HILTON COLLEGE

TRIAL EXAMINATION
AUGUST 2012

MATHEMATICS: PAPER I

Time: 3 hours 200 marks

GENERAL INSTRUCTIONS

PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY.

1. This question paper consists of 7 pages and a separate formula sheet. Please check that your paper is complete.
2. Read the questions carefully.
3. This question paper consists of 12 questions. Answer all questions.
4. Number your answers exactly as the questions are numbered.
5. You may use an approved non-programmable and non-graphical calculator, unless a specific question prohibits the use of a calculator.
6. Round off your answers to one decimal digit where necessary, unless otherwise stated.
7. All necessary working details must be shown.
8. It is in your own interest to write legibly and to present your work neatly.
9. Please note that the diagrams are **NOT** necessarily drawn to scale.

Please do not turn over this page until you are asked to do so

SECTION A : ALGEBRA AND CALCULUS

QUESTION 1

(a) Determine $\frac{dy}{dx}$. You need not simplify your answers.

(1) $y = x\tan x$ (4)

$$\frac{dy}{dx} = \tan x + x\sec^2 x$$

(2) $y = \frac{x^2+5}{\sqrt{2x+3}}$ (8)

$$\frac{dy}{dx} = \frac{2x\sqrt{2x+3} - \frac{1}{2}(2x+3)^{-\frac{1}{2}}2\left(x^2+5\right)}{2x+3}$$

(b) Determine the equation of the tangent to the curve $x^2y + y^2 = 8 + 2x$ at the point $(-1;2)$ (8)

$$x^2y + y^2 = 8 + 2x$$

$$\therefore 2xy + x^2\frac{dy}{dx} + 2y\frac{dy}{dx} = 2$$

$$\therefore \frac{dy}{dx} = \frac{2-2xy}{x^2+2y}$$

$$at \quad (-1;2) \quad \frac{dy}{dx} = \frac{6}{5}$$

$$\therefore y - 2 = \frac{6}{5}(x+1)$$

$$\therefore y = \frac{6}{5}x + 3\frac{1}{5}$$

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QUESTION 2

(a) Determine the following limits:

(1) $\lim_{x \to 0} \frac{2\sin 5x}{3x}$ (5)

$= \frac{2}{3} \lim_{x \to 0} \frac{\sin 5x}{x}$

$= \frac{2}{3} \times 5 \lim_{5x \to 0} \frac{\sin 5x}{5x}$

$= \frac{10}{3}$

(2) $\lim_{x \to \infty} \frac{2x^2 + 5x}{x(x+1)+2}$ (6)

$= \lim_{x \to \infty} \frac{2x^2 + 5x}{x^2 + x + 2}$

$= \lim_{x \to \infty} \frac{2 + \frac{5}{x}}{1 + \frac{1}{x} + \frac{2}{x^2}}$

$= 2$

(b) Consider the function

$$f(x) = \begin{cases} x^2 + 2, & x \le 2 \\ ax - 4, & x > 2 \end{cases}$$

(1) Determine the value of *a* if the function is continuous at $x = 2$. (3)

$\lim_{x \to 2^-} f(x) = 6$

so $\lim_{x \to 2^+} (ax - 4) = 6$

so $2a - 4 = 6$

$\therefore a = 5$

(2) Using your value of a from (1) determine whether or not the function is differentiable at $x = 2$ (4)

$\lim_{x \to 2^-} f'(x)$

$= \lim_{x \to 2^-} (2x)$

$= 4$

but $\lim_{x \to 2^+} f'(x) = 5$

so $\lim_{x \to 2} f'(x)$ *does not exist*

and $f(x)$ *is not differentiable at* $x = 2$

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QUESTION 3

(a) The equation $x^3 - x^2 + x + 39$ has $2 + 3i$ as one of its roots.
Solve the equation in $\mathbb{C}$. (5)

if $2+3i$ *is a root then so is* $2-3i$

and $(x-2+3i)$ *and* $(x-2-3i)$ *are factors*

as is $\left((x-2)^2 - 9i^2\right) = \left(x^2 - 4x + 13\right)$

$x^3 - x^2 + x + 39$

$= \left(x^2 - 4x + 13\right)(x+3)$

$\therefore x = -3$ *or* $2 \pm 3i$

(b) Simplify, giving your answers in standard $a + bi$ form. Show working!

(1) $(2+3i)(4-5i)$ (3)

$= 8 + 2i - 15i^2$

$= 23 + 2i$

(2) $\dfrac{5+5i}{2-i}$ (4)

$= \dfrac{5+5i}{2-i} \times \dfrac{2+i}{2+i}$

$= \dfrac{10+15i+5i^2}{4-i^2}$

$= \dfrac{5+15i}{5}$

$= 1 + 3i$

(c) Decompose $\frac{7x^2+12x+8}{(x+3)(x^2+x+1)}$ into partial fractions. (8)

$=\frac{A}{x+3}+\frac{Bx+C}{x^2+x+1}$

$by \quad \text{cov}er \quad up \quad method, \quad A=5$

$5(x^2+x+1)+(Bx+C)(x+3)$

$5+B=7 \quad so \quad B=2 \quad and \quad C=1$

$\therefore \frac{5}{x+3}+\frac{2x+1}{x^2+x+1}$

(d) Solve:

(1) $|x|^2-4|x|-12=0$ (6)

$\therefore (|x|-6)(|x|+2)=0$

$\therefore x=\pm 6 \quad or \quad x=\pm 2$

(2) $\frac{e^x}{e^x-1}=5$ (to 2 decimal places) (5)

$\frac{e^x}{e^x-1}=5$

$\therefore e^x=5e^x-5$

$\therefore 4e^x=5$

$\therefore e^x=\frac{5}{4}$

$\therefore x=\ln\frac{5}{4}\approx 0.22$

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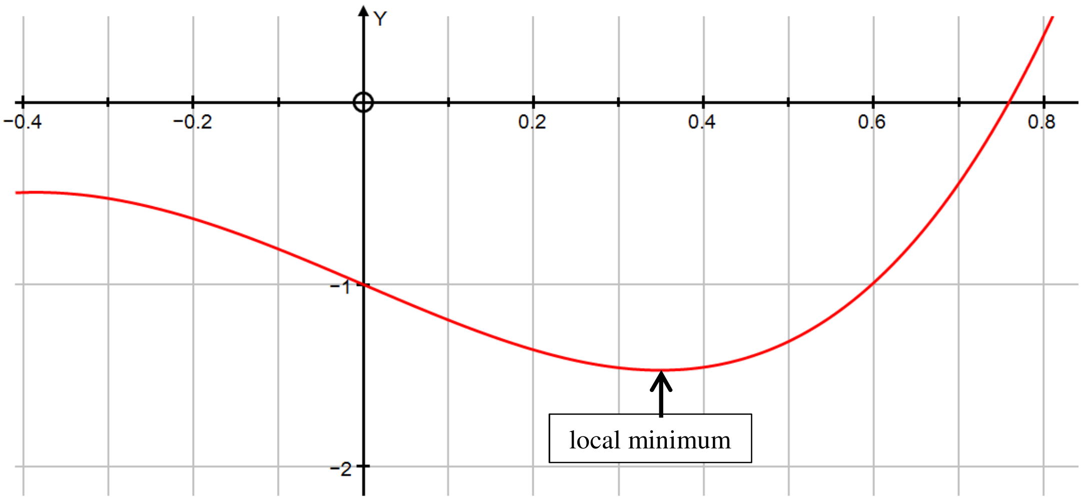
QUESTION 4

(a) A portion of graph of $f(x) = x^4 + 5x^3 - 2x - 1$ is shown below:



Use the Newton-Raphson method with an initial approximation of 1 to find the x-coordinate of the local minimum shown above. Give your answer correct to 4 decimal places. (10)

$$f(x) = x^4 + 5x^3 - 2x - 1$$

$$f'(x) = 4x^3 + 15x^2 - 2 = 0$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\therefore x_{n+1} = x_n - \frac{4x_n^{\ 3} + 15x_n^{\ 2} - 2}{12x_n^{\ 2} + 30x_n}$$

$$x_0 = 1$$

$$x_1 = \frac{25}{42}$$

$$to \quad 4\,d.p. \quad x = 0.3492$$

(b) Consider the function $f(x)=\frac{x^2+2x}{-4x+8}$

(1) Determine the coordinates of the intercepts with both axes. (6)

$f(x)=\frac{x^2+2x}{-4x+8}$

$y-int \quad (0;0)$

$x-intercepts:$

$x^2+2x=0$

$\therefore x(x+2)=0$

$\therefore x=0 \quad or \quad -2$

(2) Determine the equation of the oblique asymptote. (10)

$f(x)=\frac{x^2+2x}{-4x+8}$

$\therefore x^2+2x=(-4x+8)\left(-\frac{1}{4}x-1\right)+R$

$\therefore \frac{x^2+2x}{-4x+8}=-\frac{1}{4}x-1+\frac{R}{-4x+8}$

$so, \quad the \quad oblique \quad asymptote \quad is \quad y=-\frac{1}{4}x-1$

(3) Determine the equations of any vertical asymptotes. (6)

$vertical \quad asymptotes \quad when \quad -4x+8=0$

$\therefore x=2$

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QUESTION 5

(a) Use a Riemann sum to determine the area between the curve $y = x^2 + x$ and the x axis on the interval $[1;4]$ (13)

$$Area = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x_i$$

$$\Delta x_i = \frac{3}{n}$$

$$x_i = 1 + \frac{3i}{n}$$

$$f(x_i) = \left(1+\frac{3i}{n}\right)^2 + \left(1+\frac{3i}{n}\right) = 2 + \frac{9i}{n} + \frac{9i^2}{n^2}$$

$$f(x_i)\Delta x_i = \frac{3}{n}\left(2+\frac{9i}{n}+\frac{9i^2}{n^2}\right) = \frac{6}{n}+\frac{27i}{n^2}+\frac{27i^2}{n^3}$$

$$\therefore Area = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x_i$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{6}{n}+\frac{27i}{n^2}+\frac{27i^2}{n^3}\right)$$

$$= \lim_{n\to\infty}\left(6\sum_{i=1}^{n}\left(\frac{1}{n}\right)+\frac{27}{n^2}\sum_{i=1}^{n} i+\frac{27}{n^3}\sum_{i=1}^{n} i^2\right)$$

$$= \lim_{n\to\infty}\left(6+27\left(\frac{n^2}{2n^2}+\frac{n}{2n^2}\right)+27\left(\frac{n^3}{3n^3}+\frac{n^2}{2n^3}+\frac{n}{6n^3}\right)\right)$$

$$= \lim_{n\to\infty}\left(6+\frac{27}{2}+\frac{27}{2n}+9+\frac{27n^2}{2n^3}+\frac{27n}{6n^3}\right)$$

$$= 28.5 \quad units^2$$

(b) Determine the following integrals, showing all working details:

(1) $\int_{-1}^{3} 2x^2 + 5x\,dx$ (6)

$$= \left[\frac{2x^3}{3} + \frac{5x^2}{2}\right]_{-1}^{3}$$

$$= \left(18 + \frac{75}{2}\right) - \left(-\frac{2}{3} + \frac{5}{2}\right)$$

$$= 38\frac{2}{3}$$

(2) $\int \sin 3x \cos 2x\,dx$ (8)

$$= \frac{1}{2}\int \left(\sin(5x) + \sin x\right) dx$$

$$= -\frac{\cos(5x)}{10} - \frac{\cos x}{2} + c$$

(3) $\int \left(3x^2 + 5\right)\sqrt{x^3 + 5x + 7}\,dx$ (8)

$$= \frac{2\left(x^3 + 5x + 7\right)^{\frac{3}{2}}}{3} + c$$

(4) $\int x \sin x\,dx$ (8)

$$= -x\cos x + \int cosx\,dx$$

$$= -x\cos x + \sin x + c$$

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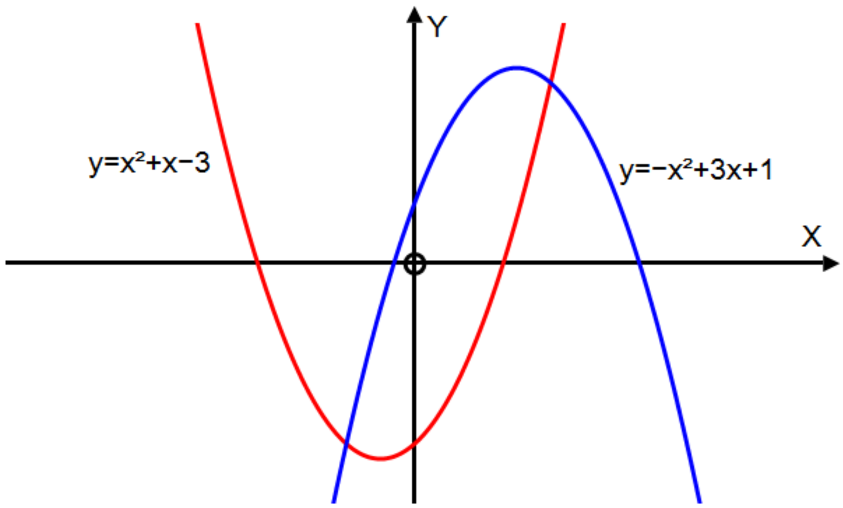
QUESTION 6

(a) Determine the area enclosed between the two curves shown below: (14)



points of intersection

$x+x-3=-x^2+3x+1$

$\therefore 2x^2-2x-4=0$

$\therefore x^2-x-2=0$

$\therefore (x-2)(x+1)=0$

$\therefore x=2 \quad or \quad -1$

$$\therefore Area=\int_{-1}^{2} -x^2+3x+1-\left(x^2+x-3\right)dx$$

$$=\int_{-1}^{2} -2x^2+2x+4\,dx$$

$$=\left[-\frac{2x^3}{3}+x^2+4x\right]_{-1}^{2}$$

$$=\left(-\frac{16}{3}+4+8\right)-\left(\frac{2}{3}+1-4\right)$$

$$=9 \quad units^2$$

(b) Determine the volume when the area enclosed between the x-axis and the curve $y = x^2 - 4$ is rotated about the x-axis. (14)

limits of integration are -2 and 2

$$V = \pi \int_{-2}^{2} \left(x^2 - 4\right)^2 dx$$

$$= \pi \int_{-2}^{2} x^4 - 8x^2 + 16\, dx$$

$$= \pi \left[\frac{x^5}{5} - \frac{8x^3}{3} + 16x \right]_{-2}^{2}$$

$$= 34.13\pi$$

$$= 107.2 \quad units^3$$

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QUESTION 7

(a) Convert $120°$ to radians. (2)

$$\frac{2\pi}{3}$$

(b) Convert $\frac{7\pi}{6}$ radians to degrees. (2)

$210°$

(c) Determine the general solution to $\sec^2 x - \sec x - 2 = 0$ in radians. (8)

$$\therefore \left(\sec x - 2\right)\left(\sec x + 1\right) = 0$$

$$\therefore \sec x = 2 \quad or \quad \sec x = -1$$

$$\therefore \cos x = \frac{1}{2} \quad or \quad \cos x = -1$$

$$\therefore x = \frac{\pi}{3} + 2\pi k \quad or \quad \frac{5\pi}{3} + 2\pi k \quad or \quad \pi + 2\pi k \quad with \quad k \in \mathbb{Z}$$

(d) Given that $\dfrac{1}{\tan x\sqrt{1-\cos^2 x}} = \csc x \cot x$

(1) Prove the identity (8)

$$LHS = \frac{1}{\tan x\sqrt{1-\cos^2 x}}$$
$$= \frac{1}{\frac{\sin x}{\cos x}\sqrt{\sin^2 x}}$$
$$= \frac{\cot x}{\sin x}$$
$$= \csc x \cot x$$
$$= RHS$$

(2) Use this result to determine $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{\tan x\sqrt{1-\cos^2 x}}\,dx$ (8)

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{\tan x\sqrt{1-\cos^2 x}}\,dx$$
$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \csc x \cot x\,dx$$
$$= -\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} -\csc x \cot x\,dx$$
$$= -\left[\csc x\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$
$$= -(1-2)$$
$$= 1$$

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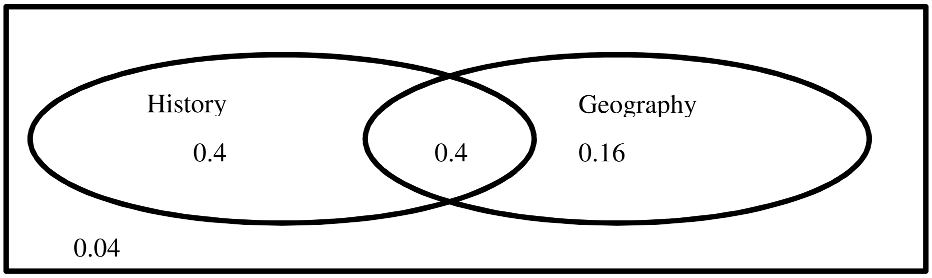
SECTION B : STATISTICS

QUESTION 8

The probability that a boy studies Geography is 0.56 while the probability that he studies History is 0.8. If the probability that he studies both is 0.4 then determine:



a. The probability that he studies only History? (4)

40%

b. The probability that he studies neither Geography nor History? (4)

4%

c. The probability that he studies Geography or History. (4)

96%

d. $P(History \mid Geography)$ (4)

$$P(History \mid Geography) = \frac{P(History \cap Geography)}{P(Geography)}$$

$$= \frac{0.4}{0.56} = 71.4\%$$

e. Are the events "studying History" and "studying Geography" independent? Justify your answer. (4)

No, $P(History \cap Geography) \neq P(History) \times P(Geography)$

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QUESTION 9

A bag contains 1 yellow bead, 3 red beads and 5 green beads.

a. Three beads are removed without replacement.

(1) Give the probability that one bead of each colour is drawn. (7)

$$\frac{\binom{1}{1}\binom{3}{1}\binom{5}{1}}{\binom{9}{3}}$$
$$=17.9\%$$

(2) Give the probability distribution for the number of red beads. (10)

$$P(R=r) = \begin{cases} \dfrac{\binom{6}{3-r}\binom{3}{r}}{\binom{9}{3}}, & r\in[0;3] \\ 0, & elsewhere \end{cases}$$

b. A bead is drawn, its colour noted and it is replaced. This is done three times.

(1) Give the probability of getting at least one green bead. (7)

$$P(at \quad least \quad one \quad green)=1-P(no \quad greens)$$
$$=1-\left(\frac{4}{9}\right)^3$$
$$=91.2\%$$

(2) Give the probability distribution for the number of green beads. (10)

$$P(R=r) = \binom{3}{r}\left(\frac{5}{9}\right)^r\left(\frac{4}{9}\right)^{3-r}, \quad r\in[0;3]$$
$$0, \quad elsewhere$$

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QUESTION 10

(a) The masses of tins of baked beans are normally distributed with a mean of 410g and a standard deviation of 3g.

Determine:

(1) The probability of a randomly selected tin weighing more than 412g. (6)

$$\begin{aligned} &P(x>412)\\ &=P\left(z>\frac{2}{3}\right)\\ &=1-P\left(z<\frac{2}{3}\right)\\ &=1-(0.5+0.2486)\\ &=25.1\% \end{aligned}$$

(2) The probability that a randomly selected tin weighs between 408g and 411g. (10)

$$\begin{aligned} &P(408<x<411)\\ &=P\left(-\frac{2}{3}<z<\frac{1}{3}\right)\\ &=P\left(z<\frac{2}{3}\right)+P\left(z<\frac{1}{3}\right)\\ &=0.2486+0.1293\\ &=37.8\% \end{aligned}$$

(b) In a sample of 100 boys 13 are found to be left-handed. Find a 95% confidence interval for the proportion of left handers in the population. (8)

$$P\left(P-1.96\sqrt{\frac{P(1-P)}{n}}<\pi<P+1.96\sqrt{\frac{P(1-P)}{n}}\right)=0.95$$

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$$\therefore P\left(0.13-1.96\sqrt{\frac{0.13\times0.87}{100}}<\pi<0.13+1.96\sqrt{\frac{0.13\times0.87}{100}}\right)=0.95$$

$$\therefore P(0.064<\pi<0.196)=0.95$$

QUESTION 11

To test the effect of alcohol on reaction time, a sample of 14 students was selected. They were asked to perform a test before alcohol and after. The test was scored out of 100. A lower score indicates a lower reaction time.

The results are as follows:

Student	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Initial score (I)	90	87	93	86	81	84	70	87	86	91	89	88	90	92
Final score (F)	75	72	76	70	68	73	56	80	73	79	79	69	77	80

(a) Determine the correlation coefficient. (6)

0.882

(b) What does this tell us about the nature and the strength of the linear relationship between I and F? (3)

There is a relatively strong positive correlation between the initial and final scores.

(c) Determine the equation of the least squares regression line of F on I. (6)

$F = 0.9806x - 11.676$

(d) Two further students Jon and Bob are tested and their initial scores are 85 and 64 respectively.

(1) Predict their final scores. (4)

Jon 71.7 and Bob 51.1

(2) Which prediction in d(1) would be more reliable? Why? (3)

Jon's prediction is more reliable as we are extrapolating with Bob.

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