assessment matters

GRADE 12 EXAMINATION
NOVEMBER 2008

ADVANCED PROGRAMME MATHEMATICS

Time: 3 hours | 300 marks

PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY

1. This question paper consists of 19 pages and an insert of 4 pages (i – iv) containing formula sheets. Please check that your question paper is complete.

2. This question paper consists of **four** modules, of which **two** must be answered.

   MODULE 1: CALCULUS AND ALGEBRA (200 marks) is compulsory.

   Choose **ONE** of the **THREE** Optional Modules:

   MODULE 2: STATISTICS (100 marks) OR
   MODULE 3: FINANCE AND MODELLING (100 marks) OR
   MODULE 4: MATRICES AND GRAPH THEORY (100 marks)

3. Non-programmable and non-graphical calculators may be used, unless otherwise indicated.

4. All necessary calculations must be clearly shown and writing should be legible.

5. Diagrams have not been drawn to scale.

6. Write all your answers in the separate Answer Book provided.

MODULE 1 CALCULUS AND ALGEBRA

QUESTION 1

1.1 (a) Factorise $x^3 - 1$ (1)

(b) Hence solve $x^3 - 1 = 0$ for $x \in \boldsymbol{C}$ (5)

1.2 If $1-\sqrt{2}$ and $1+\sqrt{2}$ are both zeros of $f(x) = x^4 - 2x^3 + 4x^2 - 10x - 5$, factorise $f(x)$ fully for $x \in \boldsymbol{C}$. (10)

16 marks

QUESTION 2

2.1 Solve for x:

(a) $\log_{0,1}(x-20) + \log 2x = 1$ (4)

(b) $\dfrac{e^x}{e^x - 1} = 5$, correct to 2 decimal places (4)

(c) $|x|^2 - 4|x| - 12 = 0$ (6)

2.2 The intensity of sound, D, measured in decibels (dB) is given by the formula $\mathrm{D} = 10 \log\left(\dfrac{\mathrm{L}}{10^{-16}}\right)$ where L is the power of the sound in watts per square centimetre $(\mathrm{W/cm^2})$ and $10^{-16}\ \mathrm{W/cm^2}$ is the power of sound just below the threshold of hearing.

Find the power of the sound L (in $\mathrm{W/cm^2}$) experienced by the audience seated in front of an orchestra , measured at 107 dB. (6)

20 marks

QUESTION 3

Jodi was rushed for time during his Preliminary Advanced Programme Mathematics examination and only got part of the way answering the following question.

'Prove by the principle of mathematical induction that $x^n - y^n$ is divisible by $x - y$ for all $n \in N$ '

His partial solution follows.

Complete his proof, starting at Step 3, and write the final conclusion.

He wrote …

Step 1

Prove statement true for n = 1

LHS: $= x^1 - y^1$ which is clearly divisible by $x - y$

Step 2

Assume statement true for n = k

LHS: $x^k - y^k = p(x - y)$

Step 3

Prove true for $n = k + 1$ …

12 marks

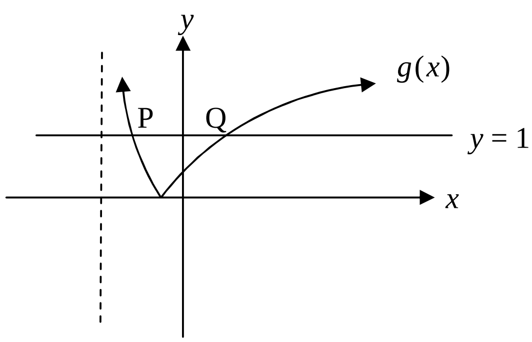
QUESTION 4

4.1 Given $f(x)=\dfrac{e^{2x}-e^{x}}{3e^{x}}$

(a) Simplify $f(x)$ to the form $f(x) = ke^{x} + m$. (2)

(b) Hence sketch $f(x)$, clearly indicating intercepts and asymptotes. (4)

4.2 The sketch shows the curve of $g(x) = |\log(2x+3)|$ intersecting the line $y = 1$ at P and Q.



Determine the x-coordinates of the points P and Q, the points of intersection of $g(x)$ and the line $y=1$. (6)

12 marks

QUESTION 5

Given the function: $f(x)=\begin{cases} 3-ax^2 & \text{if} \quad x \geq 1 \\ -4x+5 & \text{if} \quad x<1 \end{cases}$

5.1 Find the value of a if f is continuous at $x = 1$. (5)

5.2 Assuming that $a = 2$, prove that f is differentiable at $x = 1$. (9)

14 marks

QUESTION 6

$$g(x) = \frac{2x^2 - 5x + 2}{2x^2 - x - 1}$$

6.1 Determine the x-intercepts of the graph of g. (2)

6.2 Determine the equations of any vertical and horizontal asymptotes. (7)

6.3 Find, and simplify, an expression for $g'(x)$, the derivative of $g(x)$. (8)

6.4 Establish if g has any turning points (local maxima or minima). (3)

6.5 Sketch the graph of g, showing all x- and y- intercepts and asymptotes. (7)

27 marks

QUESTION 7

7.1 Find $f'(x)$ if $f(x) = \sqrt{x + \sqrt{x}}$ (6)

7.2 Prove that $\frac{d}{dx}\left(\sin x.\cos(a - x)\right) = \cos(a - 2x)$ (7)

Your working must clearly establish that you have proved the identity.

13 marks

QUESTION 8

The coordinates of each point $(x; y)$ of a graph are given as $(3t + 1 ; t^2)$

i.e. $x = 3t + 1$ and $y = t^2$ where $t \in \mathbf{R}$.

8.1 Determine an expression, in terms of t, for $\frac{dy}{dx}$ (4)

[Hint: use the fact that $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$]

8.2 Determine the coordinates of the point on the graph where the gradient is equal to 2. (5)

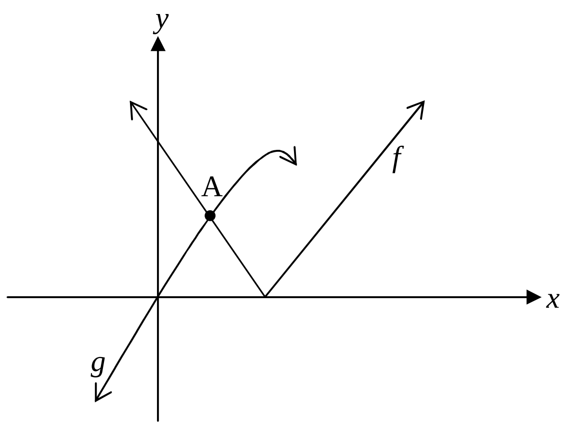
9 marks

QUESTION 9

Parts of the functions:

$f(x) = |x-1|$ and $g(x) = x(x-2)(x-3)$

are shown, intersecting at A.



9.1 Without calculation, state the number of solutions to the equation

$$|x-1| = x(x-2)(x-3) \qquad (2)$$

9.2 Give a rough estimate of the *x*-coordinate at A. (1)

9.3 Use Newton's method to find the *x*-coordinate at A, correct to 6 decimal places. (10)

13 marks

QUESTION 10

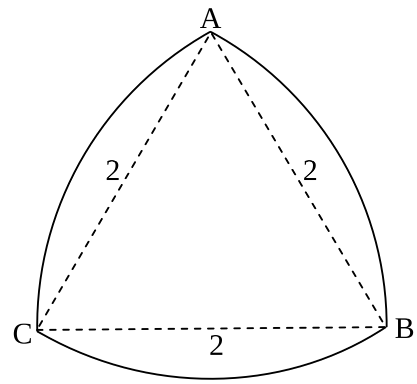
A cubic curve of the form, $y = ax^3 + bx^2 + cx + d$, has a point of inflection at (2; – 22).
The graph passes through the origin and has a gradient of – 3 at the origin.

10.1 Show that $b = -6a$. (5)

10.2 By finding the values of a, b, c and d, determine the equation of the graph. (9)

14 marks

QUESTION 11



A coin is designed by starting with an equilateral triangle ABC of side 2 cm. With centre A, an arc of a circle is drawn joining B to C. Similar arcs with centres B and C join C to A and A to B respectively.

11.1 Find the perimeter of the coin. (3)

11.2 Find the area of the face of the coin. (8)

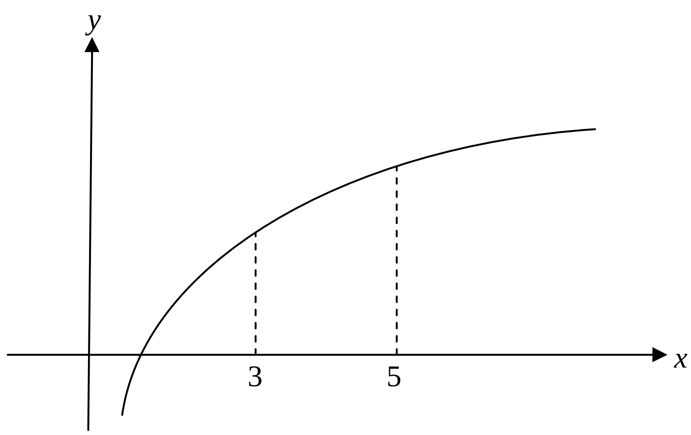
11 marks

QUESTION 12

12.1 Without the use of a calculator, determine the value of the integral correct to 2 decimal places.

$$\int_2^3 \frac{x}{\sqrt{x^2-1}}\,dx$$ (10)

12.2



The graph of $y = \ln x$ is shown.

(a) Determine an approximation (greater than the true value) for the area enclosed by the graph, the x-axis and the lines $x = 3$ and $x = 5$, by using 5 rectangles of equal width. (10)

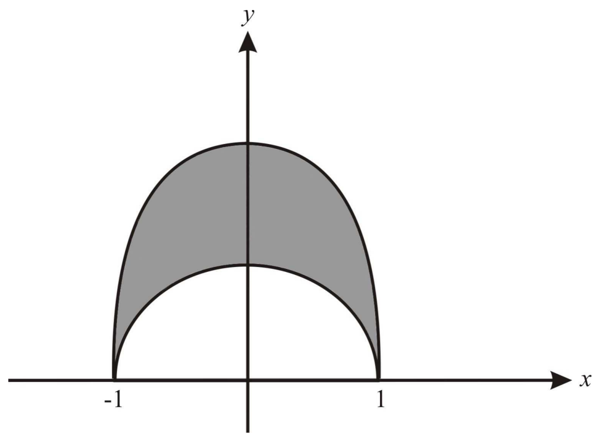
(b) You are now given that:

$$\int \ln x \; dx = x.\ln x - x + C \quad \text{where } C \text{ is a constant.}$$

Use this result to find the exact area of the region defined in part (a), correct to 3 decimal places. (6)

26 marks

QUESTION 13



The area enclosed between the parts of two curves $x^2 + y^2 = 1$ and $4x^2 + y^2 = 4$ is rotated by 2π radians about the x-axis.

Find the volume of the solid formed.

13 marks

Total for Module 1: 200

MODULE 2 STATISTICS

QUESTION 1

The birth weights of babies in South Africa are normally distributed and have a mean of 3 118 g and a standard deviation of 850,5 g. The babies of 40 women who received monthly prenatal care have a mean birth weight of 3 345 g.

1.1 Write down a suitable null hypothesis (H_0) and an alternate hypothesis (H_1) for testing whether monthly prenatal care makes a difference to the birth weight of babies. (4)

1.2 Use a 5% level of significance to test the hypothesis in 1.1 above. (7)

1.3 What minimum birth weight, greater than the mean, would cause us to reject (H_0) in favour of (H_1) when using a 5% level of significance to test the hypothesis in 1.1 above? (4)

15 marks

QUESTION 2

The following data were collected during experimental conditions to find the effect of temperature, (x° C) on the pH, (y) of soya milk.

Temperature x°C	4	9	17	24	32	40	46	57	63	69	72	78
pH y	6,85	6,75	6,74	6,63	6,68	6,52	6,54	6,48	6,36	6,33	6,35	6,29

2.1 By observing the data, explain what is revealed about the relationship between x and y. (2)

2.2 (a) Use your calculator to evaluate the correlation coefficient r, for this data. (4)

(b) What does the value of r, found above, tell us about the strength of a straight-line relationship? (1)

2.3 Determine the equation of the least squares regression line of y on x, by using an appropriate formula.
You may use the following information:

$$\sum x = 511; \quad \sum y = 78,52; \quad \sum x^2 = 28949; \sum y^2 = 514,17; \quad \sum xy = 3291,88$$
$$\bar{x} = 42,58; \quad \bar{y} = 6,54$$
(10)

2.4 (a) Estimate the pH of soya milk at 95°C. (3)

(b) Indicate, with a reason, how reliable this estimate might be. (2)

22 marks

QUESTION 3

Mr I.E. Bean, the headmaster of a local school, gathered data from an observational study carried out on the 700 learners in his school.

3.1 He found that 498 learners were in favour of a change in the timetable. Find an approximate 90% confidence interval for the proportion of learners who were **not** in favour of a change. (10)

3.2 He found the 95% confidence interval for the time in minutes that each learner studied each week day, to be [105,89; 110,11].

(a) Find the average time in minutes studied per learner. (2)

(b) Find the standard deviation for this data. (6)

3.3 He found that 55% of his learners travelled to school by bus. If he randomly selected 10 learners, find the probability that at least two of them travelled to school by bus. (9)

27 marks

QUESTION 4

4.1 Mary and Eve are in a group of 8 girls. In how many ways can the 8 girls be seated on a bench if Mary and Eve must not sit next to each other? (3)

4.2 Given that $P(A) = 0{,}4$; $P(B) = 0{,}5$ and $P(A \cap B) = 0{,}1$

(a) find $P(A \mid B)$ (1)

(b) find $P(A' \mid B)$ (1)

(c) show mathematically whether the events A and B are independent or not. (3)

4.3 A sample of 3 counters is drawn randomly from a bag containing 8 red and 4 black counters, without replacement. Let the random variable X = number of black counters drawn.

(a) Write down the probability mass function. (6)

(b) Hence, or otherwise, find the probability that two black counters are drawn. (5)

4.4 A probability density function for X, the delay in hours, for a flight from Cape Town is given by:

$$f(x) = \begin{cases} 0{,}2 - 0{,}02x \quad ; & 0 \le x \le 10 \\ 0 \quad \text{elsewhere} & \end{cases}$$

(a) Find the probability that the delay will be between two and six hours. (8)

(b) Find the median of this distribution. (9)

36 marks

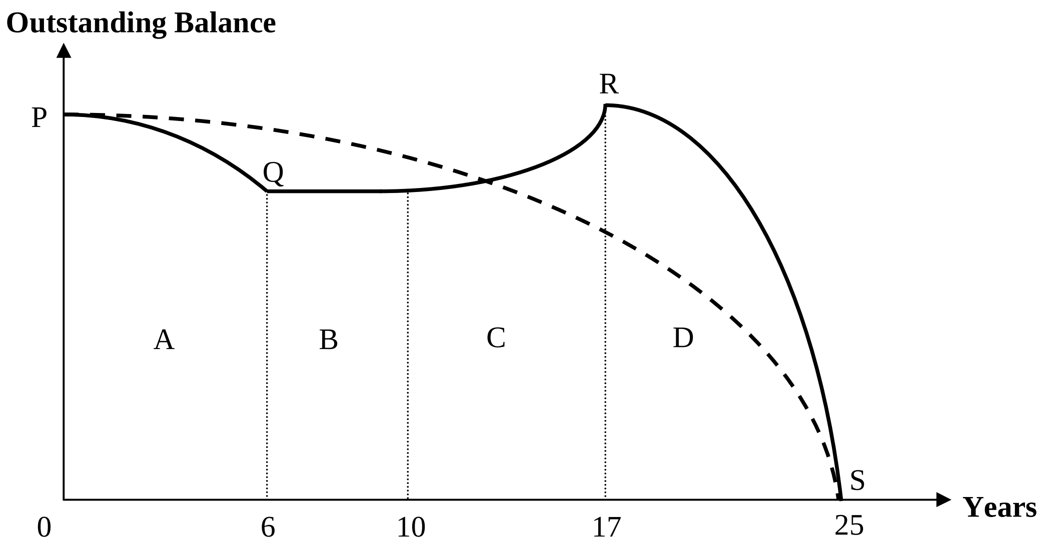
Total for Module 2: 100

MODULE 3 FINANCE AND MODELLING

QUESTION 1

1.1 A vehicle, bought for P rand, depreciates on a reducing balance at 20% per annum for the first five years and 40% per annum for the following two years. What is the average annual rate of depreciation over the seven years? (6)

1.2



Sanjay takes out a loan, P, with the intention of paying it back with monthly instalments over a period of 25 years. The continuous curve PQRS represents Sanjay's Outstanding Balance over the 25 year period. The ideal curve of Outstanding Balance is given as the dotted line PS.

With reference to the time periods labelled A, B, C and D, interpret the shape of the curve and suggest possible reasons why Sanjay's curve of Outstanding Balance did not follow the recommended path. (8)

14 marks

QUESTION 2

Philippa wins R6 000 000 on the lottery. She decides to invest the money, give up working and travel. She draws R55 000 per month from her winnings, **starting in four months' time**. The interest rate on the investment is 9½% per annum, interest being compounded monthly.

2.1 Calculate for how many years Philippa's investment will support her. (10)

Two years after winning the lottery, Philippa decides to return to work and spend what is left of her investment on a townhouse.

2.2 Calculate the value of the townhouse that Philippa will be able to afford. (12)

22 marks

QUESTION 3

Bongiwe makes **three** monthly deposits of R25 000 at the beginning of each month for three months, **starting immediately**. She then leaves her money in the bank without making any further deposits.

Three months after Bongiwe makes her first payment, Dudu starts a monthly annuity.

The interest rate on both accounts is 16% per annum, compounded monthly.

3.1 Calculate the value of Bongiwe's investment 5 years from now. (8)

3.2 Determine the size of Dudu's monthly payment such that both investments will be worth the same amount 5 years from now. (9)

17 marks

QUESTION 4

4.1 A recursive formula $T_{k+1} = q.T_k - 3T_{k-1}$ produces the sequence which begins:

p, 6, 30, 132, ...

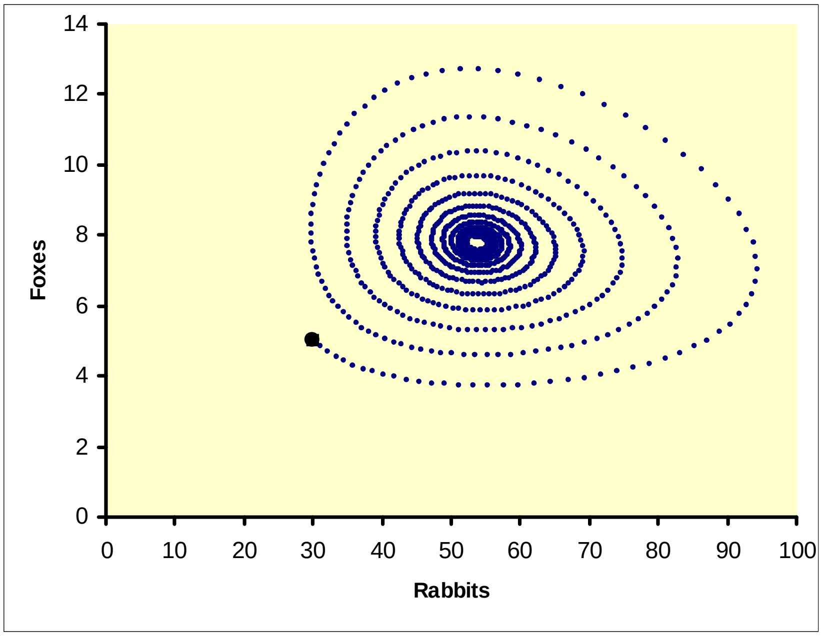
Determine the values of p and q. (8)

4.2 Frikkie starts with an investment of R100. Each month he increases his deposit by 10% of the previous month's deposit. The interest on his investment is 14% per annum compounded monthly. Write a recursive rule for the investment. (7)

15 marks

QUESTION 5

The Predator-Prey model describes the behaviour of, for example, Foxes and Rabbits in an enclosed environment. The graph of the growth/ decline of some rabbits and foxes in a particular environment is given below.



5.1 (a) Estimate the initial number of foxes and rabbits. (2)

(b) Does the graph suggest that either animal will eventually become extinct? Explain what is happening over a long period of time. (4)

(c) Estimate the eventual number of foxes and rabbits i.e. the **equilibrium point.** (2)

(d) Give the domain and range on which the population of rabbits is increasing and foxes decreasing. (4)

5.2 The formulae governing the number of rabbits and foxes are given as:

$$R_{n+1} = R_n + a.R_n.\left(1 - \frac{R_n}{K}\right) - b.R_n.F_n \quad \text{and} \quad F_{n+1} = F_n + f.b.R_n.F_n - c.F_n$$

(a) Explain the presence of the term bR_nF_n and interpret the value K. (3)

(b) Calculate the equilibrium point for the above model given the following parameters:

$a = 0{,}64 \qquad b = 0{,}008 \qquad c = 0{,}048 \qquad K = 400 \qquad f = 0{,}12$

(8)

23 marks

QUESTION 6

Shaun is not the cleanest student in the 'res', and a colony of ants has decided to target the steady supply of crumbs on his floor. Being more astute than he is hygienic, Shaun predicts that the colony cannot increase in number indefinitely as there is only a limited supply of crumbs. Hence the number of ants will tend to a bearable limit.

6.1 Is the growth of ants a Malthusian or a Logistic Model? Give a reason. (2)

6.2 Assume that the growth model is given by:

$$P_{n+1} = P_n + rP_n\left(1 - \frac{P_n}{2500}\right) \text{ where } n \text{ is in days.}$$

and with $P_0 = 250$ and $P_1 = 375$.

(a) What is the limit of the number of ants? (2)

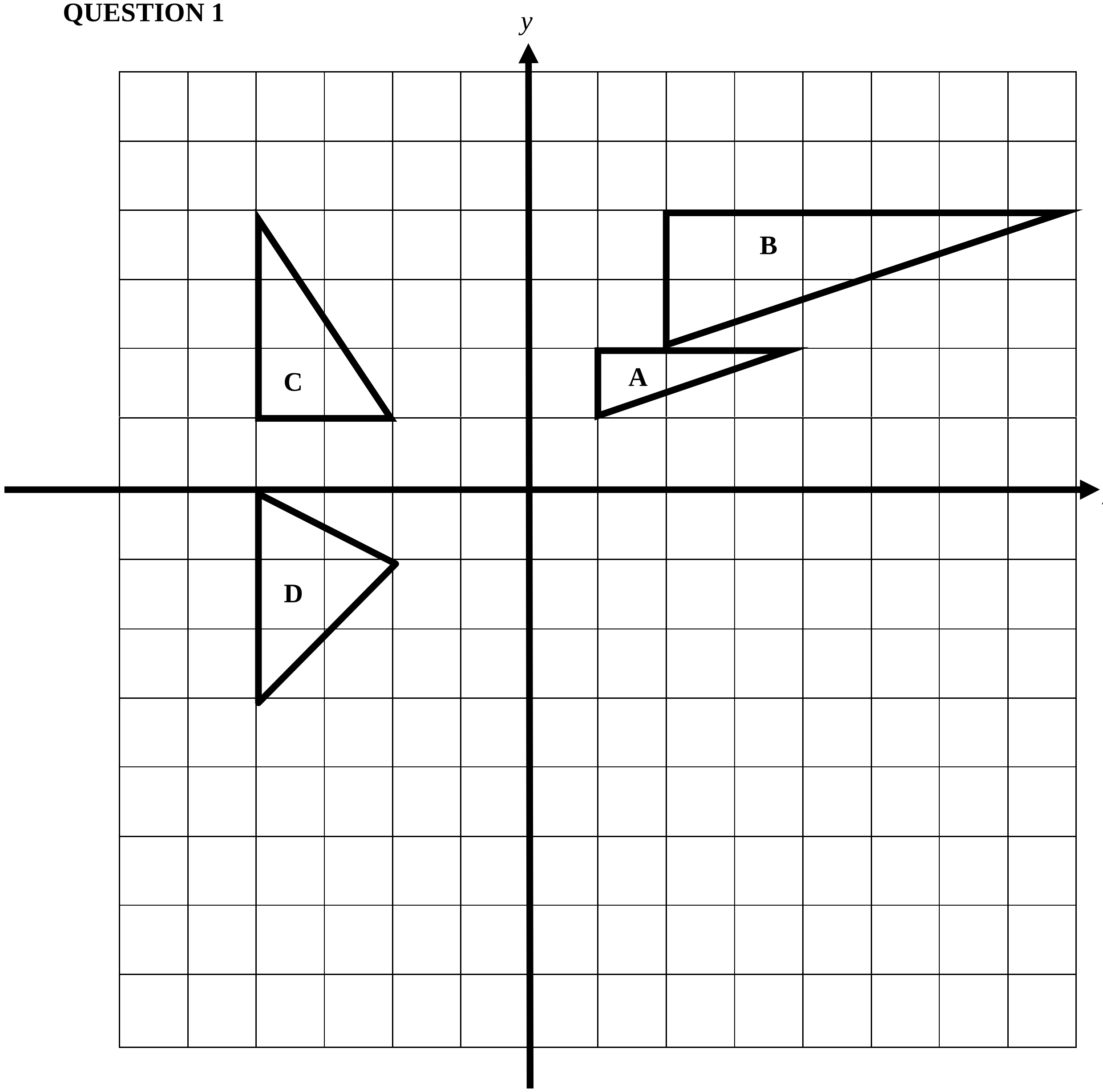
(b) Determine the value of r. (5)

9 marks

Total for Module 3: 100

MODULE 4 MATRICES AND GRAPH THEORY

QUESTION 1



1.1 Describing the transformation, write down the matrix which maps:

(a) shape A on to shape B. (4)

(b) shape C on to shape D. (5)

1.2 (a) Describe two transformations which, when combined, will map shape C onto shape B. (5)

(b) Find the single transformation matrix which maps C onto B. (6)

1.3 Determine the image of the point (10 ; 6) under a rotation of 60° anticlockwise. Leave your answer in surd form. (6)

26 marks

QUESTION 2

2.1 Determine the inverse of the matrix

$$\begin{pmatrix} 4 & 2 & -1 \\ 2 & -3 & 0 \\ 0 & 2 & -3 \end{pmatrix}$$

(10)

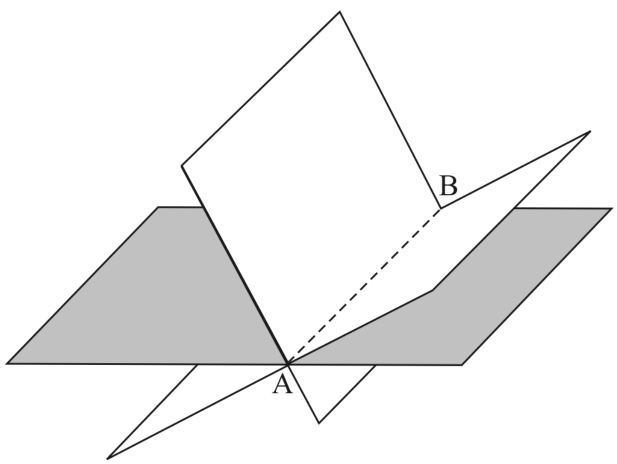
2.2 Using the answer in 2.1, or otherwise, solve the following set of equations:

$$\begin{aligned} 4x + 2y - z &= 3 \\ 2x - 3y &= 4 \\ 2y - 3z &= 7 \end{aligned}$$

(6)

16 marks

QUESTION 3



Three non-parallel planes are shown intersecting along a common line AB.

Determine which set(s) of equations might represent the situation illustrated in the diagram.
Note: It is not necessary to solve each set of equations.

(1) $$\begin{aligned} 2x + 4y - z &= 2 \\ 4x + 8y - 2z &= 6 \\ x - 3y + z &= 10 \end{aligned}$$

(2) $$\begin{aligned} 2x + 4y - z &= 5 \\ 4x + 7y - 2z &= 9 \\ x - 3y + z &= -1 \end{aligned}$$

(3) $$\begin{aligned} 5x + 4y - z &= 5 \\ 4x + 7y - 2z &= 9 \\ x - 3y + z &= -1 \end{aligned}$$

Clearly justify your answer.

10 marks

QUESTION 4

A *simple* graph is one in which any two vertices are directly connected by at most one arc and no vertex is directly connected to itself.

A *connected* graph is one in which every vertex is connected, directly or indirectly, to every other vertex.

A *simply connected* graph is one that is both simple and connected.

A simply connected graph is drawn with 6 vertices and 9 edges.

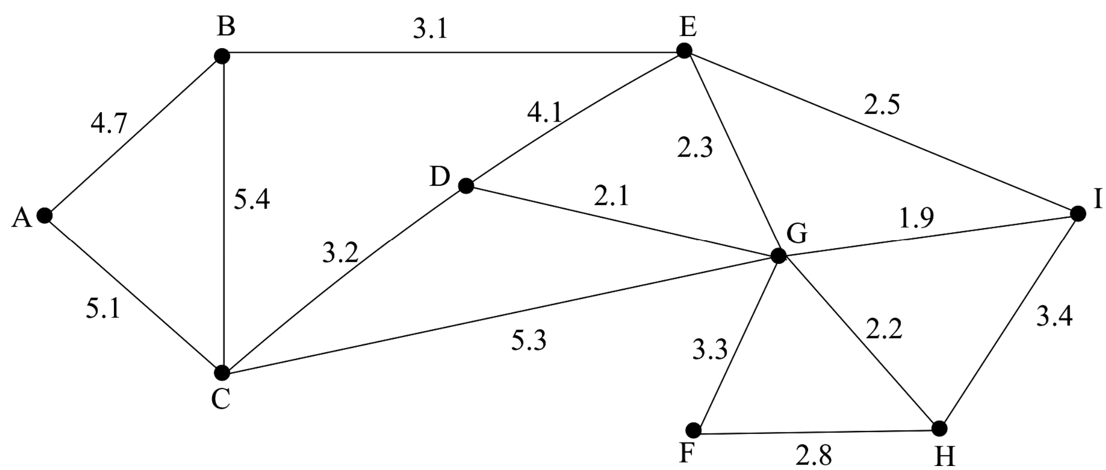
4.1 What is the sum of the orders of the vertices? (2)

4.2 Explain why if the graph has two vertices of order 5 it cannot have any vertices of order 1. (3)

4.3 Draw an example of a simply connected graph with 6 vertices and 9 edges in which one of the vertices has order 5 and all the orders of the vertices are odd numbers. (5)

10 marks

QUESTION 5



The network above shows the 'major' dirt roads that are to be graded by a local council in the Karoo. The number on each edge is the length of the road in kilometers.

5.1 List the vertices that have an odd order. (2)

5.2 Starting and finishing at A, find a route of minimum length that covers every road at least once. You should clearly indicate which, if any, roads will be travelled twice. (14)

5.3 State the total length of your shortest route. (4)

5.4 There is a 6,4 km long minor road (not shown on the network) between B and D. Decide whether or not it is sensible to include BD as part of the main grading route. Give reasons for your answer. (6)

26 marks

QUESTION 6

The management of a large theme park asks for your help to network five computers, one computer at the office and one each at the four entrances. Laying computer cable is expensive so they require you to find the minimum total length of cable required to network the computers.

The adjacency matrix shows the shortest distance, in metres, between the various sites.

	Office	**Entrance 1**	**Entrance 2**	**Entrance 3**	**Entrance 4**
Office	–	1514	488	980	945
Entrance 1	1514	–	1724	2446	2125
Entrance 2	488	1724	–	884	587
Entrance 3	980	2446	884	–	523
Entrance 4	945	2125	587	523	–

6.1 Starting at entrance 2, demonstrate the use of Prim's algorithm and hence find a minimum spanning tree. You must communicate your method fully, indicating the order in which you selected the edges of the network. (10)

6.2 Calculate the minimum total length of the cable required. (2)

12 marks

Total for Module 4: 100

Total: 300 marks