 09

1. Age

2. $r = -0,97$

3. $y = A + bx$ (calculator values: $A = 27,03$; $b = -0,71$) use these to check only

$$b = \frac{n\sum(xy) - \sum x \sum y}{n(\sum x^2) - (\sum x)^2}$$

$$= \frac{7(1496) - (169)(69)}{7(4319) - (169)^2} = -0,71$$

y passes through $(\bar{x}; \bar{y}) = (24,14; 9,86)$

$\therefore 9,86 = A - 0,71(24,14)$ (using rounded figures)

$\therefore A = 27,00$

$\therefore$ equation of least squares regression line: $y = 27 - 0,71x$

2.4) $y(28) = 27 - 0,71(28) = 7,12$

5. r is very close to -1 $\therefore$ strong negative correlation
i.e. strong negative linear relationship, meaning we are very confident using our straight line to approximate incidents for an age group.

.1 $\bar{X} \pm 1,96 \frac{\sigma}{\sqrt{n}}$

width is divided by $\sqrt{n}$ $\therefore$ if n increases by a factor of 4, this width would be halved.

3.2. No, we still divide by $\sqrt{n}$ regardless of percentage.

3.3 $P\left[\bar{X} - 1,96 \frac{0,8}{\sqrt{100}} \leq \mu \leq \bar{X} + 1,96 \frac{0,8}{\sqrt{100}}\right] = 0,95$

$\therefore$ 95% C.I. for μ is $[1,4432 ; 1,7568]$ cars per unit

there are 8000 units, $\therefore$ the 95% C.I. for number of cars in this housing complex is (rounded off to nearest car):

$[11546 ; 14054]$

4.1 P(at least 2 sharing Bday) = 1 − P(no shared Bdays)

for n people : $= 1 - \frac{(365)(364)(363)(362)\cdots(365-n+1)}{(365)^n}$

$= 1 - \frac{{}_{365}P_{23}}{(365)^n} = 50,73\%$

4.2 This time the probability of someone not sharing your birthday remains $\frac{364}{365}$ for each person

$\therefore$ P(someone sharing your Bday) $= 1 - \left(\frac{364}{365}\right)^n$

$\therefore 1 - \left(\frac{364}{365}\right)^n > 0,5$

$\therefore n > \frac{\log(0,5)}{\log\left(\frac{364}{365}\right)} = 252,7 \quad \therefore n \geq 253$ people

Grd 12 learner 01

5.1 $P(M_2) = P(U_1)P(M_2|U_1) + P(M_1)P(M_2|M_1) + P(L_1)P(M_2|L_1)$
$= (0,1)(0,48) + (0,4)(0,7) + (0,5)(0,5)$
$= 0,578$
$= \underline{57,8\%}$

5.2 looking for $P(L_1|M_2)$ from formula $= \dfrac{P(L_1 \cap M_2)}{P(M_2)}$

from 5.1 and formula: $P(M_2|L_1) = \dfrac{P(L_1 \cap M_2)}{P(L_1)}$

$\therefore P(L_1 \cap M_2) = P(M_2|L_1)\,P(L_1)$
$= (0,5)(0,5)$
$= 0,25$

$\therefore P(L_1|M_2) = \dfrac{0,25}{0,578}$
$= 0,4325$
$= \underline{43,25\%}$

$P(X \geq 2) = 1 - P(X < 2) = 1 - \left\{ \dfrac{\lambda^0}{0!} e^{-\lambda} + \dfrac{\lambda^1}{1!} e^{-\lambda} \right\}$
$\lambda = 0,5(5) = 2,5$
$\therefore P(X \geq 2) = \underline{0,7127 = 71,27\%}$

2.1 $\int_{-\infty}^{\infty} f(x)\,dx = 1$ for any valid pdf.

$\therefore c\int_0^1 x^2\,dx = 1 \qquad \therefore c\left[\tfrac{1}{3}x^3\right]_0^1 = 1$
$\therefore c = 3$

.2 $\int_{0,1}^{0,5} 3x^2\,dx = \left[x^3\right]_{0,1}^{0,5} = 0,124 = 12,4\%$