St John's College

Preliminary Examinations
July 2016
Mathematics Paper 2

Examiner: **D Clogg**
Moderator: **D Grigoratos**

Time: **3 hrs**
Marks: **150**

Name: MEMO

Teacher: **JJ** **DC** **BT** **DG** **GE** **GvH** **LC**

PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY

1. This question paper consists of 22 pages. An **Information Sheet is provided separately.** Please check that your paper is complete.
2. Read the questions carefully.
3. Answer ALL the questions on the question paper. Note that there is space for additional working at the end of the paper.
4. You may use an approved non-programmable and non-graphical calculator, unless otherwise stated.
5. Round off your answers to one decimal digit where necessary.
6. All the necessary working details must be clearly shown. Equations may not be solved solely with a calculator.
7. It is essential that you present your work neatly and logically.

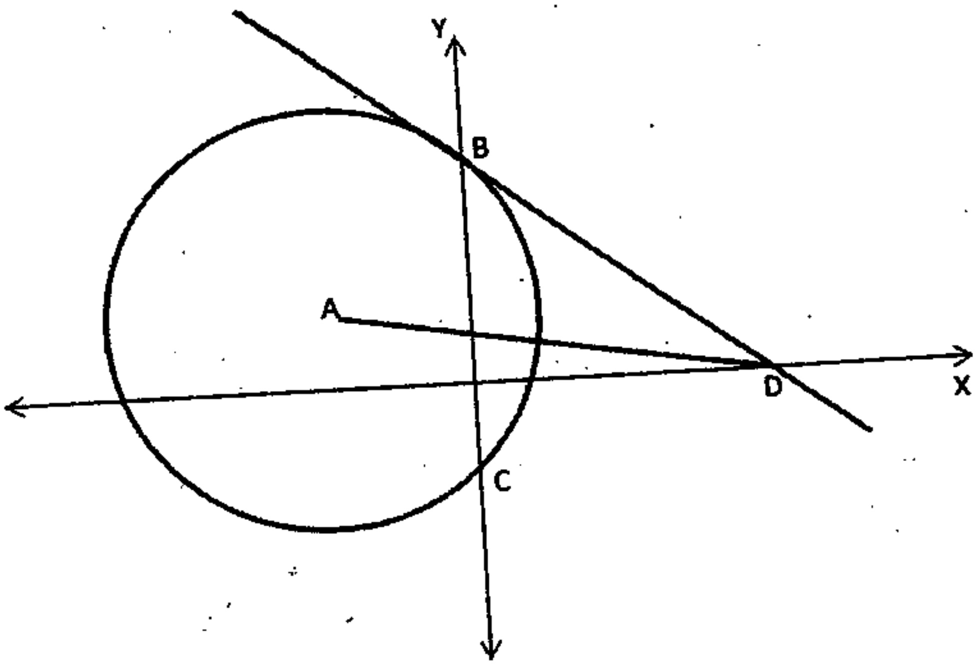
Qu	1	2	3	4	5	6	7	8	9	10	11	Tot
Out of	20	15	6	32	6	7	9	15	14	10	16	150
Mark												

SECTION A

Question 1: [20 marks]

The equation defining the circle below is $x^2 + 6x + y^2 - 4y = 12$.

B and C are the y-intercepts of the circle.



1.1 Determine the coordinates of A, the centre of the circle, and the length of the radius.

$x^2 + 6x + (3)^2 + y^2 - 4y + (2)^2 = 12 + 9 + 4$

$(x+3)^2 + (y-2)^2 = 25$ ✓

$\therefore A(-3;2)$ ✓

$r = 5$ units ✓

1.2 Determine the length of BC

$y^2 - 4y - 12 = 0$ ✓
$(y-6)(y+2) = 0$ ✓
$y = 6$ or $y = -2$ ✓
$\therefore BC = 8$ units ✓

1.3 Determine the equation of the tangent BD to the circle at point B, given that $B(0;6)$. (4)

$m_r = 4/3$ ✓
$\therefore m_t = -3/4$ ✓

$y = -\frac{3}{4}x + 6$ ✓

1.4 If $x = k$ is a tangent to the circle, determine the value(s) of k. (3)

$k = 2$ ✓ or $k = -8$ ✓

1.5 Calculate the size of $B\hat{D}A$. (2)

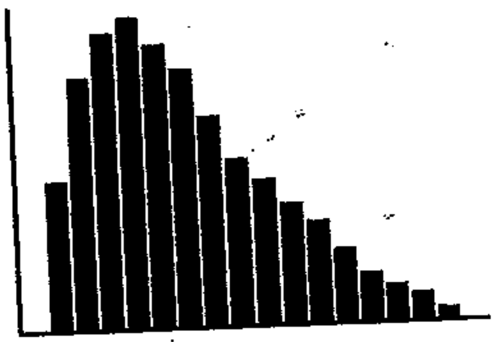
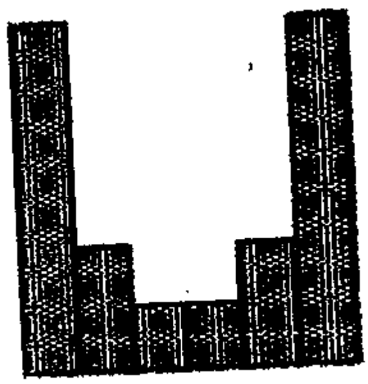
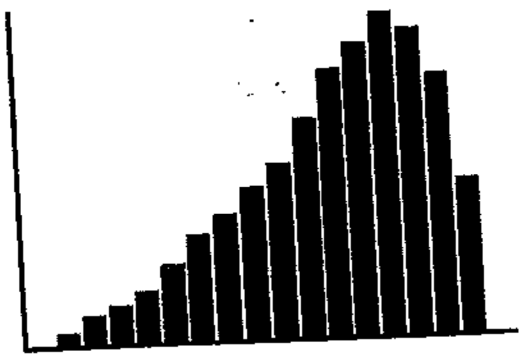
$m_{BO} = -3/4$
$\tan B\hat{O}x = -3/4$ ✓
$B\hat{O}x = 143{,}1°$ ✓
$O(8;0)$ ✓
$m_{AO} = -2/11$
$\tan A\hat{O}x = -2/11$ ✓
$A\hat{O}x = 169{,}7°$ ✓
$\therefore B\hat{O}A = 26{,}6°$ ✓

(6)

Question 2: [15 marks]

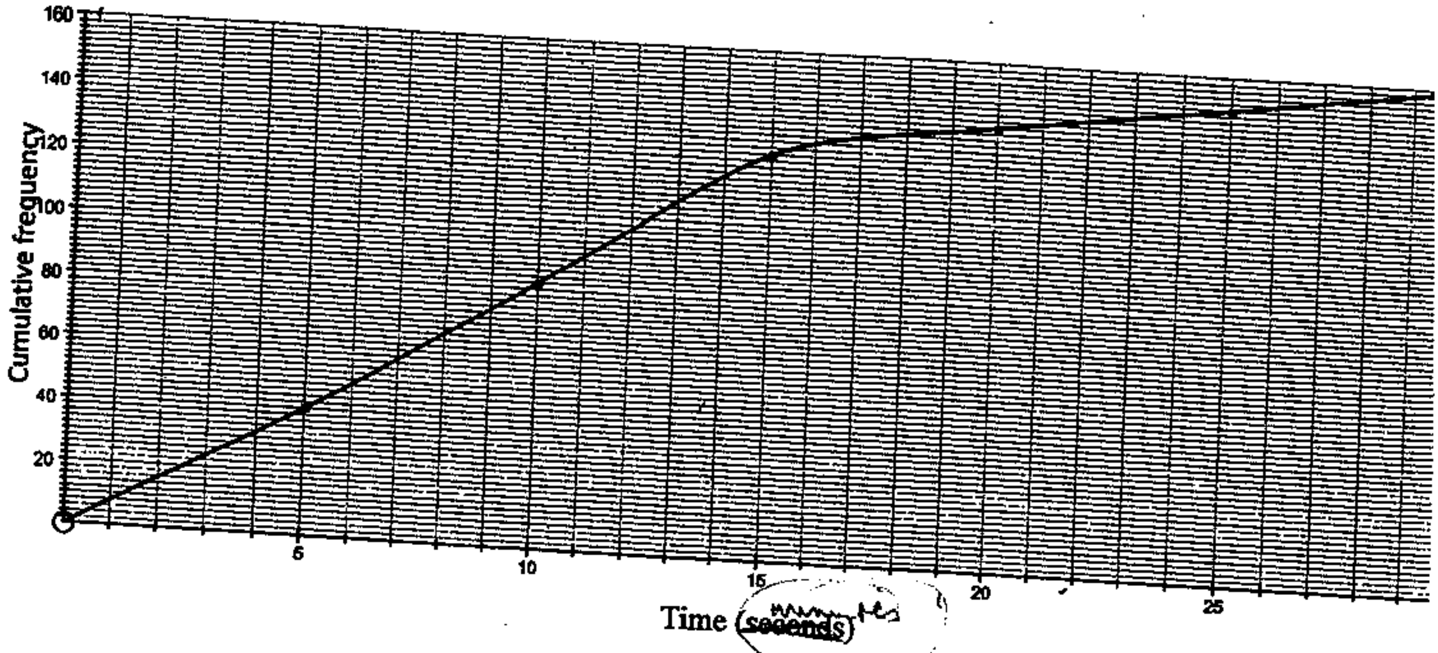
2.1 Refer to the following statements below labelled A to C and match the statements to their most expected histogram labelled D to F.

A.	B.	C.
The age at death in developed countries (First world countries)	The salaries of a major corporate company.	The results of a test where some of the students had been taught and others had not been taught the section of work covered.

D. 	E. 
F. 	Fill in the answers in this block: A → F ✓ B → D ✓ C → E ✓

(3)

2.2 The diagram below shows a cumulative frequency curve for the lengths of telephone calls from a home landline during the first 6 months of the year.



State whether each of the following statements are True or False:

a) The distribution of these times is skewed. T

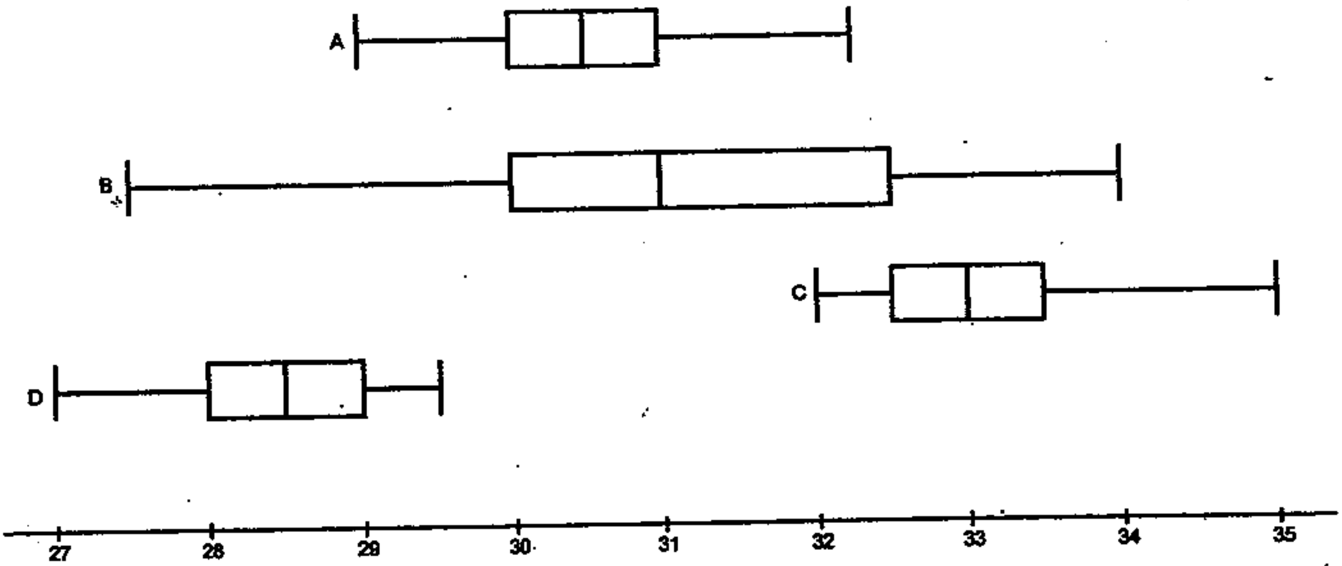
b) The majority of the calls last longer than 6 minutes. T

c) The majority of the calls last between 5 and 10 minutes. F

d) The majority of the calls are shorter than the mean length. T

(4)

2.3 A group of athletes frequently run round a cross-country course in training. The box and whisker plots below represent the times taken by athletes A, B, C and D to complete the course.



a) Compare the times taken by athletes C and D. Include variability and skewness in the analysis.

D is always faster than C ✓
C's times have a higher range ∴ more variable ✓
C's times are positively skewed ✓
D's times are negatively skewed ✓

(4)

b) Which of the athletes A or B would you pair up to race against

i) C ? Justify your choice.

Although A's slowest time is greater than C's fastest time, A will most likely win against C (note: open-ended answer)

(2

ii) D ? Justify your choice

A has little chance of beating D whereas B has a better chance of beating D ∴ athlete B would be a better choice

(

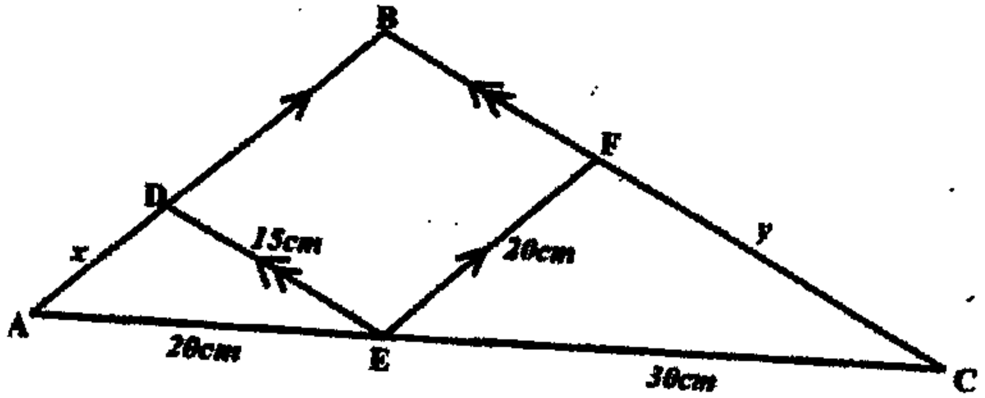
Question 3: [6 marks]

3.1 Complete the theorem that states: A line drawn parallel to one side of a triangle cuts the other two sides .....

so as to divide them proportionally

(1)

3.2 Given ΔABC with D a point on AB such that $DE \parallel BC$ and F is a point on BC such that $AB \parallel EF$. Given $DE = 15cm$, $EF = 20cm$, $EC = 30cm$ and $AE = 20cm$.



Determine, giving reasons, the values of x and y.

$DE = BF = 15$ opp sides of parm equal

$DB = EF = 20$ opp sides of parm equal

In ΔABC , $DE \parallel BC$

$\frac{AD}{DB} = \frac{AE}{EC}$ prop-int Th^m

$\frac{x}{20} = \frac{20}{30}$

$x = 13.3$ cm

In ΔABC , $EF \parallel AB$

$\frac{FC}{FB} = \frac{EC}{AE}$ prop int Th^m

$\frac{y}{15} = \frac{30}{20}$

$y = 22.5$ cm

(5)

Question 4: [32 marks]

4.1 If θ , 2θ and 3θ are the angles of a triangle evaluate, without the use of a calculator:

$\cos^2\theta + \cos^2 2\theta + \cos^2 3\theta$

$6\theta = 180°$ ∠'s in Δ

$\theta = 30°$ ✓

$\cos^2 30° + \cos^2 60° + \cos^2 90$

$\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + (0)^2$ ✓✓

1 ✓

(4)

4.2 Evaluate, without the use of a calculator:

$\sin 124°.\sin 64° + \sin 214° \sin 26°$

$\sin 56° \sin 64° - \sin 34° \sin 26°$ ✓

$\cos 34° \cos 26° - \sin 34° \sin 26°$ ✓

$\cos(34° + 26°)$ ✓

$\cos 60°$ ✓

$1/2$ ✓

(5)

4.3 a) Prove the following identity: $\frac{3\cos 2\theta+\cos\theta-2}{3\sin 2\theta-5\sin\theta}=\frac{\cos\theta+1}{\sin\theta}$

LHS : $\dfrac{3(2\cos^2\theta-1)+\cos\theta-2}{6\sin\theta\cos\theta-5\sin\theta}$ ✓

$\dfrac{6\cos^2\theta+\cos\theta-5}{\sin\theta(6\cos\theta-5)}$ ✓

$\dfrac{(6\cos\theta-5)(\cos\theta+1)}{\sin\theta(6\cos\theta-5)}$ ✓

$\dfrac{\cos\theta+1}{\sin\theta}$ ✓

$\therefore$ LHS = RHS

(5)

b) For which values of θ in the interval $[-180°; 180°]$ is the above identity not valid?

$\sin\theta = 0$ ✓ or $\cos\theta = 5/6$ ✓

$\theta = k180°$ ✓ $\theta = \pm 33{,}6° + k360°$ ✓

$\theta = \{0°; \pm 33{,}6°; \pm 180°\}$ ✓

4.4 Find the general solution of $\tan 5\theta = \tan\theta$ (5)

$5\theta = \theta + k180°$ ✓

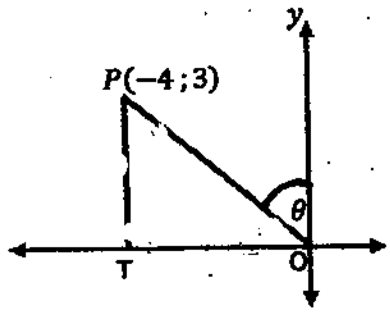
$4\theta = k180°$ ✓

$\theta = k45°; \ k \in \mathbb{Z}$ ✓

(3)

4.5 Refer to the sketch below:

Δ PTO is drawn with ~~P~~ T a point on the negative x axis and $P(-4\,;3)$ and $Y\hat{O}P = \theta$.



Use the diagram to determine the value of $\sin(90° + \theta) - \cos(90° - \theta)$.

$\cos\theta - \sin\theta$ ✓ ✓

$\frac{3}{5} - \left(\frac{-4}{5}\right)$ ✓ ✓

$\frac{7}{5}$ ✓

(5)

4.6 If $\cos\theta = k$ express the following in terms of k.

$\sin\left(\frac{\theta}{2} + 45°\right)\cos\left(\frac{\theta}{2} + 45°\right)$

$\frac{2\sin\left(\frac{\theta}{2} + 45°\right)\cos\left(\frac{\theta}{2} + 45°\right)}{2}$ ✓

$\frac{\sin 2\left(\frac{\theta}{2} + 45°\right)}{2}$ ✓

$\frac{\sin(\theta + 90°)}{2}$ ✓

$\frac{\cos\theta}{2}$ ✓

$\frac{k}{2}$ ✓

(5)

Total: Section A: 73 marks

SECTION B:

Question 5: [6 marks]

The heights of several plants (in cm) was measured at a certain stage after planting, and the following data was recorded:

x = days after planting	14	20	8	15	18	11	14
h = height (cm)	6	11	3	8	10	4	

The record of the last height has been lost, but we do know that the regression line had equation $h = 0{,}72x - 3{,}31$.

5.1 Estimate, to the nearest centimetre, what the last recorded height was.

$h = 0{,}72(14) - 3{,}31$

$= 6{,}8$ cm

(2)

5.2 Calculate the correlation coefficient for the data relating to the first 6 plants correct to four decimal places. (i.e. ignoring the last column).

$r = 0{,}9839$

(2)

5.3 Some time later another plant's height 25 days after planting was found to be 20 cm. Comment on how surprising (or not) this is in the light of your previous results.

With this model $h = 0{,}72(25) - 3{,}31 = 14{,}7$ cm and not the actual 20 cm, this is not surprising as extrapolation occurs and the graph may no longer conform to the same regression line.

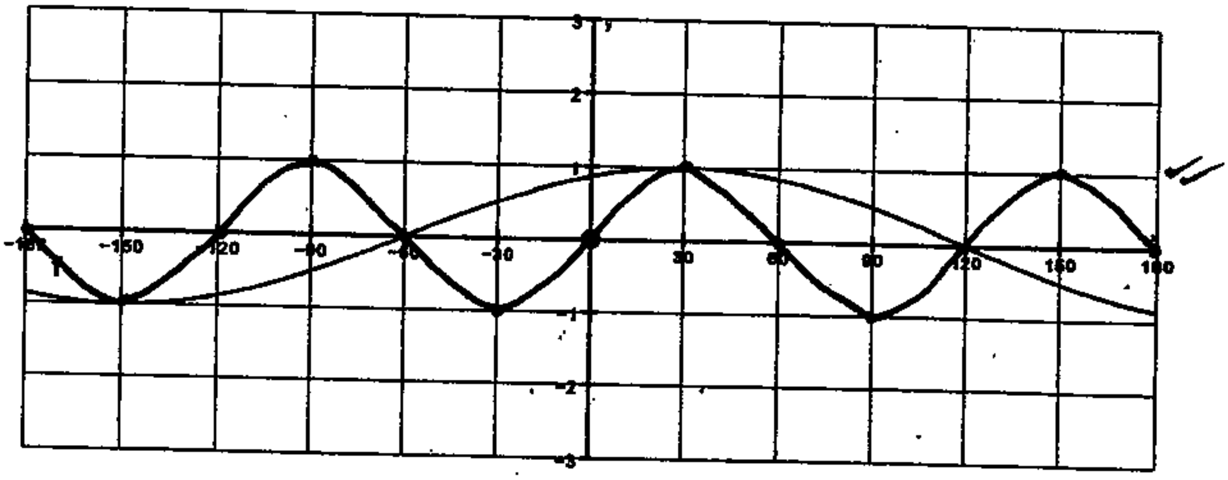
(open to logical interpretation

(2)

Question 6: [7 marks]

The graph of $f(x) = \cos(x - 30°)$ where $x \in [-180°; 180°]$ is sketched below.

6.1 Sketch the graph of $g(x) = \sin 3x$, on the same set of axes.



(2)

6.2 Using the graphs above, determine the value(s) of x for $x \in [0°; 180°]$ for which:

$f(x) > g(x)$

$x \in [0°; 30°) \cup (30°; 120°)$ +1 for correct brackets

(3)

6.3 Write down:

a) the period of h if $h(x) = g(3x)$

$40°$

(1)

b) the new equation of $f(x)$ if it is shifted 45° to the left.

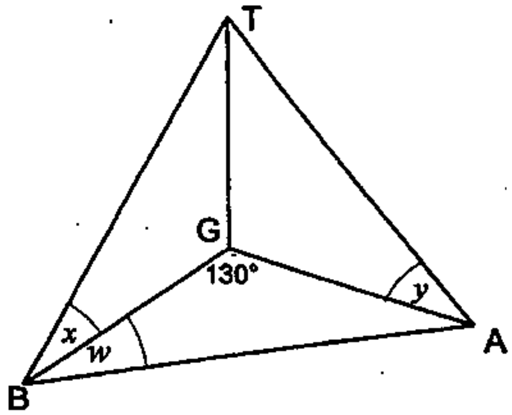
$f(x) = \cos(x + 15°)$

(1)

Question 7: [9 marks]

A, B and G are points on the base of a cut-out rectangular ~~prism~~ pyramid. TG is a vertical side of the prism. The angles of elevation of T from B and A are x and y respectively.

$G\hat{B}A = w$ and $B\hat{G}A = 130°$.



7.1 Prove that $\sin w = \frac{\tan x \cdot \sin(50° - w)}{\tan y}$

In $\triangle BGT$ In $\triangle GTA$

$BG = \frac{TG}{\tan x}$ ✓ $GA = \frac{TG}{\tan y}$ ✓

In $\triangle BGA$

$$\frac{\sin w}{\frac{TG}{\tan y}} = \frac{\sin(50 - w)}{\frac{TG}{\tan x}}$$ ✓

$\tan y \sin w = \tan x \sin(50 - w)$ ✓

$\therefore \sin w = \frac{\tan x \sin(50 - w)}{\tan y}$ ✓

(5)

7.2 Now find the volume of the above ~~prism~~ pyramid TBAG if $GT = 30cm$, $x = 40°$ and $y = 60°$. Note that the volume of a prism is calculated by the formula:

$V = \frac{1}{3}AH$ where A is the area of the base and H is the height of the prism.

$BG = 35.8$ $GA = 17.3$ ✓

$V = \frac{1}{3}\left[\frac{1}{2}(35.8)(17.3)\sin 130°\right](30)$ ✓ ✓

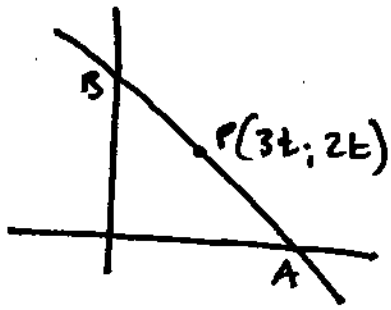
$= 2372,2 \text{ cm}^3$ ✓

(4)

Question 8: [15 marks]

8.1 The line with a gradient of -2 passing through the point $P(3t\,;2t)$ intersects the x-axis at A and the y-axis at B.

a) Find the area of ΔAOB in terms of t.



$y - 2t = -2(x - 3t)$

$y = -2x + 8t$ ✓

$\therefore A(4t;0)$ ✓ $B(0;8t)$ ✓

$A = \frac{1}{2}(4t)(8t)$ ✓

$= 16t^2$ ✓

(5)

b) The line through P perpendicular to AB intersects the x-axis at C.

Show that the midpoint of line PC has co-ordinates $(t\,;t)$.

$m_{PC} = \frac{1}{2}$ ✓

$y - 2t = \frac{1}{2}(x - 3t)$

$y = \frac{1}{2}x + \frac{t}{2}$ ✓

$\therefore C(-t;0)$ ✓

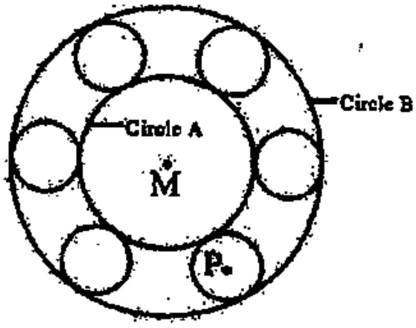
Mid-pt of PC $\left(\frac{3t-t}{2};\frac{2t+0}{2}\right)$ ✓

$= (t;t)$ ✓

(5)

8.2 The figure above shows the cross-section of a wheel bearing. The smaller circles represent ball-bearings which roll between two larger concentric circles, A and B, which both have centre M.

The equation of circle A is $x^2 + y^2 + 3x - 6y = 9$.



P is the centre of one of the small ball-bearings, each of which has a diameter of 2 units. Find the equation which best describes the path of P as it rolls between the two larger circles.

Write the answer in the form $(x - a)^2 + (y - b)^2 = r^2$

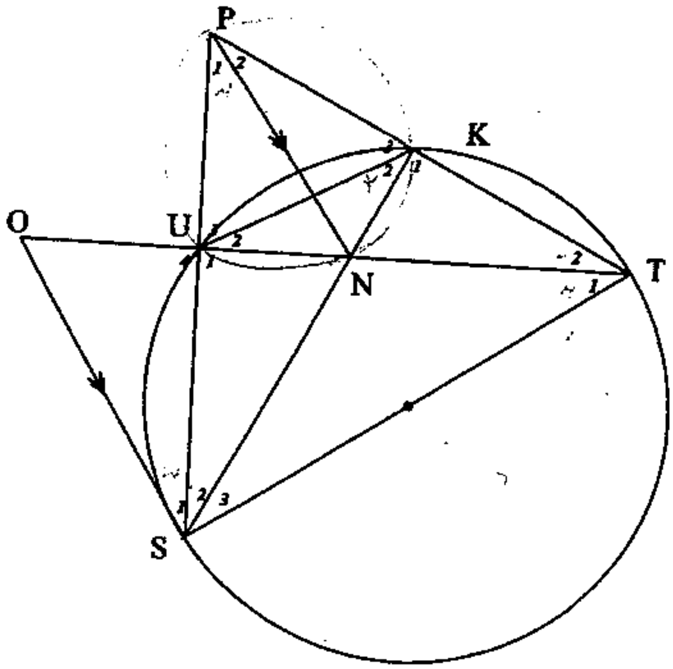
Circle M : $\left(x + \frac{3}{2}\right)^2 + (y-3)^2 = \frac{81}{4}$ $\therefore r = \frac{9}{2}$

$\therefore$ Circle P : $\left(x + \frac{3}{2}\right)^2 + (y-3)^2 = \frac{121}{4}$

(5)

Question 9: [14 marks]

Refer to the diagram below, ST is a diameter of the circle.
OS || PN, TO bisects $S\hat{T}P$.



Prove that

9.1 PUNK is a cyclic quadrilateral

$\hat{U}_1 = 90°$ ∠ in semi circle ✓

$\hat{K}_1 = 90°$ ∠ in semi circle ✓

$\therefore \hat{K}_{2+3} = 90°$ ∠'s on str line ✓

$\therefore \hat{U}_1 = \hat{K}_{2+3}$ ✓

$\therefore$ PUNK is a cyclic quad ext ∠ = opp int ∠ ✓

(5)

9.2 SO is a tangent to circle KUST

$\hat{T}_1 = \hat{T}_2 = x$ TO bisects $S\hat{T}P$ ✓

$\hat{T}_1 = \hat{K}_2 = x$ ∠'s in same seg ✓

$\hat{K}_2 = \hat{P}_1 = x$ ∠'s in same seg ✓

$\hat{P}_1 = \hat{S}_1 = x$ alt ∠'s PN || OS ✓

$\therefore \hat{S}_1 = \hat{T}_1 = x$ ✓

$\therefore$ OS is a tangent ✓ converse : tan chord Th^m

(6)

9.3 POST is a cyclic quadrilateral

$\hat{T}_1 = \hat{T}_2 = x$ ✓

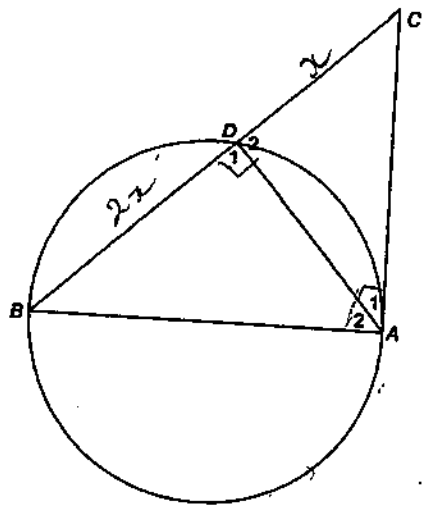
$\hat{S}_1 = \hat{T}_2 = x$ ✓

$\therefore$ POST is a cyclic quad ✓ line seg subtended by equal ∠s

(3)

Question 10: [10 marks]

10.1 In the figure, AB is a diameter of the circle and CA is a tangent to the circle at A. BC is joined and cuts the circle in D. AD is joined.



If $DC = \frac{1}{3}BC = x$, show that $\sqrt{AD^2 + AB^2 + AC^2} = x\sqrt{11}$

$\hat{A}_{1+2} = 90^\circ$ tan ⊥ rad

$\hat{D}_1 = 90^\circ$ ∠ in semi-circle

If $DC = x$ then $BC = 3x$ and $DB = 2x$

$AD^2 = BD.DC$ perp from rt ∠ vertex to hyp
$= 2x^2$

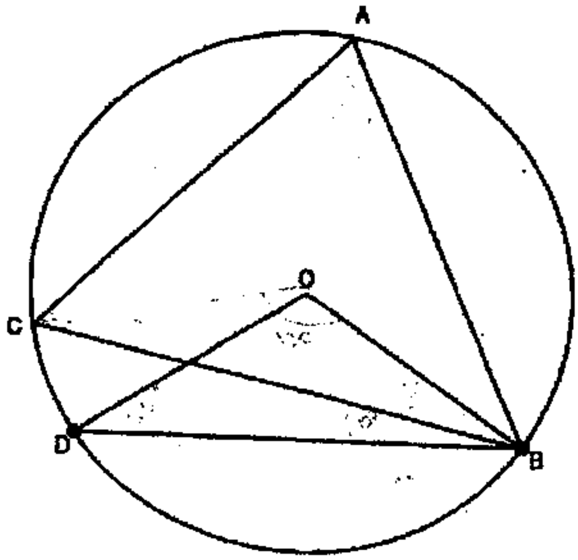
$AB^2 = BC.BD$ "
$= 6x^2$

$AC^2 = DC.BC$ "
$= 3x^2$

$\therefore \sqrt{AD^2 + AB^2 + AC^2} = \sqrt{11x^2}$
$= x\sqrt{11}$

(5)

10.2 Refer to the diagram below. O is the centre of the circle ABDC. If $B\hat{A}C = 60°$ and $C\hat{B}D = 20°$,



Determine $O\hat{D}B$

Construct line OC ✓

$B\hat{O}C = 120°$ ✓ ∠ at centre

$O\hat{B}C = 30°$ ✓ Isos Δ, OB = OC

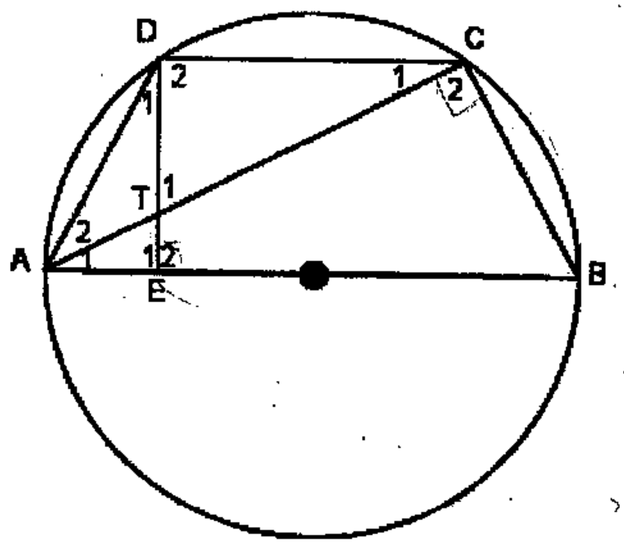
$\therefore O\hat{B}D = 50°$ ✓

$\therefore O\hat{D}B = 50°$ ✓ Isos Δ, radii OD = OB

(5)

Question 11: [16 marks]

Refer to the diagram below. AB is a diameter of circle ABCD. $DE \perp AB$ and AC is joined.



Prove that:

11.1 $\hat{T}_1 = \hat{B}$

$\hat{E}_2 = 90°$ ✓ $DE \perp AB$

$\hat{C}_2 = 90°$ ✓ ∠ in semi circle

∴ ETCB is cyclic quad ✓ ext ∠ = opp int ∠

∴ $\hat{T}_1 = \hat{B}$ ✓ ext ∠ of cyclic quad

(4)

.2 $\hat{D}_1 = \hat{C}_1$

$\hat{C}_1 + \hat{A}_2 = 180 - \hat{D}$ ✓ $\angle$'s in Δ

$\hat{B} = 180 - \hat{D}$ ✓ opp $\angle$'s of cyclic quad

$\therefore \hat{B} = \hat{C}_1 + \hat{A}_2$ ✓

$\hat{T}_1 = \hat{D}_1 + \hat{A}_2$ ✓ ext $\angle$ of Δ

$\hat{B} = \hat{T}_1$ ✓ proved above

$\therefore \hat{C}_1 = \hat{D}_1$ ✓

(6)

1.3 $DA^2 = AT.AC$

In ΔDAT and ΔDAC

$\hat{A}_2 = \hat{A}_2$ common ✓

$\hat{D}_1 = \hat{C}_1$ proved ✓

$\therefore \hat{T}_2 = \hat{D}$ $\angle$'s in Δ ✓

$\therefore \Delta ADT \parallel\!| \Delta ACD$ AAA ✓

$\therefore \frac{AD}{AC} = \frac{DT}{CD} = \frac{AT}{AD}$ ✓

$\therefore DA^2 = AT.AC$ ✓

(6)

Total: Section B: 77 marks

Total: 150 marks