Parklands College

JUNE EXAMINATION 2015

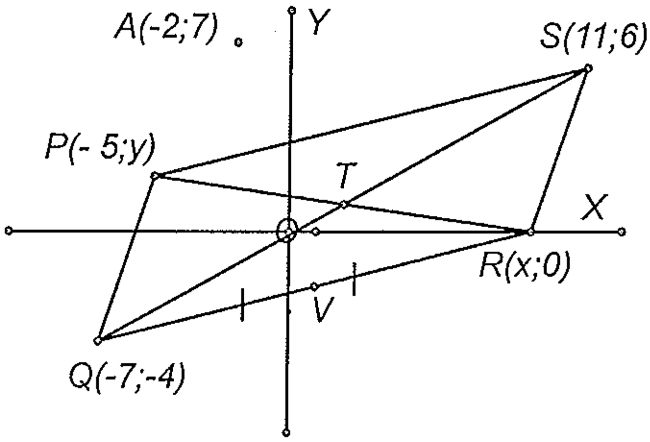
MATHEMATICS

GRADE 12 PAPER 2

SOLUTIONS

QUESTION 1

$P(-5;y)$, $Q(-7;-4)$, $R(x;0)$ and $S(11;6)$ are the vertices of a parallelogram, with T the intersection point of the diagonals.
(Note : QS does <u>not</u> pass through the origin of the axes.)



1.1. Calculate the following :

1.1.1. The coordinates of T (2)

$$T=\left(\frac{-7+11}{2};\frac{-4+6}{2}\right)=(2;1)$$

1.1.2. The values of x and y (4)

$$T=\left(\frac{-5+x}{2};\frac{y+0}{2}\right)=(2;1)$$

$$x-5=4\ ;\ y=2$$

$$x=9\ ;\ y=2$$

1.1.3. The equation of SV, a median of $\Delta\, SQR$ (5)

$$V=\left(\frac{-7+9}{2};\frac{-4+0}{2}\right)=(1;-2)$$

$$m_{SV}=\frac{(-2)-6}{1-11}=\frac{4}{5}\ ;\quad y=\frac{4}{5}x+c$$

$$(11;6)\ :\ 6=\frac{4}{5}(11)+c=\frac{44}{5}+c$$

$$c=-\frac{14}{5}=-2\frac{4}{5}\ ;\ y=\frac{4}{5}x-2\frac{4}{5}$$

1.1.4. The length of QS (in simplified surd form) (2)

$$QS^2 = (11-(-7))^2 + (6-(-4))^2 = 424$$
$$QS = \sqrt{424} = 2\sqrt{106}$$

1.1.5. The inclination of PR (4)

$$\tan\theta = m_{PR} = \frac{0-2}{9-(-5)}$$
$$\tan\theta = m_{PR} = \frac{-1}{7}$$
$$\theta = 180° - 8{,}13° = 171{,}87°$$

1.2. A line passing through the point $A(-2;7)$ bisects the area of the parallelogram.

1.2.1. Calculate the gradient of this line. (2)

$$m = \frac{1-7}{2-(-2)} = -\frac{3}{2}$$

1.2.2. Show whether this line and QS are perpendicular. (2)

$$m_{QS} = \frac{6-(-4)}{11-(-7)} = \frac{5}{9}$$
$$m \times m_{QS} = -\frac{3}{2} \times \frac{5}{9} = -\frac{5}{6} \neq -1$$

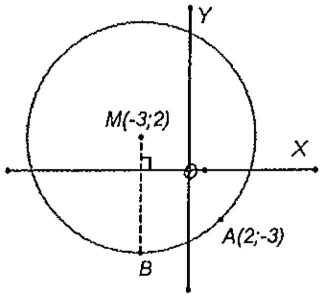
Therefore, not perpendicular

1.2.3. If A, S and $E(p;8)$ are collinear, calculate the value of p. (4)
[25]

$$\frac{8-7}{p-(-2)} = \frac{6-7}{11-(-2)}$$
$$\frac{1}{p+2} = \frac{-1}{13}$$
$$-p-2 = 13$$
$$p = -15$$

QUESTION 2

The centre of a circle is $M(-3;2)$, and $A(2;-3)$ is a point on the circle.



2.1. Determine the following :

2.1.1. The equation of the circle (3)

$(x+3)^2+(y-2)^2=r^2$

$A(2;-3): \ r^2=(2+3)^2+(-3-2)^2$
$=50$
$(x+3)^2+(y-2)^2=50$

2.1.2. The equation of the tangent to the circle at A (5)

$m_{MA}=\dfrac{-3-2}{2-(-3)}=-1$
$m_t=1$
$y=x+c$
$A(2;-3) : \ -3=2+c$
$c=-5$
$y=x-5$

2.1.3. The equation of the tangent to the circle at B (2)

$y=2-\sqrt{50}=2-5\sqrt{2}=-5{,}07$

2.2. Show whether $D(4;0)$ is a point on, inside or outside the circle with centre M. (3)

$(4+3)^2+(0-2)^2=53>50$
Therefore, outside

2.3. If the Y-axis is a tangent to another circle with centre M, write down the equation of this new circle. (2)
[15]

$$(x+3)^2+(y-2)^2=3^2=9$$

QUESTION 3

The equations of the circles with centres A and B are
$x^2+y^2-10x+24y+69=0$ and $x^2+y^2=16$ respectively.

3.1. Write the equation of circle A in centre-radius form. (4)

$$x^2-10x+y^2+24y=-69$$
$$x^2-10x+(-5)^2+y^2+24y+(12)^2=-69+(-5)^2+(12)^2$$
$$(x-5)^2+(y+12)^2=-69+25+144=100$$

3.2. Hence, determine whether the circles intersect, touch or do not intersect at all, by showing all the calculations. (4)
[8]

$$AB^2=(5-0)^2+(-12-0)^2$$
$$=169$$
$$AB=13$$

and $r_A+r_B=10+4=14$
Therefore $AB<r_A+r_B$
Therefore the circles intersect.

QUESTION 4

4.1. Given : $\sin(A+45°)+\cos(A+45°)=\sqrt{2}\cos A$

4.1.1. Prove this identity for all angles A. (3)

$$LHS=\sin A\cos 45°+\cos A\sin 45°+\cos A\cos 45°-\sin A\sin 45°$$

$$=\sin A\left(\frac{\sqrt{2}}{2}\right)+\cos A\left(\frac{\sqrt{2}}{2}\right)+\cos A\left(\frac{\sqrt{2}}{2}\right)-\sin A\left(\frac{\sqrt{2}}{2}\right)$$

$$=2\left(\frac{\sqrt{2}}{2}\right)\cos A$$

$$=\sqrt{2}\cos A=RHS$$

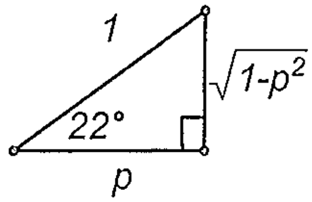
4.1.2. Hence, prove that $\sin 65° + \cos 65° = \sqrt{2}\cos 20°$, without using a calculator. (3)

$\sin(20° + 45°) + \cos(20° + 45°) = \sqrt{2}\cos 20°$ (From 4.1.1)

4.2. Prove that $\dfrac{\sin(180° - 3A)\sin(90° - A) - \cos(-3A)\sin A}{-2\sin(360° - A)\sin(180° + A) + 1} = \tan 2A$ (8)

$$\begin{aligned} LHS &= \frac{\sin 3A\cos A - \cos 3A\sin A}{-2(-\sin A)(-\sin A) + 1} \\ &= \frac{\sin(3A - A)}{1 - 2\sin^2 A} \\ &= \frac{\sin 2A}{\cos 2A} \\ &= \tan 2A = RHS \end{aligned}$$

4.3. If $\cos 22° = p$, determine the following in terms of p:



4.3.1. $\cos 44°$ (2)

$$\begin{aligned} &= \cos(2(22°)) \\ &= 2\cos^2 22° - 1 \\ &= 2p^2 - 1 \end{aligned}$$

4.3.2. $\sin 22°$ (2)

$$= \sqrt{1 - p^2}$$

4.3.3. $\cos 412°$ (4)

[22]

$$\begin{aligned} &= \cos(360° + 52°) \\ &= \cos 52° \\ &= \cos(30° + 22°) \\ &= \cos 30°\cos 22° - \sin 30°\sin 22° \\ &= \left(\frac{\sqrt{3}}{2}\right)p - \left(\frac{1}{2}\right)\sqrt{1 - p^2} \\ &= \frac{p\sqrt{3} - \sqrt{1 - p^2}}{2} \end{aligned}$$

QUESTION 5

Determine the general solution of x:

5.1. $(\cos x+\sin x)(\cos x-\sin x)=\frac{\sqrt{3}}{2}$ (4)

$\cos^2 x-\sin^2 x=\frac{\sqrt{3}}{2}$

$\cos 2x=\frac{\sqrt{3}}{2}$

$2x=\pm 30°+360°n$, $n\in Z$

$x=\pm 15°+180°n$, $n\in Z$

5.2. $sin2x=3\sin^2 x$ (7)

[11]

$2\sin x\cos x-3\sin^2 x=0$

$\sin x(2\cos x-3\sin x)=0$

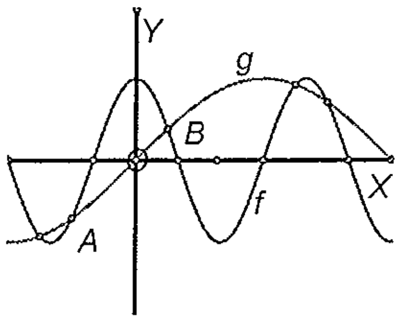
$\sin x=0$ or $3\sin x=2\cos x$

$\sin x=0$ or $\frac{\sin x}{\cos x}=\frac{2}{3}$, therefore, $\tan x=\frac{2}{3}$

$x=0°+360°n$ or $x=180°+360°n$ or $x=33{,}69°+180°n$, $n\in Z$

QUESTION 6

In the diagram, the trigonometric functions $f(x)=\cos 3x$ and $g(x)=\sin x$, $-90° \leq x \leq 180°$, are represented.



6.1. Write down the period of f. (1)

$120°$

6.2. Determine the general solution of the equation $f(x)=g(x)$. (6)

$\sin x=\cos(\pm 3x)$
$x \pm 3x = 90°+360°n$, $n \in Z$
$4x=90°+360°n$ or $-2x=90°+360°n$, $n \in Z$
$x=22{,}5°+90°n$ or $x=-45°-180°n$, $n \in Z$

6.3. Determine the x-coordinates of the points A and B. (2)

$x_A=-45°$

$x_B=22{,}5°$

6.4. Hence, determine the values of x for which $f(x) \geq g(x)$, if $-60° \leq x \leq 60°$. (2)

$-45° \leq x \leq 22{,}5°$

6.5. If the graph of f is shifted horizontally by $15°$, determine the new equation. (1)

[12]

$y=\cos(3x-45°)$

QUESTION 7

7.1. Sketches may be used for the following:

7.1.1. Show that, in any triangle ΔABC : $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$. (2)

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$2bc\cos A = b^2 + c^2 - a^2$$

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

7.1.2. Hence, show that $c\cos A + a\cos C = b$. (4)

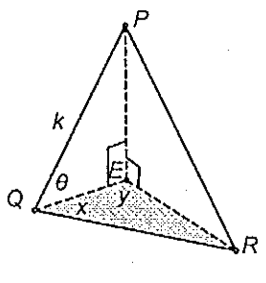
$$c\cos A + a\cos C = c\left(\frac{b^2+c^2-a^2}{2bc}\right) + a\left(\frac{a^2+b^2-c^2}{2ab}\right)$$

$$= \left(\frac{b^2+c^2-a^2}{2b}\right) + \left(\frac{a^2+b^2-c^2}{2b}\right)$$

$$= \frac{b^2+c^2-a^2+a^2+b^2-c^2}{2b}$$

$$= \frac{2b^2}{2b} = b$$

7.2. In the diagram, Q, E and R points on the same horizontal plane. PE is perpendicular to this plane. $E\hat{Q}R = x$ and $Q\hat{E}R = y$, and the angle of elevation from Q to P is θ. $PQ = k\ units$.



7.2.1. Express QR in terms of QE, x and y. (2)

$$\frac{QR}{\sin y} = \frac{QE}{\sin[(180° - (x+y)]}$$

$$QR = \frac{QE.\sin y}{\sin(x+y)}$$

7.2.2. Hence, prove that $QR = \dfrac{k\sin y.\cos\theta}{\sin(x+y)}$. (3)

$$\frac{QE}{k} = \cos\theta$$
$$QE = k\cos\theta$$

$$QR = \frac{k\cos\theta.\sin y}{\sin(x+y)}$$

7.2.3. Hence, if $RQ = RE$, prove that $QR = \dfrac{k\cos\theta}{2\cos x}$ (2)

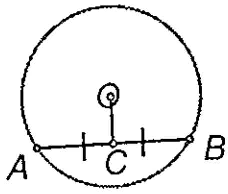
[13]

$$QR = \frac{k\cos\theta.\sin x}{\sin(2x)}.$$
$$= \frac{k\cos\theta.\sin x}{2\sin x\cos x}$$
$$= \frac{k\cos\theta}{2\cos x}$$

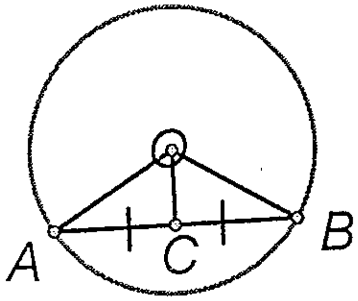
QUESTION 8

8.1. O is the centre of the circle.

Redraw the diagram and prove the theorem which states that the line segment which joins the centre of the circle and the midpoint of a chord, is perpendicular to the chord.



(6)



Constr : AO, BO

Proof : In ΔAOC, ΔBOC :

(1) $AO = BO$ (radii)
(2) $AC = BC$ (given)
(3) OC common side

Therefore $\Delta AOC \equiv \Delta BOC$ $(S\,S\,S)$

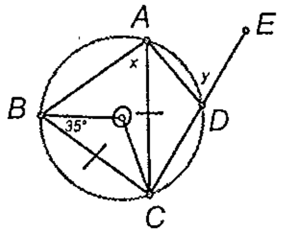
Therefore $A\hat{C}O = B\hat{C}O$

$$= \frac{180^\circ}{2} \quad \text{(angles on a straight line)}$$

$$= 90^\circ$$

8.2. In each diagram, O is the centre of the circle. Calculate, giving reasons, the values of x, y and p.

8.2.1.



(8)

$O\hat{C}B = 35°$ ($OB = OC$; radii)

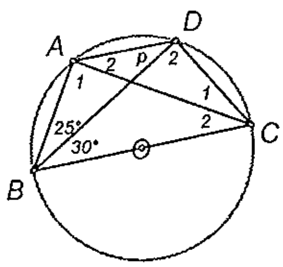
$B\hat{O}C = 180° - 2 \times 35°$ (sum of angles in $\Delta OBC = 180°$)
$= 110°$

$x = \frac{1}{2}(110°)$ (central angle = 2 X inscr. angle)
$= 55°$

$C\hat{B}A = x = 55°$ ($CB = CA$)

$y = C\hat{B}A$ (ext. angle of cyclic quad $ABCD$ = int. opp. angle)
$= 55°$

8.2.2.



(5)
[19]

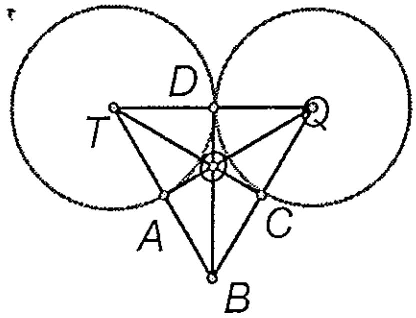
$\hat{A}_1 = 90°$ (BC diameter)

$\hat{C}_2 = 180° - (55° + 90°)$ (sum of angles in $\Delta ABC = 180°$)
$= 35°$

$p = \hat{C}_2$ (AB chord)
$= 35°$

QUESTION 9

T and Q are the centres of the circles. BD is a common tangent, AQ and CT are also tangents to the circles.



9.1. Prove that $TAOD$ is a cyclic quadrilateral. (4)

$T\hat{D}O = 90°$ (DB tangent ; TD radius)
$T\hat{A}O = 90°$ (AQ tangent ; TA radius)
Therefore $T\hat{D}O + T\hat{A}O = 90° + 90°$
$= 180°$

Therefore $TAOD$ is a cyclic quadrilateral (int. opp. angles are supplementary)

9.2. Prove that $TACQ$ is a cyclic quadrilateral. (3)

$T\hat{A}O = 90°$ (AQ tangent ; TA radius)
$T\hat{C}Q = 90°$ (CT tangent ; QC radius)

Therefore $TACQ$ is a cyclic quadrilateral (TQ subtends equal angles)

9.3. Name, without proof, one more cyclic quadrilateral which has D as a vertex. (1)

$DOCQ$ (or $DTBC$)

9.4. If a circle is drawn through the points T, A and Q, what will TQ be called in this circle ? (1)

Diameter

9.5. If AD and CD are drawn and $B\hat{T}C = x$, write down, with reasons, 3 more angles each equal to x. (5)

[14]

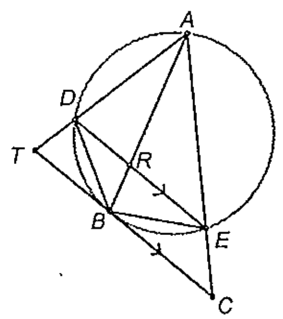
$A\hat{Q}C = A\hat{T}O = x$ (AC chord)

$A\hat{D}O = A\hat{T}O = x$ (AO chord)

$O\hat{D}C = O\hat{Q}C = x$ (OC chord)

QUESTION 10

BC is a tangent to the circumscribed circle of ΔABE, and $DE // BC$.



10.1. Prove that $BD = BE$. (4)

$B\hat{D}E = E\hat{B}C$ (BC tangent; BE chord)

$= B\hat{E}D$ (alt. angles ; $DE \parallel BC$)

Therefore $BD = BE$

10.2. If $DTBR$ is a cyclic quadrilateral, prove that $2B\hat{E}C - E\hat{B}C = 180°$. (Hint: Let $B\hat{E}C = x$ and $E\hat{B}C = y$.) (7)

[11]

$A\hat{D}B = B\hat{E}C = x$ (ext. angle of cyclic quad $ADBE$ = int. opp. angle)

$B\hat{D}E = E\hat{B}C = y$ (BC tangent ; BE chord)

Then $A\hat{D}E = x - y$

and $A\hat{R}D = \hat{T}$ (ext. angle of cyclic quad $DTBR$ = int. opp. angle)

$= A\hat{D}E$ (corr. angles ; $DE \parallel BC$)

$= x - y$

also $D\hat{A}B = D\hat{E}B = y$ (DB chord)

Therefore $2(x-y)+y = 180°$ (sum of angles in $\Delta ADR = 180°$)

Therefore $2x - 2y + y = 180°$

Therefore $2x - y = 180°$

Therefore $2B\hat{E}C - E\hat{B}C = 180°$.